TOPOLOGY PROCEEDINGS

Volume 37, 2011

Pages 331–347

http://topology.auburn.edu/tp/

Metric representations of categories of closure spaces

by

Ralph Kopperman, F. Mynard and Peter Ruse

Electronically published on October 24, 2010

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 37 (2011)
Pages 331-347

E-Published on October 24, 2010

METRIC REPRESENTATIONS OF CATEGORIES OF CLOSURE SPACES

RALPH KOPPERMAN, F. MYNARD, AND PETER RUSE

Abstract. In this paper we modify the generalized quasi-metric representation of topological spaces of [21], by weakening the properties required of the set of positive elements, to obtain a similar representation of the category of neighborhood spaces and its subcategories of closure, and of pretopological spaces. Thus we show how these notions also arise from generalized metrics.

1. Introduction

Students first meet topology through metrics, and spaces arising from "good" metrics have useful properties: *completely metrizable spaces are Baire*, and *contractions from a complete metric space to itself have unique fixed points.*

The latter result in its classical form is a key step in a proof of the inverse function theorem of multivariate calculus. It also can be extended to so-called partial metric spaces (defined later in this paper), where its interpretation becomes that certain types of algorithms that begin with partial knowledge of an object (e.g., that a number to be found is in a given interval), will converge toward complete knowledge of the object; see [24].

2010 *Mathematics Subject Classification.* 54A05, 54E99, 54B30.

Key words and phrases. Functorial representation, value lattice, (separating, resp. filtered) upper subset (with halves), set of positives, generalized quasi-metric space, neighborhood space, closure operator, closure preserving function, closure space, (pre)topological space, (co)reflective subcategory, initially dense object.

As a result, there is a long tradition of obtaining generalized metric representations of topological spaces; such work goes back to early last century (see [28]); a thorough history of the subject is given in [23]. In this paper we are interested in the axiomatization in [21], where each topology is shown to arise from a "quasimetric" into a generalization $\mathcal{V}$ of $[0,\infty]$, with a subset P of $\mathcal{V}$ that satisfies certain natural "positivity" axioms that hold for $(0,\infty] \subseteq [0,\infty]$. Here we show that weakening these positivity axioms, again in natural ways, yields a similar representation of the category of neighborhood spaces (in terminology due to [20], also called generalized topology by Á. Császár) and its subcategories of closure, and of pretopological spaces, showing how these notions also arise from generalized metrics.

To be a bit more precise, we generalize the definition of metric closure: for a metric $d : X \times X \to [0,\infty)$ we say $x \in \mathsf{cl}_d(A)$ if for each positive r there is a $y \in A$ so that $d(x,y) \leq r$ (equivalently if for each positive r, $N_r(x) \cap A \neq \emptyset$, where $N_r(x) = \{y : d(x,y) \leq r\}$). Our representation is *functorial*: if (X, d_X), (Y, d_Y) are metric spaces, then $f : X \to Y$ is a closure preserving map (that is, for each $A \subseteq X$, $f[\mathsf{cl}(A)] \subseteq \mathsf{cl}(f[A])$) if and only if, for each positive r and $x \in X$ there is a positive s such that if $d_X(x,y) \leq s$ then $d_Y(f(x), f(y)) \leq r$. Two key problems, most easily discussed in the subclass of topological spaces, must be overcome:

(1) If we insist on the usual conventions that d is valued in $[0,\infty)$ and that r is positive means $r > 0$, then for each $x \in X$, $\{N_{1/n}(x) : n \in \mathbb{N}\}$ forms a countable base about x. We avoid such cardinal restrictions by allowing a class of other objects in which our metrics can take values, and defining positivity in each object.

(2) If we insist that d be symmetric (for each x, y, $d(x,y) = d(y,x)$), it then turns out that the topology arising from d must be completely regular (as shown in [21]), so we must drop some metric axioms.

To take care of (1), the type of codomain for our "generalized metrics", or metric substitutes, is chosen to allow as many of the usual constructions as possible; so $[0,\infty]$, $[0,1]$ (with truncated addition) and $\{0,1\}$, and their powers should be among the objects in which our metrics can take values. We also need to have limits in this space of values (to allow completions of generalized metric spaces – where the distance between limits is the limit of the distances); it is useful for such an object to be a complete lattice (unlike $[0,\infty)$).

We choose to have our metric go into what we call a value lattice. (Other possibilities are given in the "Concluding remarks" section). Before we can define these, we must first define a *continuous lattice*: it is a complete lattice $\mathcal{V}$ (that is, all subsets have suprema, thus infima as well), and for each element $q \in \mathcal{V}$, $\Downarrow q = \{p \ll q\}$ is directed, and q is its supremum, where $p \ll q$ means: if $q \leq \bigvee D$ and D is directed, then for some $r \in D$, $p \leq r$. Continuous lattices are discussed in detail in [18] and in [1]. A *value lattice* is a system $(\mathcal{V}, +, \leq, 0, \infty)$, such that:

$(\mathcal{V}, \leq)$ is a poset with least element 0 and greatest ∞,

$(\mathcal{V}, \geq)$ is a continuous lattice,

$(\mathcal{V}, +)$ is a commutative semigroup with identity 0, and

for each $r \in \mathcal{V}$ and $S \subseteq \mathcal{V}$, $r + \bigwedge S = \bigwedge\{r + s : s \in S\}$.

Among value lattices are $[0, 1]$ and $\{0, 1\}$, and products of value lattices are value lattices.

As a result of its preservation of infima, $+$ is order preserving: if $a \leq b$ then for each c, $c+a = c+\bigwedge\{a, b\} = \bigwedge\{c+a, c+b\} \leq c+b$. In particular for each $c, a \in \mathcal{V}$, $c = c + 0 \leq c + a \leq c + \infty = \infty$ (this last equality is due to the fact that ∞ is the largest element of $\mathcal{V}$, so $\infty = 0 + \infty \leq c + \infty \leq \infty$).

Now we get to the central issue of positivity. Given a poset $(\mathcal{V}, \leq)$, we must carefully consider the properties of subset P of positive elements of $\mathcal{V}$ (that is, "$x \in P$" means "x is positive"). A generalized metric and value lattice do not by themselves give rise to a closure operator. The use of "$r > 0$" in the definition of metric closure is subtle, and the facts $r > 0 \Rightarrow (\exists t > 0)(t + t \leq r)$ and $r, s > 0 \Rightarrow (\exists t > 0)(t \leq r \& t \leq s)$ are central to many metric arguments in topology, but it turns out, not for anologous ones involving neighborhood spaces. We now list some properties of the set of positive numbers that are needed for the usual arguments:

Given a poset $(\mathcal{V}, \leq)$, a subset $P \subseteq \mathcal{V}$ is an *upper subset* if $p \leq q$ and $p \in P \Rightarrow q \in P$; an upper subset of a value lattice is *separating* if whenever $a, b \in \mathcal{V}$ and $a \leq b + r$ for each $r \in P$ then $a \leq b$. The set P is *filtered* if whenever $p, q \in P$, there is an $r \in P$ such that $r \leq p$ & $r \leq q$, and P and *has halves* if for each $t \in P$ there is an $s \in P$ such that $s + s \leq t$. The set P is called a *set of positives* if it is a filtered upper set which has halves.

