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by

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ON TOPOLOGICALLY INDUCED B-CONVERGENCES

DIETER LESEBERG

Abstract. In 1966 Lodato asked for an axiomatization of the following binary "nearness relation" on the power set of a set X: there exists an embedding of X into a topological space Y such that subsets A and B are near in X iff their closures meet in Y.

Then he gave an answer in terms of what later became known as Lodato proximity spaces. Afterwards, in 1975, Bentley generalized this theorem to bunch-determined nearness spaces. In this regard, recall that each topology on a set X, given by a closure operator cl, defines a compatible Leader proximity on X by declaring B to be near to A, provided B meets the closure of A. In 1964 Doitchinov introduced the notion of supertopological space in order to construct a unified theory of topological and proximity spaces. As an application he showed that the compactly generated Hausdorff-extensions of a given topological space are closely related to a special class of supertopologies called "b-supertopologies". But all structures mentioned above are special cases of the so-called "b-convergence spaces"; moreover, uniform convergence structures in the sense of Preuss can also be handled by this concept. Consequently, the results mentioned above can be recovered working in the realm of this new type of space.

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1. Basic concepts

As usual, PX denotes the power set of a set X, and we use $\mathcal{B}^X \subseteq PX$ to denote a collection of *bounded* subsets of X, also known as B-sets. Moreover, $\boldsymbol{FIL}(X \times X)$ denotes the set of all filters on $X \times X$, including the nullfilter.

Definition 1.1. For a set X we call a pair $(\mathcal{B}^X, \tau)$ consisting of a B-set $\mathcal{B}^X$ and a function $\tau : \mathcal{B}^X \longrightarrow P(\boldsymbol{FIL}(X \times X))$ a *b-convergence* (on X), the triple $(X, \mathcal{B}^X, \tau)$ a *b-convergence space*, and τ a *b-convergence operator* (on $\mathcal{B}^X$), if the following axioms are satisfied:

(bc1) $B' \subseteq B \in \mathcal{B}^X$ implies $B' \in \mathcal{B}^X$;
(bc2) $\emptyset \in \mathcal{B}^X$;
(bc3) $x \in X$ implies $\{x\} \in \mathcal{B}^X$;
(bc4) $x \in X$ implies $\dot{x} \times \dot{x} \in \tau(\{x\})$;
(bc5) $\tau(\emptyset) = \{P(X \times X)\}$;
(bc6) $B \in \mathcal{B}^X$, $\mathcal{U} \in \tau(B)$ and $\mathcal{U} \subseteq \mathcal{V} \in \boldsymbol{FIL}(X \times X)$ imply $\mathcal{V} \in \tau(B)$.

(Here $\dot{x}$ denotes the filter generated by the set $\{x\}$.) In general, for filters $\mathcal{F}$ and $\mathcal{G}$, their *cross product* is defined by

$$\mathcal{F} \times \mathcal{G} := \{\, R \subseteq X \times X \mid \exists F \in \mathcal{F}\, \exists G \in \mathcal{G}.\ R \supseteq F \times G \,\} .$$

If $\mathcal{U} \in \tau(B)$ for some $B \in \mathcal{B}^X$, we say the uniform filter $\mathcal{U}$ *b-converges* to B.

A b-convergence $(\mathcal{B}^X, \tau)$ on X and the corresponding b-convergence space $(X, \mathcal{B}^X, \tau)$ are called *saturated*, if

(sat) $X \in \mathcal{B}^X$

in which case $\mathcal{B}^X$ coincides with PX.

Given two b-convergence spaces $(X, \mathcal{B}^X, \tau_X)$ and $(Y, \mathcal{B}^Y, \tau_Y)$, a function $f : X \longrightarrow Y$ is called *b-continuous* iff it is *bounded*, which means

(c1) $\{\, f[B] \mid B \in \mathcal{B}^X \,\} \subseteq \mathcal{B}^Y$,

and $f \times f$ *preserves b-convergence of uniform filters* in the sense that

(c2) $B \in \mathcal{B}^X$ and $\mathcal{U} \in \tau_X(B)$ imply $(f \times f)(\mathcal{U}) \in \tau_Y(f[B])$, where

$$(f \times f)(\mathcal{U}) := \{\, V \subseteq Y \times Y \mid \exists U \in \mathcal{U}.\ V \supseteq (f \times f)[U] \,\} .$$

Moreover, we denote the corresponding category by $\boldsymbol{b\text{-}CONV}$, and mention here its interesting property of being *topological*, where initial and final structures are formed as follows:

For b-convergence spaces $(Y_i, \mathcal{B}^{Y_i}, \tau_i)$, $i \in I$, and functions $f_i : X \to Y_i$, respectively, $g_i : Y_i \to Z$, define B-sets on X and Z by setting

$$\begin{aligned} \mathcal{B}^X &= \{ B \subseteq X \mid \forall i \in I.\, f_i[B] \in \mathcal{B}^{Y_i} \} \text{ and} \\ \mathcal{B}^Z &= \{ B \subseteq Z \mid \exists i \in I.\, f_i^{-1}[B] \in \mathcal{B}^{Y_i} \}. \end{aligned}$$

The corresponding functions $\tau_{in} : \mathcal{B}^X \to P(\boldsymbol{FIL}(X \times X))$ and $\tau_{fin} : \mathcal{B}^Z \to P(\boldsymbol{FIL}(Z \times Z))$ map a nonempty bounded set B to

$$\tau_{in}(B) := \{ \mathcal{U} \in \boldsymbol{FIL}(X \times X) \mid \forall i \in I.\ (f_i \times f_i)(\mathcal{U}) \in \tau_i(f_i[B]) \}$$

respectively,

$$\begin{aligned} \tau_{fin}(B) := \{ \mathcal{U} \in \boldsymbol{FIL}(Z \times Z) \mid \exists i \in I.\ \exists \mathcal{U}_i \in \tau_i(g_i^{-1}[B]). \\ (g_i \times g_i)(\mathcal{U}_i) \subseteq \mathcal{U} \} \cup \{ \dot{z} \times \dot{z} \mid z \in Z \}. \end{aligned}$$

Remark 1.2. Let us point out already now that the full subcategory $\boldsymbol{satb\text{-}CONV}$ of $\boldsymbol{b\text{-}CONV}$, whose objects are the *saturated* b-convergence spaces, contains up to isomorphism all those convergence spaces which are playing important roles in the realm of *Convenient Topology*, like semi-uniform convergence spaces, filter-merotopic spaces, symmetric topological spaces and various specializations of these.

In a second direction, referred to as a *non-symmetric Convenient Topology* by Preuss, quasi-uniform convergence spaces such as quasiuniformities and various generalizations (*e.g.*, preuniform convergence spaces), but also topological structures and the corresponding generalized spaces, *e.g.*, limit spaces, Kent convergence spaces etc., can be dealt with.

Remark 1.3. Supertopological spaces, set-convergence spaces, generalized proximities or grill-defined pre-supernear spaces, respectively, now are subsumed by the broader concept of b-convergence, in quite simple fashion. Moreover, the corresponding categories can be described by their defining properties, so that in general a common concept of convergence is being established.

Remark 1.4. We will now present two fundamental types of these properties. First, we note that each b-convergence $(\mathcal{B}^X, \tau)$ induces two underlying pre-topologies, namely

(i) $cl^{\tau}(A) := \{\, x \in X \mid \exists \mathcal{U} \in \tau(\{x\}).\ \{x\} \times A \in sec\,\mathcal{U}\,\}$,
(ii) $cl_{\tau}(A) := \{\, x \in X \mid \exists \mathcal{F} \in \boldsymbol{FIL}(X).\ A \in sec\,\mathcal{F} \wedge \mathcal{F} \times \mathcal{F} \in \tau(\{x\})\,\}$;

where in general for a set system $\mathcal{S} \subseteq P(X)$ we have

$$sec\,\mathcal{S} := \{\, T \subseteq X \mid \forall S \in \mathcal{S}.\, T \cap S \neq \emptyset\,\} \ .$$

Now let $(X, \mathcal{B}^X, \tau)$ be a b-convergence space. We will call $\mathcal{C} \in \boldsymbol{FIL}(X)$ for $B \in \mathcal{B}^X$ a B-*Cauchy filter* (in τ) iff $\mathcal{C} \times \mathcal{C} \in \tau(B)$.

