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THE SET OF PERIODS OF PERIODIC POINTS OF A TORAL AUTOMORPHISM

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Abstract. In this article, subsets of $\mathbb{N}$ that can arise as sets of periods of a continuous 2-dimensional toral automorphism are characterized. Here the torus $\mathbb{T}^2$ is viewed as $[0,1)\times[0,1)$ as a group under coordinatewise addition modulo 1.

1. Introduction

There have been a lot of papers characterizing the sets of periods for various classes of self maps, like (i) Continuous self maps of the real line $\mathbb{R}$ (See [4], [5]), (ii) Polynomials on $\mathbb{C}$ (See [1]), (iii) Totally transitive maps on I (See [1]) and (iv) Degree one maps on S^1 (See [8]). In this paper, we determine the same for continuous 2-dimensional toral automorphisms.

A dynamical system is a pair (X, f), where X is a metric space and f is a continuous map. We say that a point $x \in X$ is *periodic* if $f^n(x) = x$ for some $n \in \mathbb{N}$, where f^n is the composition of f with itself $n-1$ times. The smallest such positive integer n is called the *period* of x. Let $\mathcal{P}_n = \{x \in X \mid x \text{ is a periodic point with } \textit{period } n\}$. Let $Per(f) = \{n \in \mathbb{N} \mid \mathcal{P}_n \neq \emptyset\}$, called the *set of periods* of periodic points of f.

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Our main results prove that the set of periods of a continuous 2-dimensional toral automorphism has to be one of the following eight subsets of $\mathbb{N}$

(1) $\{1\}$
(2) $\{1, 2\}$
(3) $\{1, 3\}$
(4) $\{1, 2, 4\}$
(5) $\{1, 2, 3, 6\}$
(6) $2\mathbb{N} \cup \{1\}$
(7) $\mathbb{N} \setminus \{2\}$
(8) $\mathbb{N}$.

Here the torus $\mathbb{T}^2$ is viewed as $[0, 1) \times [0, 1)$ as a group under coordinatewise addition modulo 1.

We now have an example. (For a proof, refer to proposition 3.6.)

Example. If $T : \mathbb{T}^2 \to \mathbb{T}^2$ is defined as,

(1) $T(x, y) = (x, y)(\bmod 1)$ then $Per(T) = \{1\}$.
(2) $T(x, y) = (x, x - y)(\bmod 1)$ then $Per(T) = \{1, 2\}$.
(3) $T(x, y) = (-x + y, -x)(\bmod 1)$ then $Per(T) = \{1, 3\}$.
(4) $T(x, y) = (x - 2y, x - y)(\bmod 1)$ then $Per(T) = \{1, 2, 4\}$.
(5) $T(x, y) = (x - y, x)(\bmod 1)$ then $Per(T) = \{1, 2, 3, 6\}$.
(6) $T(x, y) = (-x + y, -y)(\bmod 1)$ then $Per(T) = 2\mathbb{N} \cup \{1\}$.
(7) $T(x, y) = (x + y, x)(\bmod 1)$ then $Per(T) = \mathbb{N} \setminus \{2\}$.
(8) $T(x, y) = (x + 2y, x + y)(\bmod 1)$ then $Per(T) = \mathbb{N}$.

Observe that, for any continuous toral automorphism T, we have $1 \in Per(T)$, as $T(\mathbf{0}) = \mathbf{0}$, where $\mathbf{0} = (0, 0) \in \mathbb{T}^2$.

We now discuss some basic known results which we use in the later sections. The proofs are given for the sake of completeness and self containment.

2. Basic results

Let $GL(2, \mathbb{Z})$ be the set of all 2×2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in \mathbb{Z}$ and $Det(A) = ad - bc = \pm 1$.

Each such matrix A gives a linear map on $\mathbb{R}^2$ by $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

We define an automorphism on the torus $T_A : \mathbb{T}^2 \to \mathbb{T}^2$ by $T_A(x_1, x_2) = (ax_1 + bx_2, cx_1 + dx_2)(\text{mod } 1)$.

Now we have,

Proposition 2.1. (See [6], [9]) *Every automorphism T_A (as defined above) on the torus is a homeomorphism.*

Proof. The map T is clearly continuous, since if $|x_1-y_1|, |x_2-y_2| < \epsilon$ then $|(T_A(x_1, x_2))_1-(T_A(y_1, y_2))_1| < (|a|+|b|)\epsilon$ and $|(T_A(x_1, x_2))_2-(T_A(y_1, y_2))_2| < (|c| + |d|)\epsilon$. (Here suffix 1 refers to the first coordinate and 2 refers to the second coordinate.)