Next we consider (2). The usual metric axioms are:

(id) $d(x, x) = 0$,

(tri) $d(x, z) \leq d(x, y) + d(y, z)$,
(sym) $d(x, y) = d(y, x)$,
(t0) $d(x, y) = 0 \Rightarrow x = y$.

Below let $\mathcal{V}$ be a value lattice; we drop the requirements of symmetry and (t0) to concentrate on *$\mathcal{V}$-quasimetrics*: functions $q : X \times X \to \mathcal{V}$ that satisfy (tri) and (id). A *$\mathcal{V}$-quasimetric space* is a quadruple $\mathcal{X} = (X, \mathcal{V}, P, q)$ where X is a set, $q : X \times X \to \mathcal{V}$ is a $\mathcal{V}$-quasimetric, and P is a separating upper subset of $\mathcal{V}$. Also, a *generalized quasimetric space* is a $\mathcal{V}$-quasimetric space for some value lattice, $\mathcal{V}$.

In [21] the usual definition of metric topology was generalized: a set T is *open* in the generalized quasimetric space $\mathcal{X} = (X, \mathcal{V}, P, q)$, if for each $x \in T$ there is an $r \in P$ such that $N_r(x) \subseteq T$; the collection of open sets is denoted $\mathcal{T}_X$. It is then shown that if P is a set of positives, then $\mathcal{T}_X$ is a topology on X, and that each topology arises from a generalized quasimetric space $\mathcal{X}$ in which P is a set of positives. (The terminology there is a bit different; [21] used "value semigroups" in place of value lattices, but all the value semigroups that were needed there were powers of $[0, \infty]$, and so were value lattices as well.)

It was shown in [21] that a topology $\mathcal{T}$ is completely regular if and only if it is $\mathcal{T}_X$ for some $\mathcal{X}$ such that q satisfies (sym), and that $\mathcal{T}$ is T_0 if and only if q satisfies the quasimetric version of (t0): $q(x, y) = q(y, x) = 0 \;\Rightarrow\; x = y$. Therefore, these axioms have special roles and are not part of the definition of our categories of generalized quasimetric spaces.

A *continuous function* from a generalized quasimetric space $(X, \mathcal{V}_X, P_X, q_X)$ to another, $(Y, \mathcal{V}_Y, P_Y, q_Y)$ is an $f : X \to Y$ such that for each $x \in X$ and $s \in P_Y$, there is an $r \in P_X$ such that

$$q_X(x, y) \leq r \;\Rightarrow\; q_Y(f(x), f(y)) \leq s.$$

Below, we are interested in four categories:

The first is **QM**, the category of all generalized quasimetric spaces and continuous maps.

The others are full subcategories of **QM**, (recall that a full subcategory **D** of a category **C**, is one such that each object of **D** is an object of **C**, the **D**-maps between two objects **D** are precisely their **C**-maps). Thus the maps of the following categories are the continuous maps, and their objects are now described:

the objects of $\mathbf{QM}_{Fil}$ are the generalized quasimetric spaces such that P is filtered,

the objects of $\mathbf{QM}_{\frac{1}{2}}$ are the generalized quasimetric spaces such that P has halves, and

the objects of $\mathbf{QM}_{Fil,\frac{1}{2}}$ are the generalized quasimetric spaces with P a set of positives.

We also discuss some other representations below. For example, certain generalized quasimetric spaces come from generalized partial metrics, which have been developed by Steve Matthews (see [24]) to model the partial nature of knowledge produced by a computer program in finite time. In [22] it is shown that all topologies arise from generalized partial metric into value lattices. This option is further discussed in the "Concluding remarks" section.

Thus for topological spaces, generalized quasimetrizability becomes a unifying property; special properties hold for some topological spaces and maps because they satisfy special conditions in terms of generalized quasimetrics.

2. Categories of spaces with closure operators and of generalized quasimetric spaces giving rise to spaces with closure operators

We say that a category $\mathbf{C}$ *has generalized quasimetric representation* if there is a full subcategory $\mathbf{S}$ of $\mathbf{QM}$ that is category equivalent to $\mathbf{C}$, i.e., there exists a functor $F : \mathbf{S} \to \mathbf{C}$ that is full and faithful (that is, for each pair $A, B \in \mathbf{S}$, $F : H(A, B) \to H(F(A), F(B))$ is 1-1 and onto) and such that for every $\mathbf{C}$-object C, there exists an $\mathbf{S}$-object S such that $F(S)$ is isomorphic to C.

The goal of this note is to give generalized quasimetric representations for several categories of spaces with a closure operator. Specifically, let $\mathbf{NGB}$ denote the category whose objects are *neighborhood spaces*: pairs, (S, cl_S), S a set, with a function $\mathsf{cl}_S : 2^S \to 2^S$, satisfying the following:

cl_S is *increasing*: $A \subseteq B \Rightarrow \mathsf{cl}_S A \subseteq \mathsf{cl}_S B$

cl_S is *expansive*: each $A \subseteq \mathsf{cl}_S A$, and

cl_S is *grounded*: $\mathsf{cl}_S \emptyset = \emptyset$.

Any $\mathsf{cl}_S : 2^S \to 2^S$, satisfying these rules is called a *closure operator*. The morphisms $f : (X, \mathsf{cl}_X) \to (Y, \mathsf{cl}_Y)$ of $\mathbf{NGB}$ are the *closure preserving* functions:

$$\forall A \subseteq X : f[\mathsf{cl}_X A] \subseteq \mathsf{cl}_Y(f[A]), \text{ equivalently,}$$
$$\forall B \subseteq Y : f^{-1}[\mathsf{cl}_Y B] \subseteq \mathsf{cl}_X(f^{-1}[B]).$$

Let **CLS** denote the full subcategory of **NGB** whose objects are *closure spaces,* that is, sets with closure operators (S, cl_S) that are *idempotent*: $(\mathsf{cl}_S)^2 = \mathsf{cl}_S$. Let **PRT** be the full subcategory of **NGB** whose objects are *pretopological spaces,* that is, sets with closure operators (S, cl_S) that are *additive*: for every $A, B \in 2^X$, $\mathsf{cl}(A \cup B) = \mathsf{cl}A \cup \mathsf{cl}B$. Let **TOP** be the full subcategory of **NGB** whose objects are *topological spaces,* that is, sets with idempotent and additive closure operators. Thus $\mathbf{TOP} = \mathbf{PRT} \cap \mathbf{CLS}$. [20] covers the basics on neighborhood spaces. The *generalized topologies* of Á. Császár are equivalent to neighborhood spaces, except that they need not be grounded. He extensively studied these structures and their role in topology. Therefore a wealth of information on neighborhood spaces can be found in the papers [3]–[16]. [17] is a good source for an introduction on closure spaces and their use. Examples of how such general closures (equivalently, generalized topology) come into play in theoretical chemistry and biology can be found in e.g., [25], [26], [27].