Definition 1.5. A b-convergence space $(X, \mathcal{B}^X, \tau)$ is called

(i) a *b-filter space*, if $B \in \mathcal{B}^X$ and $\mathcal{U} \in \tau(B)$ implies the existence of a B-Cauchy filter $\mathcal{C}$ in τ with $\mathcal{C} \times \mathcal{C} \subseteq \mathcal{U}$;
(ii) *pointed* iff $B \in \mathcal{B}^X \setminus \{\emptyset\}$ implies

$$\tau(B) = \bigcup \{\, \tau(\{x\}) \mid x \in B\,\} \ .$$

Remark 1.6. Here we note that every b-filter space $(X, \mathcal{B}^X, \tau)$ satisfies $cl^{\tau}(A) \subseteq cl_{\tau}(A)$ for each $A \subseteq X$. The full subcategory ***pb-CONV*** of ***b-CONV***, whose objects are the pointed b-convergence spaces, forms a strong topological universe in which ***TOP*** and ***UNIF*** can be fully embedded, see [11].

2. Some important isomorphisms

Example 2.1. For a preuniform convergence space (X, J_X) the triple $(X, P(X), \tau_{J_X})$, where

$$\tau_{J_X}(\emptyset) := \{P(X \times X)\} \ ;$$
$$\tau_{J_X}(B) := J_X \quad \text{for } B \in P(X) \setminus \{\emptyset\} \, .$$

is a *preuniform* b-convergence space. $(X, \mathcal{B}^X, \tau)$ is called *preuniform*, provided $(\mathcal{B}^X, \tau)$ is saturated and *steady* in the sense that

(st) $B, B' \in \mathcal{B}^X \setminus \{\emptyset\}$ implies $\tau(B) = \tau(B')$.

Conversely, for a uniform b-convergence space $(X, \mathcal{B}^X, \Omega)$ setting $I_X^{\Omega} := \Omega(X)$ yields a preuniform convergence space (X, I_X^{Ω}).

Theorem 2.2. *The full subcategory* ***ub-CONV*** *of* ***b-CONV***, *whose objects are the preuniform b-convergence spaces, is isomorphic to the category* ***PUCONV*** *of preuniform convergence spaces and uniformly continuous maps in the sense of* [14]. □

Example 2.3. Let $(X, \mathcal{B}^X, N)$ be a *pre-supernear space*, *i.e.*, $\mathcal{B}^X$ is a B-set and $N : \mathcal{B}^X \longrightarrow P(P(P(X)))$ is a function satisfying the following conditions

(SN1) $N(\emptyset) = \{\emptyset\}$ and $\forall B \in \mathcal{B}^X .\ \mathcal{B}^X \notin N(B)$;

(SN2) $\mathcal{N}_2 \ll \mathcal{N}_1 \in N(B)$ implies $\mathcal{N}_2 \in N(B)$, where $\mathcal{N}_2 \ll \mathcal{N}_1$ iff $\forall F_2 \in \mathcal{N}_2\, \exists F_1 \in \mathcal{N}_1 .\ F_2 \supseteq F_1$;

(SN3) $x \in X$ implies $\{\{x\}\} \in N(\{x\})$;

that in addition is *grill-defined* in the sense that

(G) $\mathcal{N} \in N(B)$ implies the existence of a grill $\mathcal{G} \in \boldsymbol{GRILL}(X)$ such that $\mathcal{N} \subseteq \mathcal{G}$ and $\mathcal{G} \in N(B)$.

Then we obtain a b-filter space $(X, \mathcal{B}^X, \tau_N)$ with

$$\begin{aligned} \tau_N(\emptyset) &:= \{P(X \times X)\}; \\ \tau_N(B) &:= \{\mathcal{U} \in \boldsymbol{FIL}(X \times X) \mid \exists \mathcal{G} \in \boldsymbol{GRILL}(X).\, \mathcal{G} \in N(B) \\ &\qquad \wedge sec\,\mathcal{G} \times sec\,\mathcal{G} \subseteq \mathcal{U}\} \text{ for } B \in \mathcal{B}^X \setminus \{\emptyset\}. \end{aligned}$$

Conversely, for a b-filter space $(X, \mathcal{B}^X, \Omega)$ setting

$$\begin{aligned} M_\Omega(\emptyset) &:= \{\emptyset\}; \\ M_\Omega(B) &:= \{\mathcal{G} \in \boldsymbol{GRILL}(X) \mid sec\,\mathcal{G} \times sec\,\mathcal{G} \in \Omega(B)\} \\ &\quad \text{for } B \in \mathcal{B}^X \setminus \{\emptyset\}, \end{aligned}$$

yields a grill-defined pre-supernear space $(X, \mathcal{B}^X, M)$.

Remark 2.4. For the rest of the paper we introduce the following terminology: $\boldsymbol{b}$-$\boldsymbol{CAU}$ denotes the full subcategory of $\boldsymbol{b}$-$\boldsymbol{CONV}$, whose objects are the b-filter spaces. $\boldsymbol{PSN}$ denotes the category of pre-supernear spaces and nearness-preserving maps, while $\boldsymbol{G}$-$\boldsymbol{PSN}^\bullet$ stands for the category of grill-defined pre-supernear spaces and grill-continuous maps. Concretely, a bounded function $f : X \longrightarrow Y$ is called *grill-continuous* from $(X, \mathcal{B}^X, N)$ to $(Y, \mathcal{B}^Y, M)$, if

(gc) $B \in \mathcal{B}^X$ and $\mathcal{G} \in N(B)$ implies $sec\, f(sec\,\mathcal{G}) \in M(f[B])$.

Note that grill-continuous functions are always sn-maps.

Theorem 2.5. *$\boldsymbol{b}$-$\boldsymbol{CAU}$ and $\boldsymbol{G}$-$\boldsymbol{PSN}^\bullet$ are isomorphic.* □

Remark 2.6. In this context we should mention the closely related category $\boldsymbol{PNEAR}$ of prenearness spaces and nearness preserving maps. A prenearness structure on a set X is a subset of $\xi \subseteq P(P(X))$ subject to the following axioms:

(N1) $\emptyset \in \xi$ and $\{\emptyset\} \notin \xi$;
(N2) $\mathcal{N}_2 \ll \mathcal{N}_1 \in \xi$ implies $\mathcal{N}_2 \in \xi$;
(N3) $\mathcal{F} \in P(X)$ and $\bigcap \mathcal{F} \neq \emptyset$ implies $\mathcal{F} \in \xi$.

Example 2.7. For a filter (merotopic) space (X, Γ) we obtain a *merotopical* b-convergence space $(X, P(X), \tau_\Gamma)$ by setting

$$\begin{aligned}\tau_\Gamma(\emptyset) &:= \{P(X \times X)\} \quad \text{and} \\ \tau_\Gamma(B) &:= \{\mathcal{U} \in \boldsymbol{FIL}(X \times X) \mid \exists \mathcal{F} \in \Gamma.\ \mathcal{F} \times \mathcal{F} \subseteq \mathcal{U} \wedge B \in sec\,\mathcal{F}\} \\ &\quad \text{for each } B \in P(X) \setminus \{\emptyset\} ,\end{aligned}$$

where a saturated b-convergence space $(X, \mathcal{B}^X, \Omega)$, which is *isotone, i.e.*, $B_2 \subseteq B_1 \in \mathcal{B}^X$ implies $\Omega(B_2) \subseteq \Omega(B_1)$, is called *merotopical*, provided

- $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $\mathcal{U} \in \Omega(B)$ imply the existence of $\mathcal{C} \in \gamma_\Omega$ such that $\mathcal{C} \times \mathcal{C} \subseteq \mathcal{U}$ and $B \in sec\,\mathcal{C}$, where

$$\gamma_\Omega := \left\{ \mathcal{F} \in \boldsymbol{FIL}(X) \mid \mathcal{F} \times \mathcal{F} \in \bigcap \{ \Omega(F) \mid F \in sec\,\mathcal{F} \} \right\} .$$

Note that a merotopical b-convergence space is *always* a b-filter space.

Theorem 2.8. *The full subcategory* $\boldsymbol{mb\text{-}CONV}$ *of* $\boldsymbol{b\text{-}CAU}$ *, whose objects are the the merotopical b-convergence spaces, is isomorphic to* $\boldsymbol{FIL}$ *.* □

Remark 2.9. Since $\boldsymbol{FIL}$ and $\boldsymbol{GRILL}$ are isomorphic, it follows that $\boldsymbol{mb\text{-}CONV}$ is isomorphic to $\boldsymbol{GRILL}$ as well.