To see that T_A is invertible we note that if we write the inverse matrix $A^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$ then since $ad - bc = \pm 1$ we see that $A^{-1} \in GL(2, \mathbb{Z})$. The inverse to T_A is then the toral automorphism associated to A^{-1}, i.e. $T_{A^{-1}}$. □

On the other hand, in the following proposition we prove that every continuous automorphism on the torus is induced by a matrix from $GL(2, \mathbb{Z})$. Let $Aut(\mathbb{T}^2)$ to denote the set of all continuous automorphisms on the torus.

Proposition 2.2. (see [9]) *The above map $A \mapsto T_A$ from $GL(2, \mathbb{Z})$ to $Aut(\mathbb{T}^2)$ is surjective.*

Proof. Let $\phi : \mathbb{T}^2 \to \mathbb{T}^2$ be any continuous toral automorphism. Since ϕ is continuous at $(0, 0)$ there exists $0 < \delta < \frac{1}{2}$ such that $\phi(([0, \delta) \times [0, \delta))) \subset [0, \frac{1}{2}) \times [0, \frac{1}{2})$ and such that $\phi(X+Y) = \phi(X)+\phi(Y)$ for all $X, Y \in [0, \delta) \times [0, \delta)$, where $+$ denotes the usual addition in $\mathbb{R}^2$.

Note that, if $X \in [0, \delta) \times [0, \delta)$ then $\frac{1}{2^n}X \in [0, \delta) \times [0, \delta)$ for all $n \in \mathbb{N}$. For any $X \in [0, \delta) \times [0, \delta)$, $\phi(x) = \phi(\frac{X}{2} + \frac{X}{2}) = \phi(\frac{X}{2}) + \phi(\frac{X}{2}) = 2\phi(\frac{X}{2})$ and hence $\phi(\frac{X}{2}) = \frac{1}{2}\phi(X)$. By induction on n, we can prove that $\phi(\frac{1}{2^n}) = \frac{1}{2^n}\phi(X)$ for all $n \in \mathbb{N}$ and then by using the additivity, we can show that $\phi(\frac{m}{2^n}) = \frac{m}{2^n}\phi(X)$ for all $m \in \{1, 2, \cdots, 2^n - 1\}$. Since the set of all dyadic rationals is dense in $[0, 1]$, by the continuity of ϕ we have $\phi(\lambda X) = \lambda\phi(X)$ for all $\lambda \in (0, 1)$.

Hence $\phi|_{[0,\delta)\times[0,\delta)} = L|_{[0,\delta)\times[0,\delta)}$ for some linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$. This linear transformation induces an integer matrix A with determinant ± 1 such that $Ax = \phi(x)$ for all $x \in \mathbb{T}^2$

[The kernel of an endomorphism (different from the zero map), on a connected topological group cannot have nonempty interior]. Hence the proof. □

Remark 2.3. (See [2])

In fact the 1-1 correspondence in the above proposition is a group isomorphism.

Note that, for a toral automorphism T_A, the periodic points with period n are solutions of the congruent equation $A^n x = x (mod 1)$. The following proposition is in this direction.

Lemma 2.4. (See [7]) *If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an isomorphism then for every Riemann measurable set $S \subset \mathbb{R}^2$, $T(S)$ is Riemann measurable and*

$$Area(T(S)) = |Det(T)| Area(S).$$

Proposition 2.5. (See [3], [9])

Let $A \in GL(2, \mathbb{Z})$. Then

(1) The number of solutions of $A^n x = x$ in $[0, 1) \times [0, 1)$, is $|Det(A^n - I)|$, provided $Det(A^n - I) \neq 0$.

(2) If $Det(A^n - I) = 0$ then $A^n x = x$ has infinitely many solutions in $[0, 1) \times [0, 1)$.

Proof. (1) Suppose $Det(A^n - I) \neq 0$. Then note that the number of solutions of the equation, $A^n x = x$ in $[0, 1) \times [0, 1)$ is equal to the number of integer points in the image of $[0, 1) \times [0, 1)$ under $A^n - I$, treated as a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Note also that the image of $[0, 1) \times [0, 1)$ under $A^n - I$ is a parallelogram and hence the number of integer points in it is equal to its area, which is equal to $|Det(A^n - I)|$ by previous lemma.

(2) Note that, when $Det(A^n - I) = 0$, the system $(A^n - I) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ itself, has infinitely many solutions in $[0, 1) \times [0, 1)$. □

3. Main Results

3.1. **Period sets of hyperbolic toral automorphisms.** A continuous toral automorphism T_A,(where $A \in GL(2, \mathbb{Z})$) is said to be *hyperbolic* if A has no eigen values with absolute value 1.

