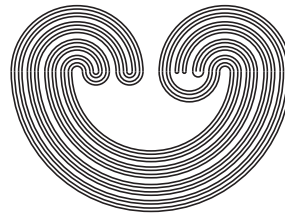

TOPOLOGY PROCEEDINGS



Volume 37, 2011

Pages 349–366

<http://topology.auburn.edu/tp/>

COMPARING LOCALLY ISOMETRIC TOPOLOGIES FOR \mathbb{R}^n

by

JON W. SHORT AND T. CHRISTINE STEVENS

Electronically published on October 24, 2010

Topology Proceedings

Web: <http://topology.auburn.edu/tp/>

Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA

E-mail: topolog@auburn.edu

ISSN: 0146-4124

COPYRIGHT © by Topology Proceedings. All rights reserved.



COMPARING LOCALLY ISOMETRIC TOPOLOGIES FOR \mathbb{R}^n

JON W. SHORT AND T. CHRISTINE STEVENS

ABSTRACT. In a previous paper, the authors showed that a large class of metrizable group topologies for \mathbb{R}^n are locally isometric. The metrics in question are defined by specifying a sequence in \mathbb{R}^n and the rate at which it converges to zero, and the corresponding topologies are always weaker than the usual topology for \mathbb{R}^n . Under rather mild restrictions on the sequences and the rate at which they converge to zero, two very different sequences will yield metrics ν and μ that make (\mathbb{R}^n, ν) and (\mathbb{R}^n, μ) locally isometric, and the completions of these groups will also be locally isometric. Since the local isometry cannot, in general, be extended to a global homomorphism, the question of whether (\mathbb{R}^n, ν) and (\mathbb{R}^n, μ) are isomorphic as topological groups remains open. In the current paper we explore the conditions that determine whether such an isomorphism exists. Our results have applications to the larger problem of determining the ways in which the topology of an arbitrary connected Lie group can be weakened, while remaining a Hausdorff topological group.

1. INTRODUCTION

We study the properties of a collection of group topologies for the additive group \mathbb{R}^n that are metrizable and weaker than the usual topology. These topologies are defined by choosing a sequence $\{v_i\}$ in \mathbb{R}^n and specifying the approximate rate $\{p_i\}$ at which it will

2010 *Mathematics Subject Classification.* Primary 22A05; Secondary 54H11.

Key words and phrases. Topological group, topological isomorphism, converging sequence, weakened Lie group.

©2010 Topology Proceedings.

converge to the identity. We refer to $\{v_i\}$ and $\{p_i\}$ as the “converging sequence” and the “rate sequence,” respectively. Using a strengthened version of a theorem in [7], we prove that the resulting topologies are always locally isometric, provided the rate sequence is the same. This raises the question, which we investigate in Sections 3 and 4, of whether the two corresponding topologies must be *globally* the same. Section 3 presents some conditions that guarantee an affirmative answer to that question, while Section 4 contains a procedure for constructing examples where the answer is negative. In Section 5, we discuss the possibility that the two groups are topologically isomorphic, even though they have distinct topologies, and we show that this reduces to the situation addressed in Sections 3 and 4. Finally, Section 6 announces results that compare topologies that share the same converging sequence but have different rate sequences.

Our investigation is related to the study of Lie groups of transformations. If L is a Lie group acting on a topological space, then that action gives rise to a topology for L that is weaker than the Lie topology and that makes L a topological group. If L is connected, then Theorem 3.2 in [9] says that the ways in which the Lie topology can be weakened and remain a Hausdorff group topology are completely determined by a certain closed abelian subgroup of L , which is called a decisive subgroup. As an abelian Lie group, that decisive subgroup must have the form $\mathbb{R}^p \times \mathbb{T}^q \times \mathbb{Z}^r \times D$, where \mathbb{T}^q is a toroid and D is finite, and it can therefore be viewed as a quotient group of some subgroup of \mathbb{R}^n . Thus an examination of the ways in which the usual topology for \mathbb{R}^n can be weakened will shed light on Lie groups of transformations. Our focus on metrizable topologies is prompted, in part, by a theorem of Gleason and Palais ([1], Corollary 7.3), which implies that every finite-dimensional metric group can be obtained by weakening the topology of some Lie group.

Group topologies that are defined by forcing a specific sequence to converge to the identity have been studied by others, including Nienhuys ([3], [4], [5]), Protasov and Zelenyuk [6], and Lukács [2]. Our work differs from theirs by not requiring the converging sequence to consist of integers and by paying explicit attention to the rate of convergence, rather than focusing, as in the study of T -sequences, on the strongest topology in which a given sequence converges to the identity.

2. BACKGROUND AND TERMINOLOGY

We will use the notation presented in this section throughout the paper. Other notation will be introduced as needed.

\mathbb{R} will denote the set of real numbers and \mathbb{R}^n the (set-theoretic) product of n copies of \mathbb{R} ; the group operation on these sets will always be addition. If $x \in \mathbb{R}^n$, then $\|x\|$ will denote the usual Euclidean norm of x .

Topologies for \mathbb{R}^n will be assumed to be group topologies, unless explicitly stated otherwise. Since we will be examining many such topologies for \mathbb{R}^n , topological statements will always mention the specific topology under consideration, and the group \mathbb{R}^n with the topology \mathcal{T} will be denoted by the pair $(\mathbb{R}^n, \mathcal{T})$.

We caution the reader to be wary of making assumptions about the nature of \mathcal{T} . If A and B are topological groups with identity element e , then we say that \mathcal{T} is a product topology on the group $A \times B$ if the projection maps from $A \times B$ to $A \times \{e\}$ and $\{e\} \times B$ are continuous. If at least one of the projections is not continuous, we say that \mathcal{T} is a *nonproduct* topology. The group topologies for \mathbb{R}^n that are considered in this paper are typically nonproduct topologies, so that understanding the properties of $(\mathbb{R}^n, \mathcal{T})$ is not as simple as assessing what happens on each factor of \mathbb{R}^n .

Central to our strategy for constructing group topologies on \mathbb{R}^n is the notion of a *groupnorm* (or simply a *norm*).

Definition 2.1. A *groupnorm* on an abelian group G is a function $\nu : G \rightarrow \mathbb{R}$ satisfying, for all $x, y \in G$,

- (i) $\nu(x) \geq 0$;
- (ii) $\nu(x) = 0$ if and only if $x = 0$;
- (iii) $\nu(x + y) \leq \nu(x) + \nu(y)$;
- (iv) $\nu(x) = \nu(-x)$.