Example 2.10. For a set-convergence space $(X, \mathcal{M}^X, q)$ (in Wyler's terminology) setting

$$\tau_q(B) := \{\mathcal{U} \in \boldsymbol{FIL}(X \times X) \mid \exists \mathcal{F} \in \boldsymbol{FIL}(X).\ \mathcal{F}\, q\, B \wedge \mathcal{F} \times \mathcal{F} \subseteq \mathcal{U}\} \quad \text{for each } B \in \mathcal{B}^X ,$$

yields a set-pointed b-filter space $(X, \mathcal{M}^X, \tau_q)$. A b-convergence space $(X, \mathcal{M}^X, \Omega)$ is called *set-pointed*, if

(sp) $B \in \mathcal{B}^X$ implies $\dot{B} \times \dot{B} \in \tau(B)$.

Conversely, for a set-pointed b-filter space $(X, \mathcal{B}^X, \Omega)$ we set

$$\mathcal{F}\, p_\Omega\, B \quad \text{iff} \quad \mathcal{F} \times \mathcal{F} \in \Omega(B) .$$

Theorem 2.11. *The full subcategory* ***SETb-CAU*** *of* ***bCAU*** *, whose objects are the set-pointed b-filter spaces is isomorphic to the category* ***SET-CONV*** *of set-convergence spaces.* □

Example 2.12. Any superneighborhood space $(X, \mathcal{M}^X, \vartheta)$ in the sense of [17] induces a *centric* set-pointed b-filter space $(X, \mathcal{M}^X, \tau_\vartheta)$ by setting

$$\tau_\vartheta(B) := \{\mathcal{U} \in \boldsymbol{FIL}(X \times X) | \vartheta(B) \times \vartheta(B) \subseteq \mathcal{U}\} \quad \text{for each } B \in \mathcal{B}^X .$$

Here, a b-convergence space $(X, \mathcal{B}^X, \Omega)$ is called *centric*, provided

(c) $B \in \mathcal{B}^X$ implies $\bigcap \tau(B) \in \tau(B)$.

Conversely, for a centric set-pointed b-filter space $(X, \mathcal{B}^X, \Omega)$ we set

$$\Theta_\Omega(B) := \{ F \subseteq X \mid F \times F \in \bigcap \Omega(B) \} .$$

Theorem 2.13. *The full subcategory* ***censetb-CAU*** *of* ***SETb-CAU*** *, whose objects are the centric set-pointed b-filter spaces, is isomorphic to the category* ***PRESTOP*** *of superneighborhood spaces and corresponding maps.* □

Definition 2.14. A centric uniform b-convergence space is called Δ*-uniform.*

Theorem 2.15. *The full subcategory* Δ*-* ***ub-CONV*** *of* ***ub-CONV*** *, whose objects are the* Δ*-uniform b-convergence spaces, is isomorphic to the category* Δ*-* ***UNIF*** *of* Δ*- uniform spaces* $(X, \mathcal{U})$ *(every* $U \in \mathcal{U}$ *contains the diagonal* $\Delta \subseteq X \times X$ *) and uniformly continuous maps.* □

Example 2.16. Given a g-proximity space $(X, \mathcal{B}^X, p)$ (a generalized proximity space in the sense of Tozzi and Wyler), we obtain a *proximal* b-filter space $(X, \mathcal{B}^X, \Omega_p)$ by setting

$$\Omega_p(\emptyset) := \{P(X \times X)\} ;$$
$$\Omega_p(B) := \{\mathcal{U} \in \boldsymbol{FIL}(X \times X) \mid sec\, p(B) \times sec\, p(B) \subseteq \mathcal{U}\}$$
$$\text{for } B \in \mathcal{B}^X \setminus \{\emptyset\}, \text{ where } p(B) := \{ A \subseteq X \mid B\, p\, A \} .$$

An isotone (see Example 2.7) and centric b-filter space $(X, \mathcal{B}^X, \tau)$ is called *proximal*, provided

(p) $B \in \mathcal{B}^X$ implies $sec\, \delta_\tau(B) \times sec\, \delta_\tau(B) = \bigcap \tau(B)$, where

$$B\, \delta_\tau\, A \quad \text{iff} \quad \exists \mathcal{F} \in \boldsymbol{FIL}(X).\, A \in sec\, \mathcal{F} \wedge \mathcal{F} \times \mathcal{F} \in \tau(B) .$$

Note that an isotone centric b-filter space is always set-pointed.

Theorem 2.17. *The full subcategory* ***PROXb-CAU*** *of* ***b-CAU****, whose objects are the proximal b-filter spaces, is isomorphic to the category* ***g-PROX*** *of generalized proximity spaces and corresponding maps.* □

Remark 2.18. Note that ***g-PROX*** is isomorphic to a full subcategory of ***G-PSN***$^\bullet$. Moreover, if $(\mathcal{B}^X, \tau)$ is saturated, then the definition of δ_τ leads us to the well-known proximities on PX in the sense of [17]. □

Example 2.19. Given a pretopological closure space $(X, {}^-)$, we obtain a pretopological b-filter space $(X, P(X), \Omega_-)$ by setting

$$\Omega_-(B) := \{\mathcal{U} \in \boldsymbol{FIL}(X \times X) \mid \exists \mathcal{F} \in \boldsymbol{FIL}(X).\, \mathcal{F} \times \mathcal{F} \subseteq \mathcal{U} \wedge B \in sec\{\bar{M} \mid M \in sec\mathcal{F}\}\} \text{ for all } B \subseteq X \ .$$

A saturated proximal b-filter space $(X, \mathcal{B}^X, \tau)$ is called *pretopological*, provided

(prt) $\mathcal{U} \in \tau(B)$ implies $\exists \mathcal{F} \in \boldsymbol{FIL}(X)$. $\mathcal{F} \times \mathcal{F} \subseteq \mathcal{U} \wedge B \in sec\{\, cl_\tau(M) \mid M \in sec\mathcal{F}\,\}$.

Theorem 2.20. *The full subcategory* ***PRTOPb-CAU*** *of* ***b-CAU****, whose objects are the pretopological b-filter spaces, is isomorphic to the category* ***PRTOP*** *.* □

Definition 2.21. Let $(X, \mathcal{B}^X, \tau)$ be a b-convergence space. For $B \in \mathcal{B}^X$ a B-Cauchy filter $\mathcal{C} \in \boldsymbol{FIL}(X)$ is called τ-*dense*, provided

- $A \subseteq X$ and $cl_\tau(A) \in sec\mathcal{C}$ implies $A \in sec\mathcal{C}$.

Definition 2.22. A b-convergence space $(X, \mathcal{B}^X, \tau)$ is called *dense*, provided

(d) $B \in \mathcal{B}^X$ and $\mathcal{U} \in \tau(B)$ implies the existence of a τ-dense B-Cauchy filter $\mathcal{C} \in \boldsymbol{FIL}(X)$ with $\mathcal{C} \times \mathcal{C} \subseteq \mathcal{U}$.

Corollary 2.23. *For a dense b-convergence space* $(X, \mathcal{B}^X, \tau)$ *the underlying closure operator* cl_τ *is topological.* □

Definition 2.24. A dense pretopological b-filter space $(X, \mathcal{B}^X, \tau)$ is called *topological*.

Theorem 2.25. *The full subcategory* ***TOPb-CAU*** *of* ***b-CAU***, *whose objects are the topological b-filter spaces, is isomorphic to the category* ***TOP***. □

Proof. For a given topological b-filter space $(X, \mathcal{B}^X, \tau)$ it is easy to verify that cl_τ as described in Remark 1.4(ii) constitutes a topological closure operator on X.

Conversely, for a topological space $(X, ^-)$ specified by a Kuratowsky-closure operator $^-$ on X, consider the triple $(X, P(X), \Omega_-)$ where Ω_- is defined as in Example 2.19.

Clearly, $(X, P(X), \Omega_-)$ is a proximal b-filter space. In order to see that it is pretopological, it suffices to show $\bar{A}$ coincides with $cl_{\Omega_-}(A)$ for any $A \subseteq X$ (see Example 2.19).

For $x \in cl_{\Omega_-}(A)$ by Remark 1.4(ii) we find $\mathcal{F} \in \boldsymbol{FIL}(X)$ with $A \in sec\mathcal{F}$ and $\mathcal{F} \times \mathcal{F} \in \Omega_-(\{x\})$. By definition of the later there exists $\mathcal{F}' \in \boldsymbol{FIL}(X)$ with $\mathcal{F} \times \mathcal{F} \subseteq \mathcal{F}' \times \mathcal{F}'$ and $\{x\} \in sec\{\bar{M} \mid M \in sec\mathcal{F}'\}$, in other words, $x \in \bigcap\{\bar{M} \mid M \in sec\mathcal{F}'\}$. As $sec\mathcal{F}' \subseteq sec\mathcal{F}$, we obtain $x \in \bar{A}$.