Proposition 3.1. (See [3])

For 2−dimensional hyperbolic toral automorphisms, the eigen values are real and irrational.

Proof. Let A be the matrix of a toral automorphism and $(x, y)(\neq (0, 0))$ be an eigenvector of A. If λ is the corresponding eigenvalue, we have

$$ax + by = \lambda x \text{ .....(1)}$$

$$cx + dy = \lambda y \text{ .....(2).}$$

First, $x \neq 0$; for if $x = 0$, then $b = 0$ from (1). $Det(A) = \pm 1$ and $b = 0$ implies $ad = \pm 1$. Therefore $d = \pm 1$ (since $y \neq 0$) and hence from (2) we obtain $\lambda = \pm 1$, contradiction to the hyperbolicity.

Next, letting $t = \frac{y}{x}$ and using it to eliminate x and y from (1) and (2), we get $a + bt = \lambda$ and $c + dt = \lambda$.

Now eliminating λ, we have a quadratic in t, namely $bt^2 + (a - d)t - c = 0$. The discriminant of this quadratic equation is $(a + d)^2 \pm 4$, never a perfect square unless $a + d = 0$. This is because the Diophantine equation $X^2 - Y^2 = 4$ has no integer solution except when $Y = 0$. But $a + d = 0$ implies the characteristic polynomial of A is equal to $x^2 \pm 1$. This is not possible since A is hyperbolic. Hence from equation (2) it follows that eigen values are irrational. □

Let p_n denote the number of solutions(in $\mathbb{T}^2$) of the equation $A^n X = X$ and let $q_n = |Trace(A^n)|$. Then it follows from proposition 2.5 that $p_n = |Det(A^n - I)|$. It is observed in the following lemma that the sequence (q_n) follows a nice pattern for hyperbolic automorphisms.

Proposition 3.2. *Let T_A be a hyperbolic toral automorphism induced by the matrix $A \in GL(2, \mathbb{Z})$. Let t denote the trace of A and let $q_n = |Tr(A^n)|$. Then $q_{n+1} + Det(A)q_{n-1} = |t|q_n$ for all $n \geq 2$.*

Proof. Let $\lambda \in \mathbb{R}$ be an eigen value of A such that $|\lambda| > 1$. (Which exists because T_A is hyperbolic.)

Case 1: When $Det(A) = 1$

Here the eigen values are λ and $\frac{1}{\lambda}$. Now for any $n \in \mathbb{N}$, we have $q_n = \lambda^n + \frac{1}{\lambda^n} > 1$ and hence ≥ 2 as trace is an integer.

subcase 1: Both the eigen values are positive

Here $t = Trace(A) = \lambda + \frac{1}{\lambda} > 0$.

Hence $|t|q_n = tq_n = (\lambda + \frac{1}{\lambda})(\lambda^n + \frac{1}{\lambda^n}))$

$$= (\lambda^{n+1} + \frac{1}{\lambda^{n+1}}) + (\lambda^{n-1} + \frac{1}{\lambda^{n-1}})$$
$$= q_{n+1} + q_{n-1} for all n \geq 2.$$

subcase 2: Both the eigen values are negative

Here $t = Trace(A) = \lambda + \frac{1}{\lambda} < 0$ and in this case,

$$q_n = \begin{cases} \lambda^n + \frac{1}{\lambda^n} & \text{if } n\ is\ even \\ -(\lambda^n + \frac{1}{\lambda^n}) & \text{if } n\ is\ odd \end{cases}$$

Now, when $n(\geq 2)$ is even, we have

$tq_n = (\lambda + \frac{1}{\lambda})(\lambda^n + \frac{1}{\lambda^n}) = (\lambda^{n+1} + \frac{1}{\lambda^{n+1}}) + (\lambda^{n-1} + \frac{1}{\lambda^{n-1}}) = -q_{n+1} - q_{n-1}$.

Thus, $q_{n+1} + q_{n-1} = -tq_n = |t|q_n$ for all $n \geq 2$.

Similarly, when $n(\geq 2)$ is odd,

$tq_n = -(\lambda + \frac{1}{\lambda})(\lambda^n + \frac{1}{\lambda^n}) = -(\lambda^{n+1} + \frac{1}{\lambda^{n+1}}) - (\lambda^{n-1} + \frac{1}{\lambda^{n-1}}) = -q_{n+1} - q_{n-1}$.

Thus $q_{n+1} + q_{n-1} = -tq_n = |t|q_n$ for all $n \geq 2$.

Case 2: When $Det(A) = -1$

Here the eigen values are λ and $-\frac{1}{\lambda}$. In this case, either $\lambda < -1$ or $\lambda > 1$.