If ν is a groupnorm on G , then the function $d(x, y) = \nu(x - y)$ defines an invariant metric on G , and the corresponding metric topology makes G a Hausdorff topological group. Blurring the distinction between the norm ν , the metric d , and the topology it induces on G , we will denote by (G, ν) the group G with the topology induced by d , and by $\mathcal{C}(G, \nu)$ its completion in the category of topological groups.

\mathbb{Z} denotes the set of integers. If $x \in \mathbb{R}$, then $\lfloor x \rfloor$ will denote the greatest integer less than or equal to $x - 1$. Unless stated otherwise, all sums will be assumed to have only finitely many terms.

In [8] Stevens introduced the following method for constructing metrizable group topologies on \mathbb{R}^n that are weaker than the usual topology. The group topologies that arise from this construction are the subject of this paper.

Proposition 2.2. [8, Proposition 4.1] *Let $\{p_i : i \in \mathbb{N}\}$ be a nonincreasing sequence of positive real numbers which converges to zero in the standard topology on \mathbb{R} , and let $\{v_i : i \in \mathbb{N}\}$ be a sequence of nonzero elements of \mathbb{R}^n such that $\{\|v_i\|\}$ is nondecreasing and the sequence $\{p_{i+1}\|v_{i+1}\|/\|v_i\|\}$ has a positive lower bound. Then the function $\nu : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by*

$$\nu(x) = \inf \left\{ \sum |c_i|p_i + \left\| x - \sum c_i v_i \right\| : c_i \in \mathbb{Z} \right\}$$

is a groupnorm on \mathbb{R}^n such that $\nu(x) \leq \|x\|$ for all $x \in \mathbb{R}^n$ and $\nu(v_i) \leq p_i$. ν gives rise to a metrizable group topology on \mathbb{R}^n , weaker than the standard topology, in which $v_i \rightarrow 0$.

Definition 2.3. If the sequences $\{v_i\}$ and $\{p_i\}$ satisfy the hypothesis of 2.2, then $(\{v_i\}, \{p_i\})$ will be called a *sequential-norming pair SNP* for \mathbb{R}^n . If the groupnorm they induce is ν , then $(\{v_i\}, \{p_i\}, \nu)$ will be called a *sequential-norming triple SNT* for \mathbb{R}^n .

For example, $(\{(i! + \sqrt{3}, \pi i)\}, \{1/i\})$ is an SNP on \mathbb{R}^2 . If ν is the corresponding norm, then $\nu(i! + \sqrt{3}, \pi i) \leq 1/i$, and thus $(i! + \sqrt{3}, \pi i)$ converges to zero in (\mathbb{R}^2, ν) at least as fast as $1/i$ converges to zero in the usual topology for \mathbb{R} .

As mentioned in the introduction, we refer to the sequence $\{v_i\}$ as the “converging sequence” and $\{p_i\}$ as the “rate sequence.” For future reference, we note that multiplying the rate sequence by a positive constant will change the values of the norm ν but not the corresponding topology on \mathbb{R}^n , and that the topology is also unchanged if we remove finitely many terms from the converging sequence $\{v_i\}$ and the corresponding terms from the rate sequence $\{p_i\}$.

3. CHANGING THE CONVERGING SEQUENCE

In this section we compare the topologies generated by two SNTs for \mathbb{R}^n that have different converging sequences but the same rate sequence. Throughout this section, $(\{v_i\}, \{p_i\}, \nu)$ and $(\{u_i\}, \{p_i\}, \mu)$ will denote two such SNTs. *Prima facie*, one would expect that

changing the converging sequence might change the topology, and that situation can, of course, occur. For example, the two SNPs $(\{i!\}, \{1/i\})$ and $(\{i! + 1\}, \{1/i\})$ must determine different topologies for \mathbb{R} , for otherwise the constant sequence $\{i! + 1 - i!\} = \{1, 1, 1, \dots\}$ would converge to 0 in the topology that they determine, contradicting the fact that it is Hausdorff. On the other hand, it is possible to change the converging sequence without changing the topology. For example, Proposition 3.3 implies that the two SNPs $(\{i!\}, \{1/i\})$ and $(\{i! + \frac{3}{i}\}, \{1/i\})$ generate the same topology for \mathbb{R} .

Whether the two SNTs $(\{v_i\}, \{p_i\}, \nu)$ and $(\{u_i\}, \{p_i\}, \mu)$ determine the same topology or not, it is always the case that (\mathbb{R}^n, ν) and (\mathbb{R}^n, μ) are *locally* isometric, as are their completions. The first step in proving this fact, which is formally stated as Theorem 3.2, is to establish a strengthened version of the local isometry theorem (Theorem 8) that is the main result in [7].

Proposition 3.1. *Let $(\{v_i\}, \{p_i\}, \nu)$ and $(\{u_i\}, \{p_i\}, \mu)$ be SNTs on \mathbb{R}^n such that, for some real number $\epsilon > 0$,*

$$p_i \left\lfloor \frac{\|v_{i+1}\|}{\|v_i\|} \right\rfloor, p_i \left\lfloor \frac{\|u_{i+1}\|}{\|u_i\|} \right\rfloor > \epsilon$$

for all i . Then (\mathbb{R}^n, ν) is locally isometric to (\mathbb{R}^n, μ) , and $\mathcal{C}(\mathbb{R}^n, \nu)$ is locally isometric to $\mathcal{C}(\mathbb{R}^n, \mu)$.

Proof. Without loss of generality, we may assume that $\epsilon \leq 1$, and we note that Theorem 8 in [7] addresses the case where $\epsilon = 1$. In order to extend the theorem to the situation where $\epsilon < 1$, we make some minor modifications to the results in [7], which we will now briefly describe. In Lemma 13 in [7] we replace the hypothesis that $\sigma(x) < 1/3$ and $\sum |a_i|p_i, \sum |b_i|p_i < 1/3$ by the assumption that $\sigma(x) < \epsilon/3$ and $\sum |a_i|p_i, \sum |b_i|p_i < \epsilon/3$. Then it is easy to check that the conclusion of that lemma still holds. In the proof of Proposition 14 in [7], we change the choice of $k \in \mathbb{R}$ so that $0 < k \leq \epsilon/6$. In the first paragraph of the proof, it then follows that, for any $x, y \in B_{\sigma_\nu}$, we have $\sigma(x - y) < \epsilon/3$, and the remainder of the proof of Proposition 14 is valid without any changes. In the proof of Theorem 8 that appears in Section 5 of [7], we choose r so that $0 < r < \min(\epsilon/6, s/2)$. Then the fact that $\epsilon < 1$ guarantees that the open balls $B(q, r - \sigma_\nu(q))$ are mutually disjoint, and the remainder of the proof is unchanged. \square

We now prove that this strengthened version of the local isometry theorem can be applied to *any* two SNTs with the same rate sequence.