Conversely, for $x \in \bar{A}$ the filter $\mathcal{F} := sec\{F \subseteq X \mid x \in \bar{F}\}$ has the property that $M \in sec\mathcal{F}$ implies $x \in \bar{M}$, hence in particular $A \in sec\mathcal{F}$. In order to show $\mathcal{F} \times \mathcal{F} \in \Omega_-(\{x\})$, we observe that $M \in sec\mathcal{F}$ implies $x \in \bar{M}$ and thus $\{x\} \in sec\{\bar{M} \mid M \in sec\mathcal{F}\}$. Therefore $\mathcal{F} \times \mathcal{F} \in \Omega_-(\{x\})$, which shows $x \in cl_{\Omega_-}(A)$.

It remains to show that $(X, P(X), \Omega_-)$ is dense. For $B \subseteq X$ and $\mathcal{U} \in \Omega_-(B)$ there exists some $\mathcal{F} \in \boldsymbol{FIL}$ with $\mathcal{F} \times \mathcal{F} \subseteq \mathcal{U}$ and $B \in sec\{\bar{M} \mid M \in sec\mathcal{F}\}$. Setting $\mathcal{F}' := \{F \subseteq X \mid B \cap \bar{F} \neq \emptyset\}$ yields a B-Cauchy filter in Ω_- with $\mathcal{F}' \times \mathcal{F}' \subseteq \mathcal{F} \times \mathcal{F}$, since $sec\mathcal{F} \subseteq sec\mathcal{F}'$. In order to establish that $\mathcal{F}'$ is Ω_--dense, observe that $A \subseteq X$ and $cl_{\Omega_-}(A) \in sec\mathcal{F}'$ imply $\bar{A} \in sec\mathcal{F}'$, and consequently $B \cap \bar{\bar{A}} \neq \emptyset$. Since $^-$ is topological and hence idempotent, we get $B \cap \bar{A} \neq \emptyset$, which shows $A \in sec\mathcal{F}'$, as desired.

The desired bijection between topological b-filter spaces and topological spaces now follows, if we can prove $\Omega_{cl_\tau} = \tau$.

For $B \subseteq X$ consider $\mathcal{U} \in \Omega_{cl_\tau}(B)$. There exists some $\mathcal{F} \in \boldsymbol{FIL}(X)$ with $\mathcal{F} \times \mathcal{F} \subseteq \mathcal{U}$ and $B \in sec\{cl_\tau(M) \mid M \in sec\mathcal{F}\}$. In view of Example 2.16 we need to show $sec_{\delta_\tau}(B) \subseteq \mathcal{F}$. This inclusion holds iff $sec\mathcal{F} \subseteq \delta_\tau(B)$. But $M \in sec\mathcal{F}$ implies $B \cap cl_\tau(M) \neq \emptyset$. Choose x in this intersection. There exists $\mathcal{F}' \in \boldsymbol{FIL}(X)$ with $M \in sec\mathcal{F}'$ and $\mathcal{F}' \times \mathcal{F}' \in \tau(\{x\})$. Since τ is isotone, we also get $\mathcal{F}' \times \mathcal{F}' \in \tau(B)$, which shows $M \in \delta_\tau(B)$. By hypothesis we now get $\mathcal{U} \in \tau(B)$.

Conversely, let $\mathcal{U}$ be an element of $\tau(B)$. By axiom (prt) we can find $\mathcal{F} \in \boldsymbol{FIL}(X)$ with $\mathcal{F}\times\mathcal{F} \subseteq \mathcal{U}$ and $B \in sec\{\, cl_\tau(M) | M \in sec\mathcal{F}\,\}$, hence $\mathcal{U} \in \Omega_{|cl_\tau}(B)$.

At last, consider topological spaces $(X, \,^{-X})$ and $(Y, \,^{-Y})$ and a function $f : X \longrightarrow Y$. We need to establish the equivalence of the following assertions:

(i) f is continuous from $(X, \,^{-X})$ to $(Y, \,^{-Y})$;
(ii) f is b-continuous from $(X, P(X), \Omega_{-X})$ to $(Y, P(Y), \Omega_{-Y})$.

(i) $\Rightarrow$ (ii): $\mathcal{U} \in \Omega_{-X}(B)$ implies the existence of $\mathcal{F} \in \boldsymbol{FIL}(X)$ with $\mathcal{F}\times\mathcal{F} \subseteq \mathcal{U}$ and $B \in sec\{\, \bar{M}^X | M \in sec\mathcal{F}\,\}$. Then $f(\mathcal{F}) \in \boldsymbol{FIL}(Y)$ satisfies $f(\mathcal{F}) \times f(\mathcal{F}) = (f \times f)(\mathcal{F} \times \mathcal{F}) \subseteq (f \times f)(\mathcal{U})$.

Now for an element $M \in sec\, f(\mathcal{F})$ we have to verify $f[B] \cap \bar{M}^Y \neq \emptyset$. But $f^{-1}[M] \in sec\mathcal{F}$ implies $B \cap \overline{f^{-1}[M]}^X \neq \emptyset$, and therefore

$$\emptyset \neq f[B \cap \overline{f^{-1}[M]}^X] \subseteq f[B] \cap f[\overline{f^{-1}[M]}^X] \subseteq f[B] \cap \overline{f[f^{-1}[M]]}^Y \subseteq f[B] \cap \bar{M}^Y$$

which had to be shown.

(ii) $\Rightarrow$ (i): For $A \subseteq X$ and $x \in \bar{A}^X$ we have to verify $f(x) \in \overline{f[A]}^Y$. But $\bar{A}^X = cl_{\Omega_{-X}}(A)$ implies the existence of $\mathcal{F} \in \boldsymbol{FIL}(X)$ with $A \in sec\mathcal{F}$ and $\mathcal{F} \times \mathcal{F} \subseteq \Omega_{-X}(\{x\})$. By hypothesis we get $f(\mathcal{F}) \times f(\mathcal{F}) = (f \times f)(\mathcal{F} \times \mathcal{F}) \in \Omega_{-Y}(\{f(x)\})$ and $f[A] \in f(sec\mathcal{F}) \subseteq sec\, f(\mathcal{F})$. Choose $\mathcal{F}' \in \boldsymbol{FIL}(Y)$ with $\mathcal{F}' \subseteq f(\mathcal{F})$ and $f(x) \in \bigcap\{\, \bar{M}^Y \mid M \in sec\mathcal{F}'\,\}$. From $sec\, f(\mathcal{F}) \subseteq sec\mathcal{F}'$ it follows that $f(x) \in \overline{f[A]}^Y$, as desired. □

Figure 1 displays the relationships among the categories mentioned in this Section.

3. Topological extensions

By $\boldsymbol{TEXT}$ we denote the category, whose objects $(e, \mathcal{B}^X, Y)$ are specified by topological spaces $X = (X, cl_X)$ and $Y = (Y, cl_Y)$ (given by closure operators), a B-set $\mathcal{B}^X$ and a function $e : X \longrightarrow Y$ that satisfies the following conditions:

(E1) $A \subseteq X$ implies $cl_X(A) = e^{-1}[cl_Y(e[A])]$, where e^{-1} denotes the inverse image under e;
(E2) $cl_Y(e[X]) = Y$, which means that the image of X under e is *dense* in Y;

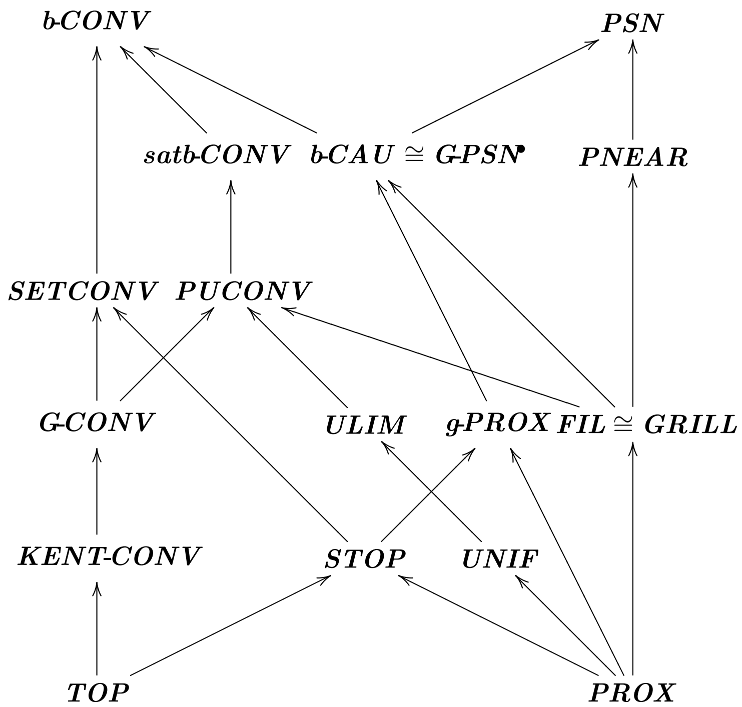

FIGURE 1. Embeddings among some categories mentioned above.