Subcase 1: When $\lambda > 1$

Then $t = \lambda - \frac{1}{\lambda} > 0$. Also, for any $n \in \mathbb{N}$, we have

$$q_n = \begin{cases} \lambda^n + \frac{1}{\lambda^n} & \text{if } n\ is\ even \\ \lambda^n - \frac{1}{\lambda^n} & \text{if } n\ is\ odd \end{cases}$$

When $n(\geq 2)$ is odd,

$|t|q_n = tq_n = (\lambda - \frac{1}{\lambda})(\lambda^n - \frac{1}{\lambda^n}) = \lambda^{n+1} - \frac{1}{\lambda^{n-1}} - \lambda^{n-1} + \frac{1}{\lambda^{n+1}} = q_{n+1} - q_{n-1}$.

Similarly, When $n(\geq 2)$ is even we get

$|t|q_n = tq_n = (\lambda - \frac{1}{\lambda})(\lambda^n + \frac{1}{\lambda^n}) = \lambda^{n+1} + \frac{1}{\lambda^{n-1}} - \lambda^{n-1} - \frac{1}{\lambda^{n+1}} = q_{n+1} - q_{n-1}$.

Subcase 2: When $\lambda < -1$

Then $t = \lambda - \frac{1}{\lambda} < 0$.

Also, observe that

$$q_n = \begin{cases} \lambda^n + \frac{1}{\lambda^n} & \text{if } n\ is\ even \\ \frac{1}{\lambda^n} - \lambda^n & \text{if } n\ is\ odd \end{cases}$$

for all n.

When $n(\geq 2)$ is odd,
$tq_n = (\lambda - \frac{1}{\lambda})(\lambda^n - \frac{1}{\lambda^n}) = \frac{1}{\lambda^{n-1}} - \lambda^{n+1} - \frac{1}{\lambda^{n+1}} + \lambda^{n-1} = q_{n-1} - q_{n+1}$.
Thus $q_{n+1} - q_{n-1} = -tq_n = |t|q_n$ for all $n \geq 2$.
Similarly, we can prove the case when n is even.
Hence the proposition. □

Lemma 3.3. *For $n \geq 3$, $n^2 - 2 > 2n$.*

Proof. Proof follows by induction on n. □

Theorem 3.4. *Let T_A be a hyperbolic toral automorphism. Then $Per(T_A)$ is either $\mathbb{N}$ or $\mathbb{N} \setminus \{2\}$.*

Proof. Let $T_A : \mathbb{T}^2 \to \mathbb{T}^2$ be hyperbolic. Note that, to prove $n \in Per(T_A)$ it is enough to prove that $p_n > p_1 + p_2 + ... + p_{n-1}$, where p_k = the number of solutions of the equation $A^k X = X$ which is equal to $|Det(A^k - I)|$, by proposition 2.5.

In fact, it is enough to prove: $p_n > p_1 + p_2 + ... + p_{n-2}$, because if x is a point of period n for T_A and $T_A^m(x) = x$ then n must divide m.

Let α be an eigen value of A with $|\alpha| > 1$. By proposition 3.1, $\alpha \in \mathbb{R}$ and hence either $\alpha < -1$ or $\alpha > 1$.

Case(1) : $\alpha > 1$

Subcase(1) : $Det(A) = 1$

Then the eigen values are $\alpha, \frac{1}{\alpha}$. Then $t = \alpha + \frac{1}{\alpha} \geq 3$ as t is an integer and $(\sqrt{\alpha} - \frac{1}{\sqrt{\alpha}})^2 > 0$. By proposition 2.5, $p_n = q_n - 2$ $\forall\, n \in \mathbb{N}$. From proposition 3.2, we have $q_{n+1} + q_{n-1} = tq_n$ $\forall\, n \geq 2$.

Adding these equations for $n = 2, 3, \dots, k$ we get,

$$q_1 + q_2 + 2(q_3 + q_4 + \dots + q_{k-1}) + q_k + q_{k+1} = t(q_2 + q_3 + \dots + q_k)$$

i.e.,

$$q_{k+1} = -q_1 + (t-1)q_2 + (t-2)[q_3 + q_4 + \dots + q_{k-1}] + (t-1)q_k$$

$\geq -q_1 + 2q_2 + q_3 + \dots + q_{k-1} + 2q_k$, since $t \geq 3$.
$= (-q_1 + 2q_2) + q_3 + \dots + q_{k-1} + 2q_k$
$> q_1 + q_2 + \dots + q_{k-1} + q_k$ (since $t \geq 3$, by lemma-3.3 we have $t^2 - 2 > 2t$. Hence $q_2 = t^2 - 2 > 2t = 2q_1$ and therefore $q_2 - q_1 > q_1$.)