Theorem 3.2. *If $(\{v_i\}, \{p_i\}, \nu)$ and $(\{u_i\}, \{p_i\}, \mu)$ are SNTs on \mathbb{R}^n , then (\mathbb{R}^n, ν) is locally isometric to (\mathbb{R}^n, μ) and $\mathcal{C}(\mathbb{R}^n, \nu)$ is locally isometric to $\mathcal{C}(\mathbb{R}^n, \mu)$.*

Proof. By Proposition 3.1, it will suffice to show that any SNT $(\{v_i\}, \{p_i\}, \nu)$ satisfies the inequality $p_i \left\lfloor \frac{\|v_{i+1}\|}{\|v_i\|} \right\rfloor > \epsilon$ for some $\epsilon > 0$ and for all but finitely many i . By the definition of SNT, we can choose a real number $C > 0$ such that $\frac{p_{i+1}\|v_{i+1}\|}{\|v_i\|} > C$ for all i , and the fact that $\{p_i\}$ is a nonincreasing sequence implies that $\frac{p_i\|v_{i+1}\|}{\|v_i\|} > C$ for all i , as well. Since $p_i \rightarrow 0$, we know that $p_i < C/4$ for all sufficiently large i . Then for all such i we have

$$p_i \left\lfloor \frac{\|v_{i+1}\|}{\|v_i\|} \right\rfloor \geq p_i \left(\frac{\|v_{i+1}\|}{\|v_i\|} - 2 \right) \geq p_i \left(\frac{C}{p_i} - 2 \right) = C - 2p_i > C/2,$$

where the first inequality holds because $\lfloor x \rfloor \geq x - 2$ for all real numbers x . Thus the desired condition holds, with $\epsilon = C/2$. \square

From Theorem 3.2 we see that any two SNTs on \mathbb{R}^n with the same rate sequence will determine topologies that have the same *local* topological properties. This naturally raises the question of whether those topologies are *globally* the same. This question does not have a straightforward solution, and many cases must be resolved by using *ad hoc* methods to show that some sequence converges in one of the topologies but not in the other. There are, however, several conditions that will assure that the ν -topology and the μ -topology are the same, and we devote the remainder of this section to them. In the next section, we will present a strategy for creating SNTs that definitely determine distinct topologies.

We begin by observing that, if $(\{v_i\}, \{p_i\}, \nu)$ and $(\{u_i\}, \{p_i\}, \mu)$ determine the same topology, then clearly $\nu(u_i) \rightarrow 0$ and $\mu(v_i) \rightarrow 0$. That this condition is not sufficient to assure the equality of the two topologies is demonstrated by the SNTs $(\{i!\}, \{1/i\}, \nu)$ and $(\{i! + \frac{1}{\sqrt{i}}\}, \{1/i\}, \mu)$ for \mathbb{R} . If we let $v_i = i!$ and $u_i = i! + \frac{1}{\sqrt{i}}$, then the two converging sequences differ by $1/\sqrt{i}$, which converges to zero in the usual topology and thus in both (\mathbb{R}, ν) and (\mathbb{R}, μ) .

It follows easily that $\nu(u_i) \rightarrow 0$ and $\mu(v_i) \rightarrow 0$. We will now show that the partial sums of the series

$$\sum_{i=1}^{\infty} u_{i^2} = \sum_{i=1}^{\infty} (i^2)! + \frac{1}{i}$$

form a Cauchy sequence in (\mathbb{R}, μ) but not in (\mathbb{R}, ν) . Given any $\epsilon > 0$, we can choose $k \in \mathbb{N}$ so that $\sum_{i=k}^l \frac{1}{i^2} < \epsilon$ for all $l \geq k$, and thus $\mu\left(\sum_{i=k}^l u_{i^2}\right) \leq \sum_{i=k}^l \frac{1}{i^2} < \epsilon$ for all $l \geq k$. On the other hand, the fact that $\sum_{i=1}^{\infty} \frac{1}{i}$ diverges implies that there is an $l \geq k$ such that $\sum_{i=k}^l \frac{1}{i}$ is more than $1/3$ from the nearest integer. It follows that, for such an l , $\nu\left(\sum_{i=k}^l (i^2)! + \frac{1}{i}\right) \geq 1/3$, so that the sequence of partial sums is not ν -Cauchy, and μ and ν determine different topologies for \mathbb{R} .

We note that in the preceding example the difference between the converging sequences is $\frac{1}{\sqrt{i}}$, which goes to zero in the usual topology much more slowly than the rate sequence $\{1/i\}$. Our first result says that the SNTs $(\{v_i\}, \{p_i\}, \nu)$ and $(\{u_i\}, \{p_i\}, \mu)$ on \mathbb{R}^n will determine the same topology if the difference $\|v_i - u_i\|$ between the two converging sequences is “not too big,” compared with the rate sequence $\{p_i\}$.

Proposition 3.3. *If $(\{v_i\}, \{p_i\}, \nu)$ and $(\{u_i\}, \{p_i\}, \mu)$ are SNTs on \mathbb{R}^n and there is a real number $K > 0$ such that $\|v_i - u_i\| \leq Kp_i$ for all $i \in \mathbb{N}$, then ν and μ determine the same topology for \mathbb{R}^n .*

Proof. Recall that sums are assumed to have only finitely many terms, unless otherwise specified. Given $\epsilon > 0$, let $\delta < \frac{\epsilon}{K+1}$. If $\nu(x) < \delta$, then x can be written as $x = \sum a_i v_i + y$, where $a_i \in \mathbb{Z}$, $y \in \mathbb{R}^n$, and $\sum |a_i| p_i + \|y\| < \delta$. If we rewrite x as

$$x = \sum a_i u_i + y + \sum a_i (v_i - u_i),$$

then it follows from the definition of μ that

$$\begin{aligned} \mu(x) &\leq \sum |a_i| p_i + \|y\| + \sum a_i (v_i - u_i) \\ &\leq \sum |a_i| p_i + \|y\| + K \sum |a_i| p_i \\ &< \delta + K\delta = (K+1)\delta < \epsilon, \end{aligned}$$

and thus the μ -topology is weaker than the ν -topology. A similar argument shows the reverse inclusion, and thus the two topologies for \mathbb{R}^n are the same. \square

For example, we can apply Proposition 3.3 to the two SNTs $(\{i!\}, \{1/i\}, \nu)$ and $(\{i! + \frac{2i}{i^2+3}\}, \{1/i\}, \mu)$. Since $\frac{2i}{i^2+3} \leq \frac{2}{i}$, the metrics ν and μ determine the same topology on \mathbb{R} .