Consider $QN : \mathbf{QM} \to \mathbf{NGB}$ defined on objects by $QN((X, \mathcal{V}, P, q)) = (X, \mathsf{cl}_{q,P})$ where $x \in \mathsf{cl}_{q,P} A$ if for every $r \in P$, $q(x, y) \leq r$ for some $y \in A$.

It is often useful to think in terms of interiors of sets, where these are defined by $\mathsf{int}(A) = X \setminus \mathsf{cl}(X \setminus A)$. Of course, $x \in \mathsf{int}_{q,P}(A)$ if and only if, for some $r \in P$, $N_r(x) \cap (X \setminus A) = \emptyset$, that is, if and only if, for some $r \in P$, $N_r(x) \subseteq A$.

Lemma 2.1. *QN takes values in* **NGB***, and the restrictions of QN to $\mathbf{QM}_{Fil}$, to $\mathbf{QM}_{\frac{1}{2}}$, and to $\mathbf{QM}_{Fil,\frac{1}{2}}$ take values in* **PRT***,* **CLS** *and* **TOP** *respectively.*

Proof. If $(X, \mathcal{V}, P, q)$ is a generalized quasimetric space then $\mathsf{cl}_{q,P} : 2^X \to 2^X$ is surely grounded and expansive, and it is increasing because for each $a \in X,\ r \in P,\ q(a, a) = 0 \leq r$.

If moreover P is filtered, then $\mathsf{cl}_{q,P}$ is additive. Indeed, for every $\mathcal{V}$-quasimetric space, since $QN((X, q_X))$ is a neighborhood space and $A, B \subseteq A \cup B$, $\mathsf{cl}_{q,P} A \cup \mathsf{cl}_{q,P} B \subseteq \mathsf{cl}_{q,P}(A \cup B)$ is true. For the reverse set inclusion, assume that $x \notin \mathsf{cl}_{q,P} A \cup \mathsf{cl}_{q,P} B$. Then there are $r, s \in P$ such that $N_r(x) \cap A = \emptyset$ and $N_s(x) \cap B = \emptyset$.

Since P is filtered, for some $t \in P,\ t \le r, s$, and so $N_t(x) \cap (A \cup B) \subseteq (N_r(x) \cap A) \cup (N_s(x) \cap B) = \emptyset$. Thus $x \notin \mathsf{cl}_{q,P}(A \cup B)$. In this case, $QN((X, q_X))$ is additive.

If P has halves, then $\mathsf{cl}_{q,P}$ is idempotent. Indeed, if $x \in \mathsf{cl}_{q,P}(\mathsf{cl}_{q,P}A)$ then for every $r \in P$, find $s \in P$ so that $s + s \le r$. Then $N_s(x) \cap \mathsf{cl}_{q,P}A \neq \emptyset$. Hence there is $y \in \mathsf{cl}_{q,P}A$ such that $q(x, y) \le s$. Moreover, there is $a \in A$ such that $q(y, a) \le s$. Hence $q(x, a) \le q(x, y) + q(y, a) \le s + s \le r$. Hence $N_r(x) \cap A \neq \emptyset$ so $x \in \mathsf{cl}_{q,P}A$. Thus if P has halves, $QN((X, q_X))$ is a closure space, and if this P is also filtered then $QN((X, q_X))$ is a topological space. □

The map QN extends naturally to a *concrete* functor; that is, all objects are **SET**-based, and the functor leaves both the sets and **SET**-maps unchanged. To see this, note that if $y \in f(\mathsf{cl}_{q_X}A)$ then there is $x \in \mathsf{cl}_{q_X}A$ such that $y = f(x)$. If $s \in P_Y$, by continuity of f, there is an $r \in P_X$ such that $q_X(x, z) \le r \Rightarrow q_Y(f(x), f(z)) \le s$. Since $x \in \mathsf{cl}_{q_X}A$, there exists $z \in N_r(x) \cap A \neq \emptyset$. Then $f(z) \in N_s(y) \cap f(A)$, showing $y \in \mathsf{cl}_{q_Y}(f(A))$. Hence continuous maps are closure preserving. Composition is also preserved, because a composition of closure preserving maps is closure preserving.

Lemma 2.2. *QN : $\mathbf{QM} \to \mathbf{NGB}$ is a full and faithful functor. Moreover, its restrictions QN : $\mathbf{QM}_{Fil} \to \mathbf{PRT}$, QN : $\mathbf{QM}_{\frac{1}{2}} \to \mathbf{CLS}$ and QN : $\mathbf{QM}_{Fil,\frac{1}{2}} \to \mathbf{TOP}$ are full and faithful.*

Proof. Let $f \in \mathrm{Hom}_{\mathbf{NGB}}(QN(X), QN(Y))$. Since QN preserves the **SET**-map of a morphism, we only need to check that $f : (X, \mathcal{V}_X, P_X, q_X) \to (Y, \mathcal{V}_Y, P_Y, q_Y)$ is continuous if $f : QN(X) \to QN(Y)$ is closure preserving. But if f is not continuous there are $x_0 \in X,\ s_0 \in P_Y$ such that for every $r \in P_X$ there is $x_r \in N_r(x_0)$ so that $f(x_r) \notin N_{s_0}(f(x_0))$. Let $A = \{x_r : r \in P_X\}$. By definition, $x_0 \in \mathsf{cl}_{QN(X)}A$. But $f(A) \cap N_{s_0}(f(x_0)) = \emptyset$, so $f(x_0) \notin \mathsf{cl}_{QN(Y)}(f(A))$. Hence $f : QN(X) \to QN(Y)$ is not closure preserving.