Therefore, $q_{k+1} - 2 > (q_1 - 2) + (q_2 - 2) + \dots + (q_{k-1} - 2) + (q_k - 2)$
i.e, $p_{k+1} > p_1 + p_2 + ... + p_k$ for all $k \geq 2$.
Hence $k \in Per(T_A)$ for all $k \geq 3$ and hence $Per(T_A) \supset \mathbb{N} \setminus \{2\}$.

Subcase 2 : $Det(A) = -1$

Then eigen values are $\alpha, \frac{1}{\alpha}$. Then $t = \alpha - \frac{1}{\alpha} > 0$ and hence $t \geq 1$, as t is an integer. Using proposition 2.5,

$$p_n = \begin{cases} q_n - 2 & \text{if } n \text{ is even} \\ q_n & \text{if } n \text{ is odd} \end{cases}$$

From proposition 3.2, we have $q_{n+1} - q_{n-1} = tq_n$ for all $n \geq 2$.

Adding these relations for $n = 2, 3, ..., k$, we get,

$$\begin{aligned} q_{k+1} &= q_1 + (t+1)q_2 + t(q_3 + q_4 + ... + q_{k-1}) + (t-1)q_k \\ &> q_1 + q_2 + ... + q_{k-1} \text{ (since } t \geq 1.) \end{aligned}$$

Hence $p_{k+1} > p_1 + p_2 + ... + p_{k-1}$, for all $k \geq 2$.

Which implies that $Per(T_A) \supset \mathbb{N} \setminus \{2\}$.

Case (2) : $\alpha < -1$

Subcase 1 : $Det(A) = 1$

Therefore the eigen values are $\alpha, \frac{1}{\alpha}$. Then $-t = -\alpha - \frac{1}{\alpha} \geq 3$ as $(\sqrt{-\alpha} - \frac{1}{\sqrt{-\alpha}})^2 > 0$ and t is an integer. Using proposition 2.5, we get,

$$p_n = \begin{cases} q_n - 2 & \text{if } n \text{ is even} \\ q_n + 2 & \text{if } n \text{ is odd} \end{cases}$$

From proposition 3.2, we have $q_{n+1} + q_{n-1} = -tq_n$ for all $n \geq 2$. Adding these relations for $n = 2, 3, ..., k$, we get,

$$q_1 + q_2 + 2(q_3 + q_4 + ... + q_{k-1}) + q_k + q_{k+1} = -t(q_2 + q_3 + ... + q_k).$$

Hence

$$\begin{aligned} q_{k+1} &= -q_1 - (t+1)q_2 - (t+2)(q_3 + q_4 + ... + q_{k-1}) - (t+1)q_k. \\ &\geq -q_1 + 2q_2 + q_3 + q_4 + ... + q_{k-1} + 2q_k, \text{ as } -t \geq 3. \\ &> q_1 + q_2 + ... + q_k. \end{aligned}$$

Therefore $p_{k+1} > p_1 + p_2 + ... + p_{k-1}$ for all $k \geq 2$. Hence $Per(T_A) \supset \mathbb{N} \setminus \{2\}$.

Subcase 2 : $Det(A) = -1$

Here the eigen values are $\alpha, \frac{-1}{\alpha}$. Then $t = \lambda - \frac{1}{\lambda} < 0$ and hence $-t \geq 1$, since t is an integer. Using proposition 2.5,

$$p_n = \begin{cases} q_n - 2 & \text{if } n \text{ is even} \\ q_n & \text{if } n \text{ is odd} \end{cases}$$

From proposition 3.2, we have $q_{n-1} - q_{n+1} = tq_n$, for all $n \geq 2$.

Adding these relations for $n = 2, 3, ...k$, we get,

$$\begin{aligned} q_{k+1} &= q_1 + (1-t)q_2 - t(q_3 + q_4 + ... + q_{k-1}) - (t+1)q_k \\ &> q_1 + q_2 + \ldots + q_{k-1} \text{ (Since } -t \geq 1 \text{, we have } 1 - t > 1.) \end{aligned}$$

Which implies that $p_{k+1} > p_1 + p_2 + ... + p_{k-1}$ for all $k \geq 2$.
Hence $Per(T_A) \supset \mathbb{N} \setminus \{2\}$. $\square$

Remark 3.5. From the above proposition, it is clear that for a hyperbolic automorphism T_A, the period set $Per(T_A)$ is either $\mathbb{N}$ or $\mathbb{N} \setminus \{2\}$. In proposition 3.6 we prove that for certain class of hyperbolic automorphisms $Per(T_A)$ is $\mathbb{N}$ and for some other class of automorphisms, it is $\mathbb{N} \setminus \{2\}$.