Although the preceding proposition's hypothesis about the relative sizes of $\|v_i - u_i\|$ and p_i is sufficient to make ν and μ determine the same topology, that condition is by no means necessary. It is possible for v_i and u_i to be quite far apart and still have $(\{v_i\}, \{p_i\}, \nu)$ and $(\{u_i\}, \{p_i\}, \mu)$ determine the same topology. One way to accomplish this is to re-index or "shift" a converging sequence. For example, the SNTs $(\{i!\}, \{1/i\}, \nu)$ and $(\{(i+10)!\}, \{1/i\}, \mu)$ determine the same topology for the real numbers. As the next proposition demonstrates, such "shifts" do not change the topology, provided the rate sequence does not converge to zero too rapidly.

Proposition 3.4. *Let $(\{v_i\}, \{p_i\}, \nu)$ be an SNT and suppose there exist $k \in \mathbb{N}$, $C \in \mathbb{R}$ such that $\frac{p_i}{p_{i+k}} \leq C$ for all i . If $u_i = v_{i+k}$, then $(\{u_i\}, \{p_i\}, \mu)$ is an SNT that defines the same topology as ν .*

Proof. To show that $(\{u_i\}, \{p_i\}, \mu)$ is an SNT, it suffices to observe that $\{\frac{p_{i+1}\|u_{i+1}\|}{\|u_i\|} : i \in \mathbb{N}\}$ has a positive lower bound, since $(\{v_i\}, \{p_i\}, \nu)$ is an SNT and

$$\frac{p_{i+1}\|u_{i+1}\|}{\|u_i\|} = \frac{p_{i+1}\|v_{i+k+1}\|}{\|v_{i+k}\|} \geq \frac{p_{i+k+1}\|v_{i+k+1}\|}{\|v_{i+k}\|}.$$

We also note that $C \geq 1$, since $\{p_i\}$ is a non-increasing sequence. To prove that the μ -topology is weaker than the ν -topology, let $\epsilon > 0$ be given. If $x \in \mathbb{R}^n$ and $\nu(x) < \min(\epsilon/C, p_k)$, then we can write x as $x = \sum_{i=1}^l a_i v_i + y$, where $a_i \in \mathbb{Z}$, $y \in \mathbb{R}^n$, and $\sum_{i=1}^l |a_i| p_i + \|y\| < \min(\epsilon/C, p_k)$. Thus $a_i = 0$ for all $i \leq k$, and, after letting $b_i = a_{i+k}$, we have $x = \sum_{i=k+1}^l a_i v_i + y = \sum_{i=1}^{l-k} b_i u_i + y$,

so that

$$\begin{aligned}\mu(x) &\leq \sum_{i=1}^{l-k} |b_i| p_i + \|y\| = \sum_{i=1}^{l-k} |a_{i+k}| p_i + \|y\| \\ &\leq C \left(\sum_{i=1}^{l-k} |a_{i+k}| p_{i+k} + \|y\| \right) = C \left(\sum_{i=k+1}^l |a_i| p_i + \|y\| \right) \\ &< C(\epsilon/C) = \epsilon.\end{aligned}$$

Conversely, if $\mu(x) < \epsilon$, then

$$x = \sum_{i=1}^m c_i u_i + z = \sum_{i=1}^m c_i v_{i+k} + z,$$

where $c_i \in \mathbb{Z}$, $z \in \mathbb{R}^n$, and $\sum_{i=1}^m |c_i| p_i + \|z\| < \epsilon$. It follows that

$$\nu(x) \leq \sum_{i=1}^m |c_i| p_{i+k} + \|z\| \leq \sum_{i=1}^m |c_i| p_i + \|z\| < \epsilon. \quad \square$$

Propositions 3.3 and 3.4 do not, of course, describe all possible circumstances in which two SNTs define the same topology for \mathbb{R}^n . The following example points to some additional possibilities. We let $v_i = i!$ and $p_i = 1/i$, obtaining an SNT $(\{v_i\}, \{p_i\}, \nu)$. If we let $u_i = i! = v_i$ when i is odd, and $u_i = (i-1)(i-1)! = i! - (i-1)! = v_i - v_{i-1} = v_i - u_{i-1}$ when i is even, it is easy to verify that $(\{u_i\}, \{1/i\}, \mu)$ is an SNT. Since $v_i - u_i = (i-1)!$ when i is even, we see that Proposition 3.3 does not apply, nor does Proposition 3.4. Nevertheless, using the fact that $p_{i-1} \leq 2p_i$ for all $i \geq 2$, one can show that ν and μ determine the same topology for \mathbb{R} .

4. SNTs THAT DETERMINE DISTINCT TOPOLOGIES

In the previous section, we investigated the circumstances under which two SNTs with the same rate sequence must determine the same topology for \mathbb{R}^n . Looking at the issue from a different perspective, we now present a strategy for constructing SNTs that have the same rate sequence but determine distinct topologies for \mathbb{R}^n . We begin by choosing a rate sequence (that is, a nonincreasing sequence $\{p_i\}$ of strictly positive real numbers that converges to zero in the usual topology) and a real number δ such that $0 < \delta \leq p_1$.

Using an idea suggested by the proof of Lemma 2.1.9 in [6], we will inductively choose sequences $\{v_i\}$ and $\{u_i\}$ in such a way that $(\{v_i\}, \{p_i\}, \nu)$ and $(\{u_i\}, \{p_i\}, \mu)$ are SNTs and $\mu(v_i), \nu(u_i) \geq \delta$ for all i , so that the two topologies will not be comparable.