The categories $\mathbf{QM}_{Fil}$, $\mathbf{QM}_{\frac{1}{2}}$ and $\mathbf{QM}_{Fil,\frac{1}{2}}$ are full subcategories of **QM** and the categories **PRT**, **CLS** and **TOP** are full subcategories of **NGB**, so the restrictions of QN considered in the statement are also full and faithful. □

3. Quasimetric representation of categories of spaces with closure operators

By the previous section, to show that $QN : \mathbf{QM} \to \mathbf{NGB}$, $QN : \mathbf{QM}_{Fil} \to \mathbf{PRT}$, $QN : \mathbf{QM}_{\frac{1}{2}} \to \mathbf{CLS}$ and $QN : \mathbf{QM}_{Fil,\frac{1}{2}} \to \mathbf{TOP}$ are equivalences of categories, we only need to show that each neighborhood (pretopological, closure, topological respectively) space is the image under QN of a generalized quasimetric space (such that P is filtered, has halves, or is filtered and has halves, respectively). To perform these constructions, we need to introduce some machinery.

The pretopological space $\mathbf{3}$ is the set $\mathbf{3} = \{0, 1, 2\}$ with the additive closure operator $\mathsf{cl}_\mathbf{3}$ defined for nonempty sets, by $\mathsf{cl}_\mathbf{3}(A) = \{x \in \mathbf{3} : x \leq \max(A) + 1\}$. Its subspace $\mathbf{2} = \{0, 2\}$ is identified with the *Sierpiński* space (the Sierpiński space is usually considered to be the set $\{0, 1\}$ with the topology in which $\{0\}$ is the only nontrivial closed set). Finally, the pretopological space $\mathbf{3}^*$ is the set $\mathbf{3}$ with the additive closure operator $\mathsf{cl}_{\mathbf{3}^*}$ defined for nonempty sets by $\mathsf{cl}_{\mathbf{3}^*}(A) = \{x \in \mathbf{3} : x \geq \min(A) - 1\}$.

Define $\dot{-} : \mathbf{3} \times \mathbf{3}^* \to \mathbf{3}$ by $\dot{-}(a, b) = \max\{a - b, 0\}$; we usually write $a \dot{-} b$ for $\dot{-}(a, b)$. Certainly for each $x, y, z \in$, $x \dot{-} x = 0$ and $x \dot{-} z \leq (x \dot{-} y) + (y \dot{-} z)$. Also, $\dot{-}$ is separately continuous. To see continuity in the first variable, if $x \in \mathsf{cl}_\mathbf{3}(A)$ then $x \leq \max(A) + 1$, so $x \dot{-} y \leq \max(A \dot{-} \{y\}) + 1$, that is, $x \dot{-} y \in \mathsf{cl}_\mathbf{3}(A \dot{-} \{y\})$; continuity in the second variable is shown similarly.

Recall that a subcategory $\mathbf{R}$ of a category $\mathbf{D}$ is *reflective* if for every $\mathbf{D}$-object there is a *reflection map* $r_X : X \to rX$ where rX is an $\mathbf{R}$-object such that every $\mathbf{D}$-morphism $f : X \to Y$ between a $\mathbf{D}$-object X and a $\mathbf{R}$-object Y factors uniquely through r_X, that is, there is a unique morphism $\hat{f} : rX \to Y$ so that $\hat{f} \circ r_X = f$. By associating to each $\mathbf{D}$-morphism $f : X \to Y$ the $\mathbf{R}$-morphism $R(f) : rX \to rY$ defined by $R(f) = r_Y \circ \hat{f}$, we define a functor $R : \mathbf{D} \to \mathbf{R}$ called *reflector*. The reflectors we will use are *concrete*, that is, rX and X have the same underlying set and r_X is the set-theoretic identity. As a result, rX defines the finest $\mathbf{R}$-structure on the underlying set of X which is coarser than X, and for each map, f and $R(f)$ have the same underlying $\mathbf{SET}$-map.

Dually, a subcategory **C** of **D** is *coreflective* if for each **D**-object there is a *coreflection map* $c_X : cX \to X$ where cX is a **C**-object such that every **D**-morphism $f : X \to Y$ between a **D**-object X and a **C**-object Y factors uniquely through c_X. As before, this defines the *coreflector* functor $C : \mathbf{D} \to \mathbf{C}$ in such a way that $C(f) : cX \to cY$ is a **C**-morphism whenever $f : X \to Y$ is a **D**-morphism. The coreflectors we will use are also concrete so cX defines the coarsest **C**-structure on the underlying set of X which is finer than X.

The next lemma contains many of the categorical facts we find useful. They are well known, and we include proofs only for the convenience of the reader. In particular, the fact that **PRT** is coreflective in **NGB** and that **CLS** is reflective in **NGB** can be found in [20]. It is also observed there that the restrictions of the corresponding coreflector and reflector to **CLS** and **PRT** respectively define a coreflector and reflector onto **TOP**.

Lemma 3.1. *For each subset C of X, we define $w_C : X \to \mathbf{3}$ by*

$$w_C(x) = \begin{cases} 0 & x \in \quad C \\ 1 & x \in \mathsf{cl}_X C \setminus C \, . \\ 2 & x \notin \quad \mathsf{cl}_X C \end{cases}$$

(a) Let X be a neighborhood space. Then for each subset C of X, w_C is continuous. Also, if Z is any neighborhood space and $f : Z \to X$ any function, then f is continuous if and only if $w_C f : Z \to \mathbf{3}$ is continuous for each C.

(b) Let X be a closure space. Then a subset C of X is closed if and only if $w_C : X \to \mathbf{2}$ is continuous. Also, if Z is a closure space and $f : Z \to X$ any function, then f is continuous if and only if $w_C f : Z \to \mathbf{2}$ is continuous for each closed C.

(c) The category **PRT** *is a coreflective subcategory of* **NGB** *and* **TOP** *is a coreflective subcategory of* **CLS**. *In each case, given any object X, the closure operator on cX is: $\mathsf{cl}_{cX}(A) = \bigcap\{\bigcup_{i=1}^n \mathsf{cl}_X(A_i) : A \subseteq \bigcup_{i=1}^n A_i,\ n \in \mathbb{N}\}$.*

(d) The category **CLS** *is a reflective subcategory of* **NGB** *and* **TOP** *is a reflective subcategory of* **PRT**. *In each case, for any object X, the closure in rX of any set is the smallest closed set in X containing it, that is: $\mathsf{cl}_{rX}(A) = \bigcap\{C : A \subseteq C = \mathsf{cl}_X(C)\}$. Further, a function $f : X \to Y$, $X \in$* **NGB**, *$Y \in$* **CLS** *is continuous if and only if, $f^{-1}[D]$ is closed for each set D closed in Y.*

Proof. (a) To see that w_C is continuous, note that as for any function, $w_C[\mathsf{cl}_X\emptyset] = \emptyset = \mathsf{cl}_3 w_C[\emptyset]$; if $A \subseteq C$ then $w_C[\mathsf{cl}_X A] \subseteq w_C[\mathsf{cl}_X C] = \{0,1\} = \mathsf{cl}_3 w_C[A]$ and otherwise, $w_C[\mathsf{cl}_X A] = \{0,1,2\} = \mathsf{cl}_3 w_C[A]$.