Morphisms in ***TEXT*** have the form

$$(f, g) : (e, \mathcal{B}^X, Y) \longrightarrow (e', \mathcal{B}^{X'}, Y')$$

where $f : X \longrightarrow X'$, $g : Y \longrightarrow Y'$ are continuous maps such that f is *bounded*, and the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{e} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow[e']{} & Y' \end{array} \tag{3.1}$$

If $(f, g) : (e, \mathcal{B}^X, Y) \longrightarrow (e', \mathcal{B}^{X'}, Y')$ and $(f', g') : (e', \mathcal{B}^{X'}, Y') \longrightarrow (e'', \mathcal{B}^{X''}, Y'')$ are ***TEXT***-morphisms, they can be composed component-wise, *i.e.*, $(f', g') \circ (f, g) = (f' \circ f, g' \circ g)$.

Remark 3.1. The continuity of e follows from (E1), if $e : X \twoheadrightarrow Y$ is a topological embedding.

Moreover, X is allowed to carry an arbitrary B-set that can be different from the power set PX.

Finally, we mention that such an extension is called *strict* iff $\{ cl_Y(e[A]) | A \subseteq X \}$ is a base for the closed subsets of Y. $\boldsymbol{STEXT}$ denotes the full subcategory of $\boldsymbol{TEXT}$, whose objects are the strict topological extensions.

Lemma 3.2. *For a* $\boldsymbol{TEXT}$*-object* $(e, \mathcal{B}^X, Y)$ *we obtain a pointed b-filter space* $(X, \mathcal{B}^X, \tau_e)$ *with* $cl_X = cl_{\tau_e}$ *by setting*

$$\begin{aligned} \tau_e(\emptyset) &:= \{P(X \times X)\} ; \\ \tau_e(B) &:= \{\mathcal{U} \in \boldsymbol{FIL}(X \times X) \mid \exists \mathcal{F} \in \boldsymbol{FIL}(X)\, \exists x \in B.\ \mathcal{F} \times \mathcal{F} \subseteq \mathcal{U} \\ &\qquad \wedge e(x) \in \bigcap \{ cl_Y(e[A]) \mid A \in sec\,\mathcal{F} \} \} \\ &\qquad \textit{for all } B \in \mathcal{B}^X \setminus \{\emptyset\} . \end{aligned}$$

Proof. It is easy to verify that τ_e defines a pointed b-filter operator on $\mathcal{B}^X$. Now we will show that the corresponding closure operators agree.

- $\geq$ $x \in cl_{\tau_e}(A)$ implies the existence of $\mathcal{F} \in \boldsymbol{FIL}(X)$ with $A \in sec\,\mathcal{F}$ and $\mathcal{F} \times \mathcal{F} \in \tau_e(\{x\})$. Choose $\mathcal{C} \in \boldsymbol{FIL}(X)$ such that $\mathcal{C} \times \mathcal{C} \subseteq \mathcal{F} \times \mathcal{F}$ and $e(x) \in \bigcap \{ cl_Y(e[A]) \mid A \in sec\,\mathcal{C} \}$, hence $A \in sec\,\mathcal{C}$ and $e(x) \in cl_Y(e[A])$ follows. Consequently, $x \in e^{-1}[cl_Y(e[A])]$, which shows that $x \in cl_X(A)$, as required by (E1).
- $\leq$ Conversely, $x \in cl_X(A)$ implies $e(x) \in e[cl_X(A)] \subseteq cl_Y(e[A])$, since e is continuous (see (E1)). We set

$$\mathcal{F} := sec\,\{ T \subseteq X \mid e(x) \in cl_Y(e[T]) \} .$$

Then $\mathcal{F} \in \boldsymbol{FIL}(X)$ with $A \in sec\,\mathcal{F}$ and $\mathcal{F} \times \mathcal{F} \in \tau_e(\{x\})$, because $F \in sec\,\mathcal{F}$ implies $e(x) \in cl_Y(e[F])$, and therefore $e(x) \in \bigcap \{ cl_Y(e[F]) \mid F \in sec\,\mathcal{F} \}$, which shows that $x \in cl_{\tau_e}(A)$. □

Definition 3.3. For a set X, we call a pointed b-filter convergence $(\mathcal{B}^X, \tau)$ a *LEADER b-convergence*, and the triple $(X, \mathcal{B}^X, \tau)$ a *LEADER b-convergence space*, provided

(LE1) $x \in X$ implies $x_\tau \times x_\tau \in \tau(\{x\})$, where $x_\tau := sec\,\{ T \subseteq X \mid x \in cl_\tau(T) \}$;

(LE2) $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $\mathcal{U} \in \tau(B)$ implies the existence of a B-Cauchy filter $\mathcal{M}$ in τ that satisfies $\mathcal{M} \times \mathcal{M} \subseteq \mathcal{U}$ and is B-*marginal* in the sense that

(ma0) $sec\,\mathcal{M} \neq \emptyset$;

(ma1) $\mathcal{M}$ is τ-dense;

(ma2) $\mathcal{M}$ is B-*sected*, which means $B \in sec\{\, cl_\tau(A) \mid A \in sec\,\mathcal{M}\,\}$.

Remark 3.4. We point out that a LEADER b-convergence space is dense, hence its underlying closure operator is topological.

Proposition 3.5. *The pointed b-filter space* $(X, \mathcal{B}^X, \tau_e)$ *constructed in Lemma 3.2 is in fact a LEADER b-convergence space.* □

Proof.

(LE1) Since $x_{\tau_e} \in \boldsymbol{FIL}(X)$, and for $A \in sec\, x_{\tau_e}$ we have $x \in cl_{\tau_e}(A) = cl_X(A)$, which implies $e(x) \in e[cl_X(A)] \subseteq cl_Y(e[A])$, condition (LE1) is satisfied.

(LE2) $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $\mathcal{U} \in \tau_e(B)$ implies the existence of some $\mathcal{F} \in \boldsymbol{FIL}(X)$ and some $x \in B$ such that $\mathcal{F} \times \mathcal{F} \subseteq \mathcal{U}$ and $e(x) \in \bigcap\{\, cl_Y(e[A]) \mid A \in sec\,\mathcal{F}\,\}$. The filter

$$\mathcal{M}_x := sec\{\, T \subseteq X \mid e(x) \in cl_Y(e[T])\,\}$$

satisfies $\mathcal{M}_x \subseteq \mathcal{F}$, because of $sec\,\mathcal{F} \subseteq sec\,\mathcal{M}_x$. Consequently, $\mathcal{M}_x \times \mathcal{M}_x \subseteq \mathcal{U}$, since by definition $\mathcal{M}_x \times \mathcal{M}_x \in \tau_e(B)$.

(ma0) By construction, $sec\,\mathcal{M}_x \neq \emptyset$.

(ma1) Consider $A \subseteq X$ with $cl_{\tau_e}(A) \in sec\,\mathcal{M}_x$. Then

$$e(x) \in cl_Y(e[cl_{\tau_e}(A)]) = cl_Y(e[cl_X(A)]) \subseteq cl_Y(cl_Y(e[A])) \subseteq cl_Y(e[A]),$$

which implies that (ma1) is satisfied.