Before describing the procedure for choosing the two converging sequences, we introduce some notation. If $m \in \mathbb{N}$ and $x_1, \dots, x_m \in \mathbb{R}^n$, we will let

$$S(x_1, x_2, \dots, x_m) = \left\{ \sum_{i=1}^m c_i x_i : c_i \in \mathbb{Z}, \sum_{i=1}^m |c_i| p_i \leq p_1 \right\}.$$

If $\sum_{i=1}^m c_i x_i \in S(x_1, x_2, \dots, x_m)$ has the property that $\sum_{i=1}^m |c_i| p_i \leq p_1$, then each coefficient c_i in this sum satisfies the inequality $|c_i| \leq p_1/p_i$, and thus we can think of $S(x_1, x_2, \dots, x_m)$ as a set of integral linear combinations of x_1, \dots, x_m in which the coefficients are “not too big.”

For future reference, we note several other useful properties of $S(x_1, x_2, \dots, x_m)$. The first is that this set is closed under taking additive inverses, and the second is that $S(x_1, x_2, \dots, x_m) \subseteq S(x_1, x_2, \dots, x_{m+1})$. Finally, we claim that, if $\{\|x_i\|\}$ is a non-decreasing sequence and

$$x = \sum_{i=1}^m c_i x_i \in S(x_1, x_2, \dots, x_m),$$

where $c_i \in \mathbb{Z}$ and $\sum_{i=1}^m |c_i| p_i \leq p_1$, then $\|x\| \leq p_1 \|x_m\|/p_m$. To prove this, note that

$$p_1 \geq \sum_{i=1}^m |c_i| p_i \geq \sum_{i=1}^m |c_i| p_m,$$

so that $\sum_{i=1}^m |c_i| \leq p_1/p_m$. Since $\{\|x_i\|\}$ is a non-decreasing sequence, it follows that

$$\|x\| \leq \sum_{i=1}^m |c_i| \|x_i\| \leq \left(\sum_{i=1}^m |c_i| \right) \|x_m\| \leq p_1 \|x_m\|/p_m.$$

The algorithm for choosing the converging sequences $\{v_i\}$ and $\{u_i\}$ also involves another strictly positive constant, θ . The converging sequences $\{v_i\}$ and $\{u_i\}$ will be chosen so that

$$\frac{p_{i+1}\|v_{i+1}\|}{\|v_i\|}, \frac{p_{i+1}\|u_{i+1}\|}{\|u_i\|} \geq \theta$$

for all i , thus assuring that $(\{v_i\}, \{p_i\}, \nu)$ and $(\{u_i\}, \{p_i\}, \mu)$ are SNTs. Although δ and θ can be chosen arbitrarily, subject to the conditions $\theta > 0$ and $0 < \delta \leq p_1$, larger values of these constants place greater constraints on the choice of v_i and u_i . We denote by B the open ball that is centered at the origin and has radius δ , in the usual metric.

Our procedure for choosing the sequences $\{v_i\}$ and $\{u_i\}$ is as follows. We let v_1 be any element of \mathbb{R}^n such that $\pm v_1 \notin B$. As u_1 , we choose any element of \mathbb{R}^m such that $\pm u_1 \notin S(v_1) + B$. This is certainly possible, since $S(v_1)$ is finite and thus $S(v_1) + B$ is bounded. Proceeding inductively, we assume that $v_1, \dots, v_i, u_1, \dots, u_i$ have been chosen. We then successively choose v_{i+1} and u_{i+1} to be any elements of \mathbb{R}^n such that

$$\frac{\|v_{i+1}\|p_{i+1}}{\|v_i\|}, \frac{\|u_{i+1}\|p_{i+1}}{\|u_i\|} \geq \theta,$$

and for all non-zero $c \in \mathbb{Z}$ with $|c| \leq p_1/p_{i+1}$,

$$cv_{i+1} \notin S(v_1, v_2, \dots, v_i) + S(u_1, u_2, \dots, u_i) + B$$

and

$$cu_{i+1} \notin S(v_1, v_2, \dots, v_{i+1}) + S(u_1, u_2, \dots, u_i) + B.$$

Lemma 4.1. *If the sequences $\{v_i\}$ and $\{u_i\}$ are chosen as described above, then for all $i \in \mathbb{N}$,*

- (i) $(S(v_1, v_2, \dots, v_i) + B) \cap S(u_1, u_2, \dots, u_i) = \{0\}$;
- (ii) $S(v_1, v_2, \dots, v_i) \cap (S(u_1, u_2, \dots, u_i) + B) = \{0\}$.

Proof. To prove (i) when $i = 1$, we note that $S(u_1) = \{0, \pm u_1\}$, and the choice of u_1 guarantees that $(S(v_1) + B) \cap S(u_1) = \{0\}$. To prove (ii) when $i = 1$, we assume that $t \in S(v_1) \cap (S(u_1) + B)$, whence t can be written in the form $t = a_1 v_1 = b_1 u_1 + r$, where $|a_1|$ and $|b_1|$ are integers less than or equal to 1, and $r \in B$.

Then $b_1 u_1 = a_1 v_1 - r \in S(v_1) + B$. From the choice of u_1 , it follows that $b_1 = 0$ and thus $t = a_1 v_1 = r \in B$. This contradicts the choice of v_1 unless $a_1 = 0$, and we conclude that $t = 0$.

Now assume that (i) and (ii) hold when $i = j$, and let $t \in (S(v_1, v_2, \dots, v_{j+1}) + B) \cap S(u_1, u_2, \dots, u_{j+1})$. Then t can be written as $t = \sum_{k=1}^{j+1} a_k v_k + r = \sum_{k=1}^{j+1} b_k u_k$, where $a_k, b_k \in \mathbb{Z}$ are such that

$$\sum_{k=1}^{j+1} |a_k| p_k, \sum_{k=1}^{j+1} |b_k| p_k \leq p_1,$$

and $r \in B$. Solving for $b_{j+1} u_{j+1}$, we find that

$$b_{j+1} u_{j+1} \in S(v_1, v_2, \dots, v_{j+1}) + S(u_1, u_2, \dots, u_j) + B.$$

Since $|b_{j+1}| \leq p_1/p_{j+1}$, it follows that $b_{j+1} = 0$, and thus

$$t = \sum_{k=1}^{j+1} a_k v_k + r = \sum_{k=1}^j b_k u_k.$$

Solving for $a_{j+1} v_{j+1}$ shows that

$$a_{j+1} v_{j+1} \in S(v_1, v_2, \dots, v_j) + S(u_1, u_2, \dots, u_j) + B,$$

which implies that $a_{j+1} = 0$, since $|a_{j+1}| \leq p_1/p_{j+1}$. Therefore

$$\begin{aligned} t &= \sum_{k=1}^j a_k v_k + r = \sum_{k=1}^j b_k u_k \\ &\in (S(v_1, v_2, \dots, v_j) + B) \cap S(u_1, u_2, \dots, u_j) = \{0\}, \end{aligned}$$

and (i) is true when $i = j + 1$. The proof of (ii) when $i = j + 1$ is similar. \square

We now prove that the SNTs $(\{v_i\}, \{p_i\}, \nu)$ and $(\{u_i\}, \{p_i\}, \mu)$ determine different topologies for \mathbb{R}^n .