Now suppose $f : Z \to X$ is any function. Certainly if f is continuous then each $w_C f$ is continuous, since w_C is. Conversely, if f is not continuous then there is a $D \subseteq Z$ such that $f[\mathsf{cl}_Z(D)] \not\subseteq \mathsf{cl}_X(f[D])$. Thus $w_{f[D]} f[\mathsf{cl}_Z(D)] = \{0,1,2\}$, while $\mathsf{cl}_3(w_{f[D]} f[D]) = \{0,1\} \not\supseteq w_{f[D]} f[\mathsf{cl}_Z(D)]$.

(b) First note that if C is closed then $w_C(x)$ cannot take on the value 1. The continuity of w_C was shown in the proof of (a) and thus again if $f : Z \to X$ is continuous, so is each $w_C f$. But if f is not continuous, there is a closed $D \subseteq X$ such that $f^{-1}[D]$ is not closed in Z. But then for $A = f^{-1}[D]$, $f[\mathsf{cl}_Z(A)] \not\subseteq D = \mathsf{cl}_X[D]$, so $w_D[\mathsf{cl}_Z(A)] = \{0,2\} \not\subseteq \{0\} = \mathsf{cl}_2(w_D f[A])$.

(c) The function cl_{cX} is easily seen to be a closure operator since cl_X is one. Also, for $A, B \subseteq X$, whenever $A \subseteq \bigcup_{i=1}^n A_i$, $n \in \mathbb{N}$ and $B \subseteq \bigcup_{j=1}^m B_j$, $m \in \mathbb{N}$, then $A \cup B \subseteq \bigcup_{i=1}^n A_i \cup \bigcup_{i=j}^m B_j$, so $\mathsf{cl}_{cX}(A \cup B) \subseteq \bigcup_{i=1}^n \mathsf{cl}_X(A_i) \cup \bigcup_{j=1}^m \mathsf{cl}_X(B_j)$. By the arbitrary nature of the A_i, B_j, this shows $\mathsf{cl}_{cX}(A \cup B) \subseteq \mathsf{cl}_{cX}(A) \cup \mathsf{cl}_{cX}(B)$. This shows that cl_{cX} is additive, so $cX \in$ **PRT** whenever $X \in$ **NGB** and $cX \in$ **TOP** whenever $cX \in$ **CLS**.

Recall that for all these categories, the morphisms are the continuous maps. Also if $f : X \to Y$, is continuous, and cl_X is additive then $f : X \to Y$ is continuous as well. Indeed, for each $B \subseteq Y$, $\mathsf{cl}_X(f^{-1}[B]) \subseteq f^{-1}[\mathsf{cl}_{cY} B]$. To see this, consider subsets B_i of Y such that $B \subseteq \bigcup_{i=1}^{i=n} B_i$. Then $f^{-1}[\bigcup_{i=1}^{i=n} \mathsf{cl}_Y B_i] = \bigcup_{i=1}^{i=n} f^{-1}[\mathsf{cl}_Y B_i]$ and by continuity of $f : X \to Y$, $\mathsf{cl}_X(f^{-1}[B_i]) \subseteq f^{-1}[\mathsf{cl}_Y B_i]$. Moreover, by additivity of cl_X,

$$\mathsf{cl}_X(f^{-1}[\bigcup_{i=1}^{i=n} B_i]) = \mathsf{cl}_X(\bigcup_{i=1}^{i=n} f^{-1}[B_i]) = \bigcup_{i=1}^{i=n} \mathsf{cl}_X(f^{-1}[B_i]).$$

Because $\mathsf{cl}_X(f^{-1}[B]) \subseteq \mathsf{cl}_X(f^{-1}[\bigcup_{i=1}^{i=n} B_i])$, we obtain that $\mathsf{cl}_X(f^{-1}[B]) \subseteq f^{-1}[\bigcup_{i=1}^{i=n} \mathsf{cl}_Y B_i]$, thus $\mathsf{cl}_X(f^{-1}[B]) \subseteq f^{-1}[\mathsf{cl}_{cY} B]$. This shows that in both cases, cX is our coreflection.

(d) The function cl_{rX} is easily seen to be a closure operator since cl_X is one. Also, cl_{rX} is idempotent, for if $A \subseteq X$ then $\mathsf{cl}_{rX}(\mathsf{cl}_{rX}(A)) = \mathsf{cl}_{rX}(\bigcap\{C : A \subseteq C = \mathsf{cl}_X(C)\}) \subseteq \bigcap\{\mathsf{cl}_{rX}(C) : A \subseteq C = \mathsf{cl}_X(C)\} = \mathsf{cl}_{rX}(A)$.

For each continuous map $f : X \to Y$, $X, Y \in \mathbf{NGB}$, if $D \subseteq Y$ is closed then $f[\mathsf{cl}_X(f^{-1}[D])] \subseteq \mathsf{cl}_Y(f[f^{-1}[D]]) \subseteq \mathsf{cl}_Y(D) = D$, so $\mathsf{cl}_X(f^{-1}[D])] \subseteq f^{-1}[D]$, therefore $f^{-1}[D]$ is closed. Thus if cl_Y is idempotent then for any A, $\mathsf{cl}_Y(f[A])$ is closed, so A is a subset of the closed set $f^{-1}[\mathsf{cl}_Y(f[A])]$. As a result, $f^{-1}[\mathsf{cl}_Y(f[A])]$ contains the smallest closed set containing A, $\mathsf{cl}_{cX}(A)$, and it follows that $\mathsf{cl}_{rX}(A) \subseteq \mathsf{cl}_Y(f[A])$, so $f : rX \to Y$ is continuous as well, and the case $X \in \mathbf{PRT}$ is a special case. Thus in both cases, rX is our reflection.

On the characterization of continuity, we showed in the first sentence of the last paragraph that even more generally, $f^{-1}[D]$ is closed for closed $D \subseteq Y$. Conversely, if $Y \in \mathbf{CLS}$ and $f^{-1}[D]$ is closed for each closed $D \subseteq Y$, let $A \subseteq X$: then $\mathsf{cl}_Y(f[A])$ is closed so $f^{-1}[\mathsf{cl}_Y(f[A])]$ is closed in X and clearly contains A, thus $\mathsf{cl}_X(A) \subseteq f^{-1}[\mathsf{cl}_Y(f[A])]$, so $f[\mathsf{cl}_X(A)] \subseteq \mathsf{cl}_Y(f[A])$, showing f to be continuous. $\square$

For any index set I, $\mathcal{V} = [0,\infty]^I$ is a value lattice, as a power of the value lattice $[0,\infty]$. Let $I \subseteq 2^X$; and for $C \in I$, define $q_C(x,y) = \max\{w_C(x) - w_C(y), 0\}$. Then let $q : X \times X \to \mathcal{V} = [0,\infty]^I$ be defined by $q(x,y)(C) = q_C(x,y)$. Certainly by corresponding facts for $\dot{-}$, $q_C(x,x) = 0$, and $q_C(x,z) \leq q_C(x,y) + q_C(y,z)$ for each $x, y, z \in X$; as a result, for any separating upper subset P of $\mathcal{V}$, q is a $\mathcal{V}$-quasimetric.