3.2. **The Nonhyperbolic Case.** Note that $1 \in Per(T_A)$, $\forall A \in GL(2, \mathbb{Z})$. Suppose that $T_A : \mathbb{T}^2 \to \mathbb{T}^2$ is a non-hyperbolic toral automorphism with eigen values α and β. Then $Trace(A) = |\alpha + \beta| \leq |\alpha| + |\beta| \leq 2$. That is $Trace(A) \in \{-2, -1, 0, 1, 2\}$ and $Det(A) = \pm 1$. Thus, any $A \in GL(2, \mathbb{Z})$ which is nonhyperbolic will fall under one of these 10 cases. For $A \in GL(2, \mathbb{Z})$, let $ch(A)$ denote the characteristic polynomial of A.

Proposition 3.6. *Let $A \in GL(2, \mathbb{Z})$.*

(i) If $Det(A) = -1$ and $Trace(A) = 0$ then $Per(T_A) = \{1, 2\}$
(ii) If $Det(A) = 1$ and $Trace(A) = 0$ then $Per(T_A) = \{1, 2, 4\}$
(iii) If $Det(A) = 1$ and $Trace = -1$ then $Per(T_A) = \{1, 3\}$
(iv) If $Det(A) = 1$ and $Trace(A) = 1$ then $Per(T_A) = \{1, 2, 3, 6\}$
(v) If $Det(A) = -1$ and $Trace(A) = \pm 1$ then $Per(T_A) = \mathbb{N} \setminus \{2\}$
(vi) If $Det(A) = -1$ and $Trace(A) = \pm 2$ then $Per(T_A) = \mathbb{N}$
(vii) If $Det(A) = 1$ and $Trace(A) = 2$ then $Per(T_A) = \mathbb{N}$, provided $A \neq I$, in which case $Per(T_A) = \{1\}$
(viii) If $Det(A) = 1$ and $Trace(A) = -2$ then $Per(T_A) = 2\mathbb{N} \cup \{1\}$, provided $A \neq -I$, in which case $Per(T_A) = \{1, 2\}$.

Proof. (i) Here $ch(A) = x^2 - 1$. Then by Cayley-Hamilton theorem $A^2 = I$ and $A \neq I$. Hence $Per(T_A) = \{1, 2\}$.

(ii) Here $ch(A) = x^2 + 1$. Then $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ with $a^2 + bc = -1$.

Then by Cayley-Hamilton theorem, $A^2 = -I$ and hence $A^4 = I$. Note that $p_1 = |Det(A - I)| = 2$ and $p_2 = |Det(A^2 - I)| = |Det(-2I)| = 4$. Hence $2 \in Per(T_A)$. Hence $Per(T_A) = \{1, 2, 4\}$.

(iii) Here $ch(A)$ is $x^2 + x + 1$. Then$A = \begin{pmatrix} a & b \\ c & -1-a \end{pmatrix}$ with $a^2 + a + bc = -1$. Now $p_1 = |Det(A - I)| = 3$ and $p_2 = |Det(A_2 - I)| = 3$. Hence $2 \notin Per(T_A)$. Note that $A^3 = I$. This implies $3 \in Per(T_A)$ and hence $Per(T_A) = \{1, 3\}$.

(iv) Here $ch(A)$ is $x^2 - x + 1$. Then $A = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}$ with $a - a^2 - bc = 1$. By Cayley-Hamilton theorem, $A^2 - A + I = 0$.

Now $p_1 = |Det(A - I)| = 1$. Hence A has unique fixed point namely $\mathbf{0}$, $p_2 = |Det(A^2 - I)| = 3 > p_1 = 1$ showing that $2 \in Per(T_A)$, $p_3 = |Det(A^3 - I)| = 4 > p_1 = 1$ showing that $3 \in Per(T_A)$. Again, $p_4 = |Det(A^4 - I)| = 3$ is not greater than $p_2 = 3$ and hence $4 \notin Per(T_A)$. $p_5 = |Det(A^5 - I)| = 1$ is not greater than $p_1 = 1$ and hence $5 \notin Per(T_A)$. Now, $A^6 = I$. This implies that $6 \in Per(T_A)$ and hence $Per(T_A) = \{1, 2, 3, 6\}$.

(v) **Case 1:** When $Det(A) = -1$ and $Trace(A) = 1$.