Theorem 4.2. *If $\{p_i\}$ is a rate sequence and the converging sequences $\{v_i\}$ and $\{u_i\}$ are chosen as described above, then the triples $(\{v_i\}, \{p_i\}, \nu)$ and $(\{u_i\}, \{p_i\}, \mu)$ are SNTs that determine distinct topologies for \mathbb{R}^n . In fact, the two topologies are not comparable.*

Proof. That $(\{v_i\}, \{p_i\}, \nu)$ and $(\{u_i\}, \{p_i\}, \mu)$ are SNTs follows from the fact that

$$\frac{p_{i+1} \|v_{i+1}\|}{\|v_i\|}, \frac{p_{i+1} \|u_{i+1}\|}{\|u_i\|} \geq \theta$$

for all i . To prove that the corresponding topologies are distinct, it suffices to prove that $\mu(v_i) \geq \delta$ for all i . If $\mu(v_i) < \delta$, then v_i can be written as $v_i = \sum_{k=1}^j b_k u_k + r$, where $b_i \in \mathbb{Z}$, $r \in \mathbb{R}^n$, and $\sum_{k=1}^j |b_k| p_k + \|r\| < \delta$. Then $r \in B$ and $\sum_{k=1}^j |b_k| p_k < p_1$, so that $v_i \in S(u_1, u_2, \dots, u_j) + B$. If $m = \max(i, j)$, then

$$\begin{aligned} v_i &\in S(v_1, v_2, \dots, v_i) \cap (S(u_1, u_2, \dots, u_j) + B) \\ &\subseteq S(v_1, v_2, \dots, v_m) \cap (S(u_1, u_2, \dots, u_m) + B). \end{aligned}$$

Then Lemma 4.1(ii) implies that $v_i = 0$, contradicting the way in which the sequence $\{v_i\}$ was chosen. A similar argument, using Lemma 4.1(i), demonstrates that $\nu(u_i) \geq \delta$ for all i , so that the two topologies are not comparable. \square

To illustrate Theorem 4.2, we let $v_1 = 1/2$, $u_1 = 1$, and $p_1 = 1/2$, and for $i \geq 2$ we let $v_i = (i!)^2/\sqrt{2}$, $u_i = (i+1)(i!)^2$, and $p_i = 1/i$. We claim that $(\{v_i\}, \{p_i\}, \nu)$ and $(\{u_i\}, \{p_i\}, \mu)$ determine incomparable topologies on \mathbb{R} . If we let $\delta = 1/2$, then simple computations verify that these are SNTs and that the choices of v_1 , u_1 , v_2 , and u_2 meet the criteria specified in our algorithm. To show that v_{i+1} satisfies the relevant condition when $i \geq 2$, we must show that, if c is a non-zero integer such that $|c| \leq \frac{i+1}{2}$, then $c[(i+1)!]^2/\sqrt{2} \notin S(v_1, v_2, \dots, v_i) + S(u_1, u_2, \dots, u_i) + B$. To do that, it suffices to show that v_{i+1} exceeds by at least $1/2$ the sum of the maximum element of $S(v_1, v_2, \dots, v_i)$ and the maximum element of $S(u_1, u_2, \dots, u_i)$. Since those maximum elements are, respectively, less than or equal to $v_i p_1 / p_i = \frac{i(i!)^2}{2\sqrt{2}}$ and $u_i p_1 / p_i = \frac{i(i+1)(i!)^2}{2}$, it suffices to show that

$$\frac{[(i+1)!]^2}{\sqrt{2}} > \frac{i(i!)^2}{2\sqrt{2}} + \frac{i(i+1)(i!)^2}{2} + \frac{1}{2}.$$

This is equivalent to the inequality,

$$(i!)^2[i^2(2 - \sqrt{2}) + i(3 - \sqrt{2}) + 2] > \sqrt{2},$$

which is clearly true for all $i \geq 2$.

A similar argument verifies that the condition on the choice of u_{i+1} is met. We must show that, if z is a non-zero integer such that $|z| \leq \frac{i+1}{2}$, then $z(i+2)[(i+1)!]^2 \notin S(v_1, v_2, \dots, v_{i+1}) + S(u_1, u_2, \dots, u_i) + B$. To do that, it suffices to show that

$$(i+2)[(i+1)!]^2 > \frac{(i+1)[(i+1)!]^2}{2\sqrt{2}} + \frac{i(i+1)(i!)^2}{2} + \frac{1}{2},$$

which is easily seen to be the case. Thus the SNTs $(\{v_i\}, \{p_i\}, \nu)$ and $(\{u_i\}, \{p_i\}, \mu)$ determine incomparable topologies for \mathbb{R} . We note that one of the converging sequences in this example consists primarily of irrational numbers and generates a dense subgroup of \mathbb{R} , while the other contains only integers. It can be difficult to compare such SNTs by *ad hoc* methods.

5. TOPOLOGICAL ISOMORPHISMS

In Section 3, we investigated the circumstances under which two SNTs $(\{v_i\}, \{p_i\}, \nu)$ and $(\{u_i\}, \{p_i\}, \mu)$ with the same rate sequence will determine the same topology for \mathbb{R}^n . In other words, we asked whether the identity function $(\mathbb{R}^n, \nu) \rightarrow (\mathbb{R}^n, \mu)$ is a topological isomorphism. We now consider the possibility that some function other than the identity might be a topological isomorphism between these two groups. The distinction is illustrated by the SNTs $(\{2^{2^i}\}, \{1/i\}, \nu)$ and $(\{3(2^{2^i})\}, \{1/i\}, \mu)$ for \mathbb{R} . Since any integer in the open ball of radius 1 in (\mathbb{R}, μ) that is centered at the origin must be divisible by 3, $\mu(2^{2^i}) \geq 1$ for all i , and thus the two topologies are distinct. On the other hand, the function $f(x) = 3x$ is a topological isomorphism from (\mathbb{R}, ν) to (\mathbb{R}, μ) , since the definitions of ν and μ imply that $\nu(x) \leq \mu(f(x)) \leq 3\nu(x)$ for all real numbers x .