For $C \in I$ and $t \in [0,\infty]$, we define $r_{C,t} \in \mathcal{V}$ by $r_{C,t}(C) = t$ and $r_{C,t}(D) = \infty$ if $D \neq C$. We also define four upper subsets of $\mathcal{V}$: $P_{NGB} = \{r_{C,t} \in \mathcal{V} : t \geq 1.5\}$, $P_{PRT} = \{r \in \mathcal{V} : (\exists n \in \mathbb{N}, r_1, ..., r_n \in P_{NGB})(r = r_1 \wedge ... \wedge r_n)\}$, that is, P_{PRT} is the filter generated by P_{NGB}; $P_{CLS} = \{r_{C,t} \in \mathcal{V} : t > 0\}$, and $P_{TOP} = \{r \in \mathcal{V} : (\exists n \in \mathbb{N}, r_1, ..., r_n \in P_{CLS})(r = r_1 \wedge ... \wedge r_n)\}$. Note that all these are upper sets, P_{CLS} has halves, P_{PRT} is filtered, and P_{TOP} has halves and is filtered, so it is a set of positives.

Theorem 3.2. *(a) For each neighborhood space, X, there is a generalized quasimetric space $X = (X, \mathcal{V}, P, q)$, such that $\mathsf{cl}_{q,P} = \mathsf{cl}_X$. Thus $QN : \mathbf{QM} \to \mathbf{NGB}$ is an equivalence of categories.*

(b) For each pretopological space, X, there is a generalized quasimetric space $(X, \mathcal{V}, P, q)$, such that P is filtered and $\mathsf{cl}_{q,P} = \mathsf{cl}_X$. Thus $QN : \mathbf{QM}_{Fil} \to \mathbf{PRT}$ is an equivalence of categories.

(c) For each closure space, X, there is a generalized quasimetric space $(X, \mathcal{V}, P, q)$, such that P has halves and $\mathsf{cl}_{q,P} = \mathsf{cl}_X$. Thus $QN : \mathbf{QM}_{\frac{1}{2}} \to \mathbf{CLS}$ is an equivalence of categories.

(d) For each topological space, X, there is a generalized quasimetric space $(X, \mathcal{V}, P, q)$, so that P is a set of positives and $\mathsf{cl}_{q,P} = \mathsf{cl}_X$. Thus $QN : \mathbf{QM}_{Fil,\frac{1}{2}} \to \mathbf{TOP}$ is an equivalence of categories.

Proof. We showed when $\mathcal{V}, q$, were defined in the paragraph just before this theorem, that $\mathcal{V}$ is a value lattice and q is a $\mathcal{V}$-quasimetric. By Lemma 3.1, we need only show that each of the four restrictions of QN is onto:

(a) Let $I = 2^X$ and $P = P_{NGB}$. To see that for each $C \subseteq X$, $\mathsf{cl}_X(C) = \mathsf{cl}_{q,P}(C)$, note that if $x \notin \mathsf{cl}_X(C)$ then $w_C(x) = 2$, so $N_{r_{C,1.5}}(x) \cap C = \emptyset$. On the other hand, if for some $r \in P$, $N_r(x) \cap C = \emptyset$, then $r = r_{A,t}$ for some $A \subseteq X$, $t \geq 1.5$. But this means that $w_A(x) = 2$ and $w_A[C] = \{0\}$, so $C \subseteq A$ and $x \notin \mathsf{cl}_X(A)$, thus $x \notin \mathsf{cl}_X(C)$. This shows $\mathsf{cl}_X(C) = \mathsf{cl}_{q,P}(C)$ for each $C \subseteq X$, so $\mathsf{cl}_X = \mathsf{cl}_{q,P}$.

(b) First note that for each $X \in \mathbf{NGB}$, if $\mathsf{cl}_X = \mathsf{cl}_{q,P}$ then its $\mathbf{PRT}$-coreflection is $(X, \mathsf{cl}_{q,\mathcal{F}(P)})$, where for an upper set P, $\mathcal{F}(P) = \{s : \exists n,\ r_1, ..., r_n \in P, \bigwedge_{i=1}^n r_i \leq s\}$, the filter generated by P. For: $x \in \mathsf{int}_{cX}(A) \Leftrightarrow$

$$(\exists n,\ A_1, ..., A_n \subseteq X)(x \in \textstyle\bigcap_{i=1}^n \mathsf{int}_X(A_i) \text{ and } \bigcap_{i=1}^n A_i \subseteq A)$$
$$\Leftrightarrow (\exists n,\ r_1, ..., r_n \in P)(N_{r_i}(x) \subseteq A_i) \text{ and } \textstyle\bigcap_{i=1}^n A_i \subseteq A)$$
$$\Leftrightarrow (\exists n,\ r_1, ..., r_n \in P)(\textstyle\bigcap_{i=1}^n N_{r_i}(x) \subseteq A)$$

$\Leftrightarrow (\exists n,\ r_1, ..., r_n \in P)(N_{\wedge_{i=1}^n r_i}(x) \subseteq A) \Leftrightarrow (\exists r \in \mathcal{F}(P)(N_r(x) \subseteq A)$, and these equivalences show $x \in \mathsf{int}_{cX}(A) \Leftrightarrow x \in \mathsf{int}_{q,\mathcal{F}(P)}(A)$. Thus $\mathsf{cl}_{cX} = \mathsf{cl}_{q,\mathcal{F}(P)}$ so in particular, if also $X \in \mathbf{PRT}$ and $\mathsf{cl}_X = \mathsf{cl}_{q,P}$ then $cX = X$ so $\mathsf{cl}_X = \mathsf{cl}_{q,\mathcal{F}(P)}$, and $\mathcal{F}(P)$ is filtered.