Then $ch(A)$ is $x^2 - x - 1$ and hence the eigen values are $\frac{1}{2} \pm \frac{\sqrt{5}}{2}$, showing that A is hyperbolic. Therefore, by theorem 3.4 we have $Per(T_A) \supset \mathbb{N} \setminus \{2\}$.

We now prove that $2 \notin Per(T_A)$.

From the hypothesis, it is clear that $A = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}$ for some integers a, b, c with $a^2 - a + bc = 1$.

$$\begin{aligned} \text{Therefore, } p_2 &= |Det(A^2 - I)| \\ &= |Det(A - I)||Det(A + I)| \\ &= p_1 |Det(A + I)| \\ &= p_1 \text{ (Since } |Det(A + I)| = |a^2 + a - bc| = 1.) \end{aligned}$$

Hence $2 \notin Per(T_A)$.

Case 2: When $Det(A) = -1$ and $Trace(A) = -1$.

Proof of this is similar to that of Case 1.

(vi) **Case 1:** When $Det(A) = -1$ and $Trace(A) = 2$.

Then $ch(A)$ is $x^2 - 2x - 1$ and hence the eigen values are $1 \pm \sqrt{2}$, showing that A is hyperbolic. Therefore, by theorem 3.4 we have $Per(T_A) \supset \mathbb{N} \setminus \{2\}$.

We now prove that $2 \in Per(T_A)$.

From the hypothesis, it is clear that $A = \begin{pmatrix} a & b \\ c & 2-a \end{pmatrix}$ for some integers a, b, c with $a^2 - 2a + bc = 1$.

$$\begin{aligned} \text{Therefore, } p_2 &= |Det(A^2 - I)| \\ &= |Det(A - I)||Det(A + I)| \\ &= p_1 |Det(A + I)| \\ &= p_1.2 \text{ (Since } |Det(A + I)| = 2.) \end{aligned}$$

Hence $2 \in Per(T_A)$.

Case 2: When $Det(A) = -1$ and $Trace(A) = -2$.

Proof of this is similar to that of Case 1.

(vii) Here $Ch(A)$ is $x^2 - 2x + 1$. Then $A = \begin{pmatrix} a & b \\ c & 2-a \end{pmatrix}$ with $-a^2 + 2a - bc = 1$.

Now $Det(A - I) = 0$ and therefore A has infinitely many fixed points.

By induction on n, we have $A^n - I = n(A - I)$ for all $n \in \mathbb{N}$ and hence the system $A^nX = X$ has infinitely many solutions in $\mathbb{T}^2$.

To prove $Per(T_A) = \mathbb{N}$, it is enough to prove that for every $k(> 1) \in \mathbb{N}$ there exists a point $X = (x, y) \in \mathbb{T}^2$ such that $A^kX = X$ and $A^iX \neq X$ $\forall$ $1 \leq i < k$. This is evident from the fact that the equation $A^nX = X$ is equivalent to the following conditions:

$$\left.\begin{array}{l} n[(a-1)x + by] \in \mathbb{Z} \\ n[cx + (1-a)y] \in \mathbb{Z} \end{array}\right\} \tag{3.1}$$

Suppose $k \in \mathbb{N}$ and $k > 1$.

Case 1: When $b \neq 0$ and $1 - a \neq 0$.

Then $X = (0, \frac{1}{k.gcd(b,1-a)}) \in \mathbb{T}^2$ satisfies our requirement and hence it is a point of period k.

Case: 2 When $b = 0$ and $1 - a \neq 0$.

In this case, $X = (0, \frac{1}{k(1-a)}) \in \mathbb{T}^2$ is a point of period k.

Case 3: When $b \neq 0$ and $1 - a = 0$.

Let

$$X = \begin{cases} (\frac{1}{kc}, 0) & \text{if } c \neq 0 \\ (0, \frac{1}{kb}) & \text{if } c = 0 \end{cases}$$

Then X is a point of period k.

Case 4: When $b = 0$ and $1 - a = 0$.

If $c \neq 0$ then take $X = (\frac{1}{kc}, 0)$. If $c = 0$ then $A = I$, in which case $Per(T_A) = \{1\}$.

(viii) Here, $ch(A) = x^2 + 2x + 1$. Then $A = \begin{pmatrix} a & b \\ c & -2-a \end{pmatrix}$ with $a^2 + 2a + bc = -1$.