We begin with a proposition that describes the form that any topological isomorphism between the topological groups defined by two SNTs would have to take. It applies to all SNTs, whether they have the same rate sequence or not.

Proposition 5.1. *Let \mathcal{T}_1 and \mathcal{T}_2 be two group topologies for \mathbb{R}^n , each of which is weaker than the usual topology \mathcal{U} , and let*

$$f : (\mathbb{R}^n, \mathcal{T}_1) \rightarrow (\mathbb{R}^n, \mathcal{T}_2)$$

be a continuous, non-trivial homomorphism.

- (i) *If $n = 1$, then there is a non-zero real number α such that $f(x) = \alpha x$ for all $x \in \mathbb{R}$, and f is an algebraic isomorphism;*
- (ii) *If $n > 1$, then there is a non-zero $n \times n$ matrix A such that $f(x) = Ax$ for all $x \in \mathbb{R}^n$, and f is an isomorphism if and only if A is invertible.*

Proof. To prove (i), we let $\alpha = f(1)$. Since f is homomorphism, it is easy to check that $f(j) = \alpha j$ for all integers j , whence $f(q) = \alpha q$ for all rational numbers q . Now any $x \in \mathbb{R}$ is the limit of some sequence $\{q_i\}$ of rational numbers in $(\mathbb{R}, \mathcal{U})$ and thus in $(\mathbb{R}, \mathcal{T}_1)$.

The continuity of f then implies that $f(q_i) = \alpha q_i \rightarrow f(x)$ in $(\mathbb{R}, \mathcal{T}_2)$. But $\{\alpha q_i\}$ converges to αx in $(\mathbb{R}, \mathcal{U})$ and thus also in $(\mathbb{R}, \mathcal{T}_2)$. Therefore $f(x) = \alpha x$, and f is non-trivial if and only if $\alpha \neq 0$, in which case f is an isomorphism. Turning now to (ii), let $\{x_1, \dots, x_n\}$ be the standard basis for \mathbb{R}^n , and let $y_i = f(x_i)$ for $i = 1, \dots, n$. The fact that f is a homomorphism implies that $f(qx_i) = qy_i$ for all rational numbers q , and an argument like that in part (i) shows that $f(rx_i) = ry_i$ for all $r \in \mathbb{R}$, $i = 1, \dots, n$. Since f is an additive homomorphism that preserves scalar multiplication, it must be a linear transformation, and thus it can be represented by an $n \times n$ matrix A . Then $A \neq 0$ since f is non-trivial, and f is an isomorphism if and only if A is invertible. \square

The next proposition describes the effect of applying an invertible linear transformation A to a converging sequence $\{v_i\}$.

Proposition 5.2. *If $(\{v_i\}, \{p_i\}, \nu)$ is an SNT for \mathbb{R}^n and A is an invertible $n \times n$ matrix, then $(\{A(v_i)\}, \{p_i\})$ is an SNP for \mathbb{R}^n . If ρ denotes the corresponding metric, then the function $f : (\mathbb{R}^n, \nu) \rightarrow (\mathbb{R}^n, \rho)$ defined by $f(x) = A(x)$ is a topological isomorphism.*

Proof. Since A represents an invertible linear transformation and the unit sphere in \mathbb{R}^n is compact, the set $\{\|A(x)\| : \|x\| = 1\}$ has a maximum value M and a strictly positive minimum value m , and it follows that $m\|x\| \leq \|A(x)\| \leq M\|x\|$ for all $x \neq 0$. Therefore $p_{i+1} \frac{\|A(v_{i+1})\|}{\|A(v_i)\|} \geq (\frac{m}{M}) p_{i+1} \frac{\|v_{i+1}\|}{\|v_i\|}$ for all i , so that $(\{A(v_i)\}, \{p_i\})$ satisfies the conditions for being an SNP. To prove that f is continuous, let $\epsilon > 0$ be given, and choose $\delta > 0$ so that $\max(\delta, \delta M) < \epsilon/2$. If $\nu(x) < \delta$, then x can be written as $x = \sum a_i v_i + y$, where the sum contains only finitely many non-zero terms, $y \in \mathbb{R}^n$, $a_i \in \mathbb{Z}$, and $\sum |a_i| p_i + \|y\| < \delta$. Then $f(x) = A(x) = \sum a_i A(v_i) + A(y)$ and

$$\begin{aligned} \rho(f(x)) &\leq \sum |a_i| p_i + \|A(y)\| \leq \sum |a_i| p_i + M\|y\| \\ &< \delta + \delta M < \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

To prove that f^{-1} is continuous, we repeat the argument above, with A replaced by A^{-1} . \square

Applying Proposition 5.2 to the case where $n = 1$, $f(x) = 3x$, $v_i = 2^{2^i}$, and $p_i = 1/i$, we obtain the example given in the first paragraph of this section.

Building upon these propositions, the following theorem will enable us to rephrase our question about topological isomorphisms.

Theorem 5.3. *Let $(\{v_i\}, \{p_i\}, \nu)$ and $(\{u_i\}, \{p_i\}, \mu)$ be two SNTs for \mathbb{R}^n with the same rate sequence. Then (\mathbb{R}^n, ν) is topologically isomorphic to (\mathbb{R}^n, μ) if and only if there is an invertible $n \times n$ matrix B such that $(\{v_i\}, \{p_i\}, \nu)$ and $(\{B(u_i)\}, \{p_i\}, \rho)$ determine the same topology for \mathbb{R}^n . In particular, when $n = 1$, the groups (\mathbb{R}, ν) and (\mathbb{R}, μ) are topologically isomorphic if and only if there is a real number $b \neq 0$ such that $(\{v_i\}, \{p_i\}, \nu)$ and $(\{bu_i\}, \{p_i\}, \rho)$ determine the same topology for \mathbb{R} .*

Proof. If $f : (\mathbb{R}^n, \nu) \rightarrow (\mathbb{R}^n, \mu)$ is a topological isomorphism, then by Proposition 5.1 there is an invertible $n \times n$ matrix A such that $f(x) = A(x)$ for all $x \in \mathbb{R}^n$. Let $B = A^{-1}$. By Proposition 5.2, $(\{B(u_i)\}, \{p_i\})$ is an SNP that determines a metric ρ on \mathbb{R}^n , and the function $g : (\mathbb{R}^n, \mu) \rightarrow (\mathbb{R}^n, \rho)$ that is defined by $g(x) = B(x)$ is a topological isomorphism. Since f and g are topological isomorphisms, so is their composition $g \circ f$, which equals the identity map. Therefore $(\{v_i\}, \{p_i\}, \nu)$ and $(\{B(u_i)\}, \{p_i\}, \rho)$ determine the same topology for \mathbb{R}^n .