(ma2) $A \in sec\,\mathcal{M}_x$ implies $e(x) \in cl_Y(e[A])$, hence we get $x \in e^{-1}[cl_Y(e[A])]$. According to (E1) we then have $x \in cl_X(A) = cl_{\tau_e}(A)$.
Since $x \in B$, the hypothesis shows $B \cap cl_{\tau_e}(A) \neq \emptyset$, which implies $B \in sec\{\, cl_{\tau_e}(F) \mid F \in sec\,\mathcal{M}_x\,\}$. Therefore $\mathcal{M}_x$ is B-sected. □

Theorem 3.6. *Let* ***LEb-CONV*** *denote the full subcategory of* ***b-CONV***, *whose objects are the LEADER b-convergence spaces. We obtain a functor* $F : \boldsymbol{TEXT} \longrightarrow \boldsymbol{LEb\text{-}CONV}$ *by setting*

(a) $F(e, \mathcal{B}^X, Y) := (X, \mathcal{B}^X, \tau_e)$;

(b) $F(f,g) := f$ *for a* $\boldsymbol{TEXT}$*-morphism* $(f,g) : (e, \mathcal{B}^X, Y) \longrightarrow (e', \mathcal{B}^{X'}, Y')$. □

Proof. Consider a $\boldsymbol{TEXT}$-morphism

$$(f,g) : (e, \mathcal{B}^X, Y) \longrightarrow (e', \mathcal{B}^{X'}, Y').$$

We must establish the b-continuity of

$$f : (X, \mathcal{B}^X, \tau_e) \longrightarrow (X', \mathcal{B}^{X'}, \tau_{e'}).$$

By hypothesis, f is already bounded. Now for $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $\mathcal{U} \in \tau_e(B)$ choose $\mathcal{F} \in \boldsymbol{FIL}(X)$ and $x \in B$ such that $\mathcal{F} \times \mathcal{F} \subseteq \mathcal{U}$ and $e(x) \in \bigcap\{ cl_Y(e[A]) \,|\, A \in sec\,\mathcal{F}\}$. Now $x' := f(x) \in f[B]$, and $\mathcal{F}' := f(\mathcal{F}) \in \boldsymbol{FIL}(X')$ satisfies $\mathcal{F}' \times \mathcal{F}' \subseteq (f \times f)(\mathcal{U})$. Moreover, if $A' \in sec\,\mathcal{F}'$, then $A' \in sec\,f(\mathcal{F})$ and hence $f^{-1}[A'] \in sec\,\mathcal{F}'$.

By hypothesis we have $e(x) \in cl_Y(e[f^{-1}[A']])$. Since g is continuous, we obtain

$$\begin{aligned} e'(x') = e'(f(x)) = g(e(x)) &\in g[cl_Y(e[f^{-1}[A']])] \subseteq cl_{Y'}(g[e[f^{-1}[A']]]) \\ &= cl_{Y'}(e'[f[f^{-1}[A']]]) \subseteq cl_{Y'}(e'[A']) \end{aligned}$$

due to the commutativity of Diagram 3.1. □

Lemma 3.7. *For a LEADER b-convergence space* $(X, \mathcal{B}^X, \tau)$ *and for each* $x \in X$, $\times_\tau$ *is* $\{x\}$*-marginal in* τ *with the property that* $\times_\tau \times x_\tau$ *is minimal in* $\tau(\{x\})$ *ordered by inclusion.*

Proof. We first note that x_τ is a $\{x\}$-Cauchy filter, since $(\mathcal{B}^X, \tau)$ satisfies axiom (LE1) and cl_τ is a closure operator on X.

By definition, $sec\,x_\tau \neq \emptyset$ and x_τ is $\{x\}$-sected.

To show that x_τ is τ-dense, observe that $cl_\tau(A) \in sec\,x_\tau$ implies $x \in cl_\tau(cl_\tau(A)) \subseteq cl_\tau(A)$. But since cl_τ is topological, $A \in sec\,x_\tau$ follows.

Now for $\mathcal{U} \in \tau(\{x\})$ with $\mathcal{U} \subseteq x_\tau \times x_\tau$ choose an $\{x\}$-marginal $\mathcal{M}$ in τ with $\mathcal{M} \times \mathcal{M} \subseteq \mathcal{U}$. By construction, this satisfies $\mathcal{M} \subseteq x_\tau$, which implies $sec\,x_\tau \subseteq sec\,\mathcal{M}$. On the other hand, $A \in sec\,\mathcal{M}$ implies $x \in cl_\tau(A)$, since $\mathcal{M}$ is $\{x\}$-sected, which means $A \in sec\,x_\tau$.

Hence $\mathcal{M} = x_\tau$, and therefore $\mathcal{U} = x_\tau \times x_\tau$. □

4. LEADER b-convergences and strict topological extensions

In the previous section we have found a functor F : ***TEXT*** $\longrightarrow$ ***LEb-CONV***. Now we are going to introduce a related functor from ***LEb-CONV*** to ***TEXT***.

Lemma 4.1. *Let* $(X, \mathcal{B}^X, \tau)$ *be a LEADER b-convergence space. we put*

$$\hat{X} := \{\,\mathcal{M} \in \boldsymbol{FIL}(X) \mid \exists B \in \mathcal{B}^X \setminus \{\emptyset\}.\ \mathcal{M} \text{ is } B\text{-marginal in } \tau\,\}\ ,$$

and for each $\hat{A} \subseteq \hat{X}$ *we set* $cl_{\hat{X}}(\hat{A}) := \{\,\mathcal{M} \in \hat{X} \mid \Delta(\hat{A}) \subseteq sec\,\mathcal{M}\,\}$, *where*

$$\Delta(\hat{A}) := \{\, F \subseteq X \mid \forall \mathcal{C} \in \hat{A}.\, F \in sec\,\mathcal{C}\,\}.$$

Then $\mathrm{cl}_{\hat{X}}$ *is a topological closure operator on* $\hat{X}$.

Proof. Assume $cl_{\hat{X}}(\emptyset) \neq \emptyset$ and choose a B-marginal $\mathcal{M}$ in τ with $PX = \Delta(\emptyset) \subseteq sec\,\mathcal{M}$. Since $\emptyset \in sec\,\mathcal{M}$ and $B \neq \emptyset$, we get $\emptyset = B \cap cl_\tau(\emptyset) \neq \emptyset$, a contradiction. Therefore $cl_{\hat{X}}(\emptyset) = \emptyset$.

Now consider $\mathcal{M} \in \hat{A}$ and $F \in \Delta(\hat{A})$, which implies $F \in sec\,\mathcal{M}$ and hence $\mathcal{M} \in cl_{\hat{X}}(\hat{A})$.

For $\hat{A}_1 \subseteq \hat{A}_2$ and $\mathcal{M} \in cl_{\hat{X}}(\hat{A}_1)$ we have $\Delta(\hat{A}_2) \subseteq \Delta(\hat{A}_1) \subseteq sec\,\mathcal{M}$, which shows $\mathcal{M} \in cl_{\hat{X}}(\hat{A}_2)$.

$\mathcal{M} \in cl_{\hat{X}}(\hat{A}_1 \cup \hat{A}_2)$ implies $\Delta(\hat{A}_1 \cup \hat{A}_2) \subseteq sec\,\mathcal{M}$. Assume $\mathcal{M} \notin cl_{\hat{X}}(\hat{A}_1) \cup cl_{\hat{X}}(\hat{A}_2)$, hence $\Delta(\hat{A}_1) \not\subseteq sec\,\mathcal{M}$ and $\Delta(\hat{A}_2) \not\subseteq sec\,\mathcal{M}$. Choose $F_1 \in \Delta(\hat{A}_1) \setminus sec\,\mathcal{M}$ and $F_2 \in \Delta(\hat{A}_2) \setminus sec\,\mathcal{M}$.

We claim that both $X \setminus F_1$ and $X \setminus F_2$ belong to $\mathcal{M}$. From $X \setminus (F_1 \cup F_2) = X \setminus F_1 \cap X \setminus F_2 \in \mathcal{M}$ we see $F_1 \cup F_2 \notin sec\,\mathcal{M}$. By hypothesis choose $\mathcal{C} \in \hat{A}_1 \cup \hat{A}_2$ with $F_1 \cup F_2 \notin sec\,\mathcal{C}$. If $\mathcal{C} \in \hat{A}_1$, then $F_1 \in \Delta(\hat{A}_1)$ implies $F_1 \in sec\,\mathcal{C}$, and hence $F_1 \cup F_2 \in sec\,\mathcal{C}$, a contradiction. By symmetry, $\mathcal{C} \in \hat{A}_2$ leads to a contradiction as well. Thus we have $cl_{\hat{X}}(\hat{A}_1 \cup \hat{A}_2) = cl_{\hat{X}}(\hat{A}_1) \cup cl_{\hat{X}}(\hat{A}_2)$.