(c) In general, $x \in \mathsf{int}_{q,P_{NGB}}(A)$ if and only if, for some C, $N_{r_{C,1.5}}(x) \subseteq A$. Note that

$$N_{r_{C,1.5}}(x) = \{y : q_C(x, y) \leq 1\} = \begin{cases} \mathrm{X} & \mathrm{x} \in \mathsf{cl}_X(C) \\ \mathrm{X} \backslash C & \text{otherwise} \end{cases}.$$

Since X is open in each neighborhood space, we assume with no loss of generality that $A \neq X$; then by the previous line,
$N_{r_{C,1.5}}(x) \subseteq A \Leftrightarrow X/C \subseteq A \Rightarrow X \setminus \mathsf{cl}_X(C) \subseteq A \Leftrightarrow N_{r_{\mathbf{cl}_X(C),1.5}}(x) \subseteq A$.
If $X \in \mathbf{CLS}$ then $\mathsf{cl}_X(C)$ is always closed, thus $w_C : X \to \mathbf{2}$ is continuous, so for each $t > 0$, $N_{r_{\mathbf{cl}_X(C),1.5}}(x) = N_{r_{\mathbf{cl}_X(C),t}}(x)$.

Thus in this case, $x \in \mathsf{int}_{q,P_{NBD}}(A)$ if and only if for some closed $D \subseteq X$ and $t > 0$, $N_{r_{D,t}}(x) \subseteq A$. Now let $I = \mathcal{C}$, the collection of closed sets in X, and define q' by $q'(x,y) = q(x,y)|I$; then the above says that $x \in \mathsf{int}_{q,P_{NBD}}(A)$ if and only if $x \in \mathsf{int}_{q',Q}(A)$, where $Q = P^I_{CLS}$, which has halves.

(d) This follows from (c) by the argument of (b), observing in the last sentence that if $X \in \mathbf{TOP}$ then $X \in \mathbf{CLS}$, so P can be assumed to have halves. □

4. Concluding Remarks

Here we discuss some topics related to our characterizations of the above categories of closure spaces. We first outline an alternate (more structural) proof of Theorem 3.2, and later give another representation in terms of generalized partial metrics.

The alternative proof is based on a variant of the classical embedding lemma (e.g., [19], p.115). In the classical case, we use the fact that a space X can be represented as a subspace of a power of a standard space. Here, roughly speaking, we will not require that X embeds into such a power but only that it carries the initial structure induced by a subspace of such a power. Let us explain this idea in more detail.

A common feature of the four categories **TOP**, **PRT**, **CLS** and **NGB** is that they are *simple*, that is, each contains an initially dense object. An object C of a category $\mathbf{C}$ is *initially dense* if for every object B of $\mathbf{C}$, there is an *initial source* $I \subseteq H(B,C)$ such that for each function $g : D \to B$, D any object of $\mathbf{C}$, $g \in H(D,B)$ if and only if $hg \in H(D,C)$ for each $h \in I$. In our context this means that each object B of $\mathbf{C}$ carries the coarsest structure making each $h \in I$ continuous. The pretopological space $\mathbf{3}$ is initially dense in **PRT** (e.g., [2]) as well as in **NGB** (Lemma 3.1 (a)) while the topological space $\mathbf{2}$ is initially dense in **TOP** and in **CLS** (Lemma 3.1 (b)).

Let $\mathbf{C}$ denote one of these four categories and let C_0 denote the corresponding initially dense object. Note that

$$\mathbf{3} = QN(X_3) \text{ and } \mathbf{2} = QN(X_2),$$

where $X_3 = (\{0,1,2\}, \mathbf{3}, \{1,2\}, e)$ and $X_2 = (\{0,2\}, \mathbf{2}, \{2\}, e)$. Here $\mathbf{3}$ is considered as a value lattice and $\mathbf{2}$ carries the value lattice structure induced by that of $\mathbf{3}$. In $\mathbf{3}$, the operation $+$ is defined by

$a + b = (a \oplus b) \wedge 2$ where $\oplus$ is the usual addition of real numbers, the identity is 0, its absorbing element is 2 and the order is the usual linear order. Finally $e : \mathbf{3} \times \mathbf{3} \to \mathbf{3}$ and its restriction $e : \mathbf{2} \times \mathbf{2} \to \mathbf{2}$ define generalized quasimetrics via $e(x, y) = x \dot{-} y$. Note that in $\mathbf{3}$, the set $\{1, 2\}$ is filtered and in $\mathbf{2}$, the set $\{2\}$ is a set of positives.

In other words, if $\mathbf{QM}_C$ denotes the subcategory of $\mathbf{QM}$ corresponding to $\mathbf{C}$ as in Lemma 2.1, there is Y in $\mathbf{QM}_C$ such that $QN(Y) = C_0$. Moreover, if X is an object of $\mathbf{C}$, then $(f : X \to C_0)_{f \in C(X, C_0)}$ is an initial source. Equivalently, the map $i : X \to \prod^{\mathbf{C}}_{f \in C(X,C_0)} C_0$ defined by $i(x)(f) = f(x)$ is initial (where $\prod^{\mathbf{C}}$ denotes the product in $\mathbf{C}$).

Moreover,

$$\prod_{f \in C(X,C_0)}^{\mathbf{C}} C_0 = \prod_{f \in C(X,C_0)}^{\mathbf{C}} QN(Y) = QN(\prod_{f \in C(X,C_0)}^{\mathbf{QM}_C} Y),$$

where $\prod^{\mathbf{QM}_C}$ denotes the product in $\mathbf{QM}_C$. Indeed, an inspection of the product structures reveals that $QN : \mathbf{QM}_C \to \mathbf{C}$ commutes with products (see for instance [20] for products in the four instances of $\mathbf{C}$). Hence $i : X \to QN(\prod^{\mathbf{QM}_C}_{f \in C(X,C_0)} Y)$ is initial in $\mathbf{C}$. Let $(Z, \mathcal{V}, P, d)$ denote the object $\prod^{\mathbf{QM}_C}_{f \in C(X,C_0)} Y$ of $\mathbf{QM}_C$. It is now a simple verification that the $\mathcal{V}$-quasimetric $q : X \times X \to \mathcal{V}$ defined by $q(x, y) = d(i(x), i(y))$ is such that $QN((X, \mathcal{V}, P, q)) = X$ because i is initial.

Next we consider characterizations in terms of generalized partial metrics into value lattices which are parallel to those above. A *partial metric* on a set X is a function $p : X \times X \to [0, \infty)$ which satisfies:

$p(x, x) \le p(x, y)$ (small self-distance, corresponding to $d(x, y) \ge 0$),

if $p(x, x) = p(x, y) = p(y, y)$ then $x = y$ (t0),

$p(x, y) = p(y, x)$ (sym), and

$p(x, z) \le p(x, y) + (p(y, z) - p(y, y))$ (tri).