Conversely, assume there exists an invertible $n \times n$ matrix B such that $(\{v_i\}, \{p_i\}, \nu)$ and $(\{B(u_i)\}, \{p_i\}, \rho)$ determine the same topology for \mathbb{R}^n . By Proposition 5.2, the function $h : (\mathbb{R}^n, \mu) \rightarrow (\mathbb{R}^n, \rho)$ defined by $h(x) = B(x)$ is a topological isomorphism. Since ν and ρ determine the same topology, h is also a topological isomorphism of (\mathbb{R}^n, ν) with (\mathbb{R}^n, μ) . \square

Theorem 5.3 enables us to reformulate the issue that was raised at the beginning of this section. Given an SNT $(\{v_i\}, \{p_i\}, \nu)$, it is natural to ask what other SNTs with the same rate sequence yield groups that are topologically isomorphic with (\mathbb{R}^n, ν) . According to Theorem 5.3, we can focus our attention on the situation where the isomorphism is the identity map and look for SNTs $(\{u_i\}, \{p_i\}, \mu)$ that determine the same topology as ν . By applying invertible linear transformations to such SNTs, we will obtain all the SNTs with the same rate sequence that define groups that are topologically isomorphic to (\mathbb{R}^n, ν) .

6. CHANGING THE RATE SEQUENCE

Thus far, we have been comparing the topologies determined by SNTs for \mathbb{R}^n that have the same rate sequence but different converging sequences. Here we briefly discuss the opposite situation, where the converging sequence is the same but the rate sequence is changed. When the converging sequence $\{v_i\}$ consists of integers such that each v_i divides v_{i+1} , the corresponding topologies for \mathbb{Z} and their completions have been extensively studied by Nienhuys ([3], especially Section 7). His results do not apply, however, to more general v_i .

If we wish to compare any two SNTs for \mathbb{R}^n that have the same converging sequence, it is natural to begin by looking at the ratio of the two rate sequences, since we know that multiplying the rate sequence by a positive constant does not change the topology. The following theorem says that the behavior of this ratio determines whether the two SNTs generate the same topology.

Theorem 6.1. *Let $(\{v_i\}, \{p_i\}, \nu)$ and $(\{v_i\}, \{q_i\}, \mu)$ be SNTs for \mathbb{R}^n with the same converging sequence. The ν -topology and the μ -topology are the same if and only if there exist real numbers C and K such that $0 < C \leq q_i/p_i \leq K$ for all i .*

The proof of Theorem 6.1 will appear in a subsequent paper that is currently in preparation. Although it is easy to prove that the ν -topology and the μ -topology are the same if there exist real numbers C and K satisfying the conditions in the theorem, the proof of the converse is more complicated.

As an example of Theorem 6.1, we apply it to the SNTs $(\{i! + 1\}, \{1/i\}, \nu)$ and $(\{i! + 1\}, \{1/i^2\}, \mu)$ on \mathbb{R} . The ratio of the rate sequences is i/i^2 , which has zero as its greatest lower bound, and thus the ν -topology and the μ -topology must be different. On the other hand, the SNT $(\{i! + 1\}, \{\frac{3i}{4i^2+5}\}, \rho)$ must determine the same topology for \mathbb{R} that ν does, because the ratio of the rate sequences is $\frac{3i^2}{4i^2+5}$. Since this ratio converges to $3/4$ as $i \rightarrow \infty$, it must have an upper bound and a strictly positive lower bound.

7. CONCLUDING REMARKS

As noted in Section 6, the proof of Theorem 6.1 will appear in a subsequent paper.

In [7], we introduced another method for constructing weakened topologies for \mathbb{R}^n , called an *extended norming triple* (ENT). If $n > m$, then this technique uses SNPs on \mathbb{R}^m to obtain metrizable group topologies for \mathbb{R}^n that are weaker than the usual topology. Many of the results in this paper, including Theorem 3.2, are valid for ENTs, and we plan in a subsequent paper to compare the topologies defined by different ENTs.

REFERENCES

- [1] A. Gleason and R. Palais, *On a class of transformation groups*, Amer. J. Math., **79** (1957), 631–648.
- [2] Gábor Lukács, *Almost maximally almost-periodic group topologies determined by T -sequences*, Topology Appl., **153** (2006), 2922–2932.
- [3] J. W. Nienhuys, *Not locally compact monothetic groups I, II*, Nederl. Akad. Wetensch. Proc. Ser. A 73 Indag. Math., **32** (1970), 295–326.
- [4] J.W. Nienhuys, *Construction of group topologies on abelian groups*, Fund. Math., **75** (1972), 101–116.
- [5] J.W. Nienhuys, *Some examples of monothetic groups*, Fund. Math., **88** (1975), 163–171.
- [6] I. Protasov and E. Zelenyuk, *Topologies on Groups Determined by Sequences*, Mathematical Studies Monograph Series, **4**, VTNL Publishers, L’viv, 1999.
- [7] J. W. Short and T. C. Stevens, *Weakened Lie groups and their locally isometric completions*, Topology Appl., **135** (2004), 47–61.
- [8] T. C. Stevens, *Decisive subgroups of analytic groups*, Trans. Amer. Math. Soc., **274** (1982), 101–108.
- [9] T. C. Stevens, *Weakening the topology of a Lie Group*, Trans. Amer. Math. Soc., **276** (1983), 541–549.

DEPARTMENT OF MATHEMATICS AND STATISTICS, SAM HOUSTON STATE UNIVERSITY, HUNTSVILLE, TEXAS 77341
E-mail address: jon@shsu.edu

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, SAINT LOUIS UNIVERSITY, ST. LOUIS, MISSOURI, 63103
E-mail address: stevensc@slu.edu