$\mathcal{M} \in cl_{\hat{X}}(cl_{\hat{X}}(\hat{A}))$ implies $\Delta(cl_{\hat{X}}(\hat{A})) \subseteq sec\,\mathcal{M}$. We need to show $\Delta(\hat{A}) \subseteq sec\,\mathcal{M}$. $F \notin sec\,\mathcal{M}$ implies $F \notin sec\,\mathcal{C}$ for some $\mathcal{C} \in cl_{\hat{X}}(\hat{A})$, hence we get $\Delta(\hat{A}) \subseteq sec\,\mathcal{C}$. Consequently, $F \notin \Delta(\hat{A})$, which establishes the claim. □

Theorem 4.2. *For LEADER b-convergence spaces* $(X, \mathcal{B}^X, \tau)$ *and* $(Y, \mathcal{B}^Y, \Omega)$ *let* $f : X \longrightarrow Y$ *be a b-continuous map. Define a function* $\hat{f} : \hat{X} \longrightarrow \hat{Y}$ *by setting*

$$\hat{f}(\mathcal{M}) := sec\{\, D \subseteq Y \mid f^{-1}[cl_\Omega(D)] \in sec\,\mathcal{M}\,\} \quad \text{for each } \mathcal{M} \in \hat{X} \ .$$

Then the following statements are valid:

(i) $\hat{f}$ *is a continuous map from* $(\hat{X}, cl_{\hat{X}})$ *to* $(\hat{Y}, cl_{\hat{Y}})$ *;*

(ii) *The composites* $\hat{f} \circ e_X$ *and* $e_Y \circ f$ *coincide, where* $e_X : X \longrightarrow \hat{X}$ *denotes the function defined by* $e_X(x) := x_\tau$ *for each* $x \in X$ *.* □

Proof. If $\mathcal{M} \in \hat{X}$, then $\mathcal{M} \times \mathcal{M} \in \tau(B)$ for some $B \in \mathcal{B}^X \setminus \{\emptyset\}$. We have to show that $\hat{f}(\mathcal{M}) \times \hat{f}(\mathcal{M}) \in \Omega(f[B])$. By hypothesis we have $f(\mathcal{M}) \times f(\mathcal{M}) = (f \times f)(\mathcal{M} \times \mathcal{M}) \in \Omega(f[B])$, hence there exists an $f[B]$-marginal $\mathcal{F}$ in Ω with $\mathcal{F} \times \mathcal{F} \subseteq f(\mathcal{M}) \times f(\mathcal{M})$.

It remains to verify $sec\,\hat{f}(\mathcal{M}) \subseteq sec\,\mathcal{F}$. To this end, it suffices to prove $cl_\Omega(D) \in sec\,\mathcal{F}$, provided $D \in sec\,\hat{f}(\mathcal{M})$. Now in this case any $F \in \mathcal{F}$ satisfies $F \supseteq f[M]$ for some $M \in \mathcal{M}$, hence $f^{-1}[cl_\Omega(D)] \in sec\,\mathcal{M}$. Consequently, $f^{-1}[cl_\Omega(D)] \cap M \neq \emptyset$. Choose $x \in M$ with $f(x) \in cl_\Omega(D)$. Then we get $f(x) \in F$, which implies $F \cap cl_\Omega(D) \neq \emptyset$. But now $\hat{f}(\mathcal{M}) \times \hat{f}(\mathcal{M})$ is a $f[B]$-Cauchy filter in Ω, and $\hat{f}(\mathcal{M}) \neq \emptyset$ by definition. Moreover, $\hat{f}(\mathcal{M})$ is Ω-dense, as $cl_\Omega(A) \in sec\,\hat{f}(\mathcal{M})$ implies $f^{-1}[cl_\Omega(cl_\Omega(A))] \in sec\,\mathcal{M}$, which shows $A \in sec\,\hat{f}(\mathcal{M})$.

It remains to show that $\hat{f}(\mathcal{M})$ is $f[B]$-sected, which means that $f[B] \in sec\{\, cl_\Omega(A) \mid A \in sec\,\hat{f}(\mathcal{M})\,\}$. By hypothesis, $A \in sec\,\hat{f}(\mathcal{M})$ implies $B \cap cl_\tau(f^{-1}[cl_\Omega(A)]) \neq \emptyset$, consequently $x \in cl_\tau(f^{-1}[cl_\Omega(A)])$ for some $x \in B$. Then

$$f(x) \in f[cl_\tau(f^{-1}[cl_\Omega(A)])] \subseteq cl_\Omega(f[f^{-1}[cl_\Omega(A)]]) \subseteq cl_\Omega(cl_\Omega(A)) \subseteq cl_\Omega(A)$$

follows, which shows $f[B] \cap cl_\Omega(A) \neq \emptyset$.

(i) For $\hat{A} \subseteq \hat{X}$ we must show $\hat{f}[cl_{\hat{X}}(\hat{A})] \subseteq cl_{\hat{Y}}(\hat{f}[\hat{A}])$. Given $\mathcal{M} \in cl_{\hat{X}}(\hat{A})$, assume $\hat{f}(\mathcal{M}) \notin cl_{\hat{Y}}(\hat{f}[\hat{A}])$. Choose $G \in \Delta(\hat{f}[\hat{A}])$ with $G \notin sec\,\hat{f}(\mathcal{M})$, hence $f^{-1}[cl_\Omega(G)] \notin sec\,\mathcal{M}$. By hypothesis there exists $\mathcal{C}' \in \hat{A}$ with $f^{-1}[cl_\Omega(G)] \notin sec\,\mathcal{C}'$, hence $\hat{f}(\mathcal{C}') \in \hat{f}[\hat{A}]$, which implies $G \in sec\,\hat{f}(\mathcal{C}')$. But on the other hand, $f^{-1}[cl_\Omega(G)] \in sec\,\mathcal{C}'$, which is a contradiction. Therefore $\hat{f}(\mathcal{M}) \in cl_{\hat{Y}}(\hat{f}[\hat{A}])$ is valid.

(ii) For $x \in X$ we will establish the inclusion $\hat{f}(x_\tau) \subseteq f(x)_\Omega$. To this end it satisfies to verify $sec\, f(x)_\Omega \subseteq sec\, \hat{f}(x_\tau)$. Now $M \in sec\, f(x)_\Omega$ implies $f(x) \in cl_\Omega(M)$, hence $x \in f^{-1}[cl_\Omega(M)] \subseteq cl_\tau(f^{-1}[cl_\Omega(M)])$ follows, which means $f^{-1}[cl_\Omega(M)] \in sec\, f(x_\tau)$. But now we have $M \in sec\, \hat{f}(X_\tau)$.

Since $\hat{f}(x_\tau) \times \hat{f}(x_\tau) \in \Omega(\{f(x)\})$, we get that $f(x)_\Omega \times f(x)_\Omega$ is minimal in $(\Omega(\{f(x)\}), \subseteq)$. Hence $\hat{f}(x_\tau) \times \hat{f}(x_\tau)$ coincides with $f(x)_\Omega \times f(x)_\Omega$, which shows $\hat{f}(x_\tau) = f(x)_\Omega$, and hence $\hat{f} \circ e_X = e_Y \circ f$, as desired. □

Theorem 4.3. *We obtain a functor G :* ***LEb-CONV*** $\longrightarrow$ ***STEXT*** *by setting*

(a) $G(X, \mathcal{B}^X, \tau) := (e_X, \mathcal{B}^X, \hat{X})$ *with* $X = (X, cl_X)$ *and* $\hat{X} = (\hat{X}, cl_{\hat{X}})$ *;*

(b) $G(f) = (f, \hat{f})$ *for a b-continuous map* $f : (X, \mathcal{B}^X, \tau) \longrightarrow (Y, \mathcal{B}^Y, \Omega)$ □

Proof. By earlier arguments we know that cl_X and $cl_{\hat{X}}$ are topological closure operators on their defining sets X and $\hat{X}$, respectively. Moreover, $e_X : X \longrightarrow \hat{X}$ defined by $e_X(x) = x_\tau$ for each $x \in X$ is a function from X to $\hat{X}$. Now we will establish the axioms for being a topological extension:

(E1) For $A \subseteq X$ we have to show $cl_\tau(A) = e_X^{-1}[cl_{\hat{X}}(e_X[A])]$. First, we note that

$$\Delta(e_X[A]) = \Delta(\{ x_\tau \mid x \in A \}) = \{ F \subseteq X \mid A \subseteq cl_\tau(F) \} =: A^C .$$

If $x \in cl_\tau(A)$, then $sec\, x_\tau = \{x\}^C \supseteq \Delta(e_X[A])$, which means that $e_X(x) = x_\tau \in cl_{\hat{X}}(e_X[A])$. But then $x \in e_X^{-1}[cl_{\hat{X}}(e_X[A])]$ follows. Conversely, from $x \in e_X^{-1}[cl_{\hat{X}}(e_X[A])]$ we conclude $x_\tau = e_X(x) \in cl_{\hat{X}}(e_X[A])$, which implies $A \in A^C = \Delta(e_X[A]) \subseteq sec\, x_\tau$, and hence $x \in cl_\tau(A)$.

(E2) We must show $cl_{\hat{X}}(e_X[X]) = \hat{X}$. For $\mathcal{M} \in \hat{X}$ assume $\mathcal{M} \notin cl_{\hat{X}}(e_X[X])$, hence $X^C = \Delta(e_X[X]) \nsubseteq sec\, \mathcal{M}$. Choose $F \in X^C$ with $F \notin sec\, \mathcal{M}$. Then we have $X \subseteq cl_\tau(F)$. Furthermore, $X \in sec\, \mathcal{M}$ implies $cl_\tau(F) \in sec\, \mathcal{M}$. But since $\mathcal{M}$ is τ-dense, we conclude $F \in sec\, \mathcal{M}$, a contradiction.