These were introduced by S. G. Matthews (see [24]), a computer scientist; the key difference between metrics and partial metrics is that $p(x, x) > 0$ is possible for the latter. It is natural in computing to include both objects we are trying to compute and results of the computation so far in a single space, and if a program has determined at a certain point the first n places $x_0.x_1...x_n$ of a number, x, then we know so far only the distance between x and any other number to within 10^{-n}, so it is natural to set $p(x_0.x_1...x_n, x_0.x_1...x_n) = 10^{-n}$. Note that a partial metric space in which $p(x,x) = 0$ for each x, is a metric space. In [22] this definition was extended to allow generalized partial metrics into value quantales, and it was shown, among other things, that each topology arises from a generalized partial metric.

To be more precise, given a $\mathcal{V}$-partial metric on X, and $x \in X$, $r \in \mathcal{V}$, define $N_r(x) = \{y : p(x,y) \leq p(x,x) + r\}$; also, the *topology induced by p*, τ_p is the one in which a set T is open if for each $x \in T$ there is an $r \in P$, $N_r(x) \subseteq T$. But if we define $q_p(x,y) = p(x,y) \dot{-} p(x,x)$; then q_p is easily seen to be a $\mathcal{V}$-quasimetric, and the following are clear:

- $N_r(x) = \{y : q_p(x,y) \leq r\}$,
- (X, τ_p) is the topological space in which $\mathsf{cl}_{q_p,P} = \mathsf{cl}_X$.

But reconsidering the definitions before Theorem 3.2, for $I \subseteq 2^X$; $C \in I$, define $p_C(x,y) = \max\{w_C(x), w_C(y)\}$ and let $p : X \times X \to [0,\infty]^I$ by $p(x,y)(C) = p_C(x,y)$. It is then easy to check that p is a generalized partial metric and that for each $C \in I$, $x, y \in X$, $q_C(x,y) = p_{q_C}(x,y)$. As a result, $q_p = q : X \times X \to [0,\infty]$ so $\mathsf{cl}_{q_p,P} = \mathsf{cl}_{q,P}$, so defining $\mathsf{cl}_{p,P} = \mathsf{cl}_{q_p,P}$, each characterization in Theorem 3.2 with "quasimetric" replaced by "partial metric" follows from the original.

References

[1] S. Abramsky, A. Jung, *Domain theory*, In S. Abramsky, D. M. Gabbay, T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science,* vol. III. Oxford University Press, 1994. (ISBN 0-19-853762-X)

[2] G. Bourdaud, *Espaces d'Antoine et semi-espaces d'Antoine*, Cahiers de Topologie Géom. Différentielle Catégorique, **16** (1975), 107– 133.

[3] Császár, Á., *Generalized topology, generalized continuity*, Acta Math. Hungar., **96** (2002), no. 4, 351–357.

[4] Császár, Á., *γ-connected sets*, Acta Math. Hungar., **101** (2003), no. 4, 273–279.

[5] Császár, Á., *Separation axioms for generalized topologies*, Acta Math. Hungar., **104** (2004), no. 1-2, 63–69.

[6] Császár, Á., *Extremally disconnected generalized topologies*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math., **47** (2004), 91–96 (2005).

[7] Császár, Á., *Closures of open sets in generalized topological spaces*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math., **47** (2004), 123–126 (2005).

[8] Császár, Á., *Generalized open sets in generalized topologies*, Acta Math. Hungar., **106** (2005), no. 1-2, 53–66.

[9] Császár, Á., *Monotonicity properties of operations on generalized topologies*, Acta Math. Hungar., **108** (2005), no. 4, 351–354.

[10] Császár, Á., *Ultratopologies generated by generalized topologies*, Acta Math. Hungar., **110** (2006), no. 1-2, 153–157.

[11] Császár, Á., *λ-open sets in generalized topological spaces*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **49** (2006), 59–63.

[12] Császár, Á. *Intersections of open sets in generalized topological spaces*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math., **49** (2006), 53–57.

[13] Császár, Á., *Modification of generalized topologies via hereditary classes*, Acta Math. Hungar., **115** (2007), no. 1-2, 29–36.

[14] Császár, Á., *Normal generalized topologies*, Acta Math. Hungar., **115** (2007), no. 4, 309–313.

[15] Császár, Á., *Remarks on quasi-topologies*, Acta Math. Hungar. **119** (2008), no. 1-2, 197–200.

[16] Császár, Á., *δ- and θ-modifications of generalized topologies*, Acta Math. Hungar., **120** (2008), no. 3, 275–279.

[17] M. Erné, *Closure*, in F. Mynard, E. Pearl, editors, *Beyond Topology, Contemporary Mathematics*, AMS, 2009.

[18] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott, *Continuous Lattices and Domains*, Cambridge University Press, 2003. ISBN 0-521-80338-1

[19] J. L. Kelley, *General Topology*, van Nostrand, 1955.

[20] D. C. Kent and W. K. Won, *Neighborhood Spaces*, Int. J. Math. Math. Sci., **32** No 7 (2002), 387–399.

[21] R. Kopperman, *All Topologies Come From Generalized Metrics*, Am. Math. Monthly, **95** (1988), 89–97.

[22] R. Kopperman, S. Matthews, and H. Pajoohesh, *Partial metrizability in value quantales*, Applied General Topology v. 5, No 1 (2004), 115–127.

[23] Hans-Peter Kunzi, *Nonsymmetric distances and their associated topologies: About the origins of basic ideas in the area of asymmetric topology*, in C. E. Aull and R. Lowen, eds., *Handbook of the History of General Topology*, Vol.3, Dordrecht, Kluwer (2001), 853–968.

[24] Matthews, S. G., *Partial metric topology*, Proc. 8th summer conference on topology and its applications, ed S. Andima et al., New York Academy of Sciences Annals, **728** (1994), 183–197.

[25] B. Stadler and P. Stadler, *The topology of evolutionary biology*, in *Modeling in Molecular Biology*, pages 267–286, G. Ciobanu and G. Rozenberg, editors. Springer Verlag, 2004.

[26] B. Stadler, P. Stadler, G. Wagner, and W. Fontana, *The topology of the possible: Formal spaces underlying patterns of evolutionary change*, J. Theor. Biol., **213**:241–274, 2001.

[27] G. P. Wagner and P. F. Stadler, *Quasi-independence, homology and the unity of type: a topological theory of characters*, J. Theoret. Biol., **220**(4): 505–527, 2003.

[28] Wilson, W. A., *On quasi-metric spaces,* Amer. J. Math., **53** (1931), 675–684.

Department of Mathematics, City College of New York, New York, N. Y. 10031

E-mail address: rdkcc@ccny.cuny.edu

Department of Mathematics, Georgia Southern University, Statesboro GA, 30460

E-mail address: topofred@gmail.com

Department of Mathematics, Baruch College of CUNY, New York, N. Y. 10010

E-mail address: peterruse@yahoo.com