It can be shown by induction, with the use of Cayley-Hamilton's theorem that

$$A^n = \begin{cases} \begin{pmatrix} -na-n+1 & -nb \\ -nc & na+n+1 \end{pmatrix} & \text{if } n \textit{ is even} \\ \begin{pmatrix} na+n-1 & nb \\ nc & -na-n-2 \end{pmatrix} & \text{if } n \textit{ is odd} \end{cases}$$

Now ,

$$|Det(A^n - I)| = \begin{cases} 4 & \text{if } n \textit{ is odd} \\ 0 & \text{if } n \textit{ is even} \end{cases}$$

It is immediate that, if $n \neq 1$ is odd then $n \notin Per(T_A)$. Also, when n is even, the equation $A^n X = X$ has infinitely many solutions. Now, the equation $A^n X = X$ is equivalent to

$$\left. \begin{array}{c} n[(a+1)x + by] \in \mathbb{Z} \\ n[cx - (a+1)y] \in \mathbb{Z} \end{array} \right\} \tag{3.2}$$

Now, we argue as in the previous proposition to show that $2\mathbb{N} \subset Per(T_A)$, except the case when $a+1 = b = c = 0$, in which case $A = -I$, for which $Per(T_A) = \{1, 2\}$. Thus $Per(T_A) = 2\mathbb{N} \cup \{1\}$. □

Thus we have proved,

Theorem 3.7. *Let T_A be a nonhyperbolic toral automorphism. Then $Per(T_A)$ is one of the following 7 subsets of* $\mathbb{N}$.

(1) $\{1\}$
(2) $\{1, 2\}$
(3) $\{1, 3\}$
(4) $\{1, 2, 4\}$
(5) $\{1, 2, 3, 6\}$
(6) $2\mathbb{N} \cup \{1\}$
(7) $\mathbb{N}$

3.3. Main theorem.

Theorem 3.8. *Let T be a continuous toral automorphism. Then $Per(T)$ is one of the following 8 subset of* $\mathbb{N}$

(1) $\{1\}$
(2) $\{1, 2\}$
(3) $\{1, 3\}$
(4) $\{1, 2, 4\}$
(5) $\{1, 2, 3, 6\}$
(6) $2\mathbb{N} \cup \{1\}$
(7) $\mathbb{N} \setminus \{2\}$
(8) $\mathbb{N}$

Proof. Proof follows from proposition 3.6, theorems 3.4 and 3.7. □

4. Summary

For each self map f on a set X, we associate a subset of $\mathbb{N}$ as follows:

$\mathrm{Per}(f) = \{n \in \mathbb{N} :$ There exists $x \in X$ such that $f^n(x) = x$ and $f^m(x) \neq x$ for all $m < n\}$, where f is arbitrary, but belongs to a certain nice class of function, then not all subsets of $\mathbb{N}$ may arise as the set of periods.

It is natural to ask: Which subsets of $\mathbb{N}$ arise as Per(f), for some f in that class? We answer this question, for continuous 2-dimensional toral automorphisms. For a hyperbolic toral automorphism the set $Per(T_A)$ is either $\mathbb{N} \setminus \{2\}$ or $\mathbb{N}$. In the following table the set $Per(T_A)$ is listed in terms of the minimal polynomial of the induced matrix A for nonhyperbolic automorphisms.

Minimal polynomial of A	$Per(T_A)$
$x^2 - 1, x + 1$	{1,2}
$x^2 + 1$	{1,2,4}
$x^2 + x + 1$	{1,3}
$x^2 - x + 1$	{1,2,3,6}
$x^2 - 2x + 1$	$\mathbb{N}$
$x^2 + 2x + 1$	$2\mathbb{N} \cup \{1\}$
$x - 1$	{1}

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References

[1] I.N. Baker, *Fixpoints of polynomials and rational functions*, J. London Math. Soc., **39** (1964), 615–622.

[2] Bodil Branner and Poul Hjorth, *Real and complex dynamical systems*, NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences, 464, Kluwer Academic Publishers, Dordrecht, 1995.

[3] M. Brin, G. Stuck, *Inroduction to dynamical systems*, Cambridge university press, 2002.

[4] R.L. Devaney, *An Introduction to chaotic dynamical systems*, Second ed., Addison-wesley Publishing Company Advanced Book Program, Redwood City, CA, 1989.
[5] S.N. Elaydi, *Discrete Chaos*, CHAPMAN-HALL/CRC, 2000.
[6] M. Pollicott and M. Yuri, *Dynamical systems and ergodic theory*, London Math. Society, 1998.
[7] Charles Chapman Pugh, *Real Mathematical Analysis*, Springer-Verlag, Newyork, 2002.
[8] Sesha Sai, Ph.D thesis, *Symbolic dynamics for complete classification*, University of Hyderabad, 2000.
[9] I. Subramania Pillai et al., *Set of all periodic points of a toral automorphism*, Journal of Mathematical Analysis and Applications, **366** (2010) 367–371.

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