At last we identify $\{ cl_{\hat{X}}(e_X[A]) \mid A \subseteq X \}$ as a basis for the closed subsets of $\hat{X}$. If $\hat{A} \neq \hat{X}$ is closed in $\hat{X}$, we can find some $\mathcal{M} \in \hat{X} \setminus cl_{\hat{X}}(\hat{A})$, which in turn satisfies $\Delta(\hat{A}) \not\subseteq sec\,\mathcal{M}$. Hence there exists $F \in \Delta(\hat{A})$ with $F \notin sec\,\mathcal{M}$. As $\mathcal{C} \in \hat{A}$ implies $F \in sec\,\mathcal{C}$, we obtain the inclusion $\Delta(e_X[F]) \subseteq sec\,\mathcal{C}$, and therefore $\hat{A} \subseteq cl_{\hat{X}}(e_X[F])$.

On the other hand we have $\mathcal{M} \notin cl_{\hat{X}}(e_X[F])$, since $F \notin sec\,\mathcal{M}$ implies $\Delta(e_X[F]) \not\subseteq sec\,\mathcal{M}$. This shows $cl_{\hat{X}}(e_X[F]) \subseteq \hat{A}$ as desired. □

Theorem 4.4. *Let* $F : \boldsymbol{TEXT} \longrightarrow \boldsymbol{LEb\text{-}CONV}$ *and* $G : \boldsymbol{LEb\text{-}CONV} \longrightarrow \boldsymbol{STEXT}$ *be the functors defined above. Then* $F \circ G = 1_{\boldsymbol{LEb\text{-}CONV}}$. □

Proof. First we show that $F(G(X, \mathcal{B}^X, \tau)) = (X, \mathcal{B}^X, \tau)$ is an isomorphism for any $\boldsymbol{LEb\text{-}CONV}$-object $(X, \mathcal{B}^X, \tau)$. Since $F(G(X, \mathcal{B}^X, \tau) = F(e_X, \mathcal{B}^X, \hat{X}) = (X, \mathcal{B}^X, \tau_{e_X})$, we need to check whether $\tau_{e_X}(B) = \tau(B)$ for $B \in \mathcal{B}^X \setminus \{\emptyset\}$.

Now $\mathcal{U} \in \tau_{e_X}(B)$ implies the existence of a filter $\mathcal{F} \in \boldsymbol{FIL}(X)$ and some $x \in X$ with $x_\tau = e_X(x) \in \bigcap\{ cl_{\hat{X}}(A) \mid A \in sec\,\mathcal{F} \}$ and $\mathcal{F} \times \mathcal{F} \subseteq \mathcal{U}$. Since τ is pointed and in particular satisfies (LE1), we get $x_\tau \times x_\tau \in \tau(\{x\}) \subseteq \tau(B)$. It remains to show that $x_\tau \subseteq \mathcal{F}$. But any $F \in sec\,\mathcal{F}$ by hypothesis satisfies $x_\tau \in cl_{\hat{X}}(e_X[F])$, hence we have $\Delta(e_X[F]) \subseteq sec\,x_\tau$, which implies $F \in sec\,x_\tau$. This proves the claim. We now conclude $x_\tau \times x_\tau \subseteq \mathcal{U}$, which shows $\mathcal{U} \in \tau(B)$.

Conversely, given $\mathcal{U} \in \tau(B)$, since τ is pointed, we get $\mathcal{U} \in \tau(\{x\})$ for some $x \in B$. Choose a $\{x\}$-marginal $\mathcal{M}$ in τ with $\mathcal{M} \times \mathcal{M} \subseteq \mathcal{U}$, hence $\{x\} \in sec\{ cl_\tau(F) \mid F \in sec\,\mathcal{M} \}$. Moreover, we have $x_\tau = e_X(x) \in e_X[B]$. It remains to prove that

$$x_\tau \in \bigcap\{cl_{\hat{X}}(e_X[F]) \mid F \in sec\,\mathcal{M} \} .$$

For $F \in sec\,\mathcal{M}$ consider $M \in \Delta(e_X[F])$. Since $cl_\tau(M) \supseteq F$ and by hypothesis $x \in cl_\tau(F)$, we have $x \in cl_\tau(cl_\tau(M)) \subseteq cl_\tau(M)$, which establishes $M \in sec\,x_\tau$. Consequently, $x \in cl_{\hat{X}}(e_X[F])$, as desired.

Since $F \circ G$ maps any $\boldsymbol{LEb\text{-}CONV}$-morphism

$$f : (X, \mathcal{B}^X, \tau) \longrightarrow (Y, \mathcal{B}^Y, \Omega)$$

to itself, the assertion is proved. □

Remark 4.5. Note that in case of having a *separated* LEADER b-convergence space $(X, \mathcal{B}^X, \tau)$, (which means that $x_\tau = z_\tau$ implies $x = z$), the corresponding function $e_X : X \longrightarrow \hat{X}$ is a topological embedding.

Corollary 4.6. *For a pointed b-filter space* $(X, \mathcal{B}^X, \tau)$ *the following statements are equivalent:*

- $(X, \mathcal{B}^X, \tau)$ *is a separated LEADER b-convergence space.*
- *There exists a topological space* (Y, cl_Y) *into which* X *can be densely topologically embedded in such a way that a uniform filter* $\mathcal{U}$ *on* X b-converges *to* $B \neq \emptyset$ *iff there exists a filter* $\mathcal{F} \in \boldsymbol{FIL}(X)$ *with* $\mathcal{F} \times \mathcal{F} \subseteq \mathcal{U}$ *and* $\bigcap\{\, cl_Y(A) \,|\, A \in sec\,\mathcal{F}\,\} \cap B \neq \emptyset$. □

Remark 4.7. As a consequence it follows that all separated LE-proximity spaces (X, δ) can be characterized by such an embedding into a topological space (Y, cl_Y) with $B\,\delta\,A$ iff $B \cap cl_Y(A) \neq \emptyset$.

Remark 4.8. We also note that the category ***STOP*** can be embedded into ***b-LEPROX***, the category of bounded LEADER proximity spaces (*e.g.*, for a supertopological space $(X, \mathcal{B}^X, \vartheta)$ we consider the relation $p_\vartheta \subseteq \mathcal{B}^X \times P(X)$ defined by $B\,p_\vartheta\,A$ iff $A \in sec\,\vartheta(B)$). Then, according to our main result, (separated) b-LEADER proximity spaces essentially are determined by their corresponding ***LEb***-convergence spaces, which in particular are pointed. But as shown in [11], the full subcategory ***pb-CONV***, whose objects are the pointed b-convergence spaces, constitutes a *strong topological universe*, in which the categories mentioned above can be embedded.

Figure 2 displays the full embeddings among the categories mentioned in this Section.

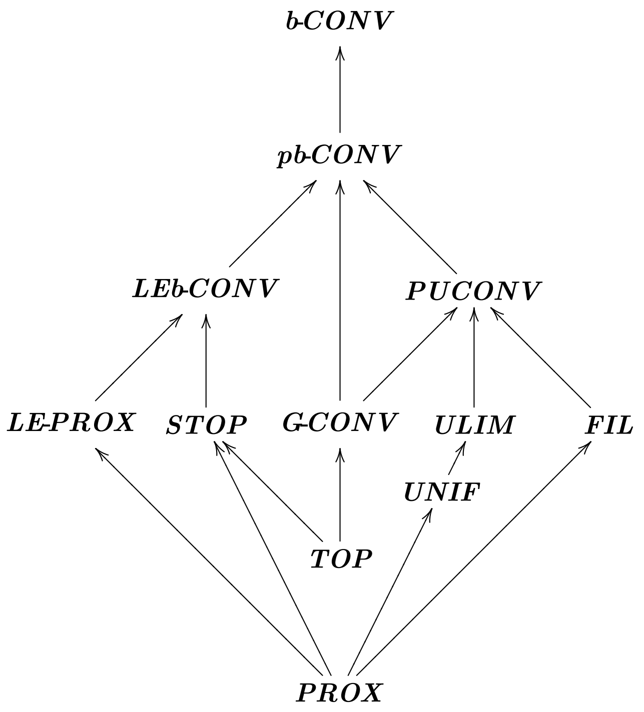

FIGURE 2. Full embeddings into the strong topological universe ***pb-CONV*** .

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