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Galo Higuera and Alejandro Illanes

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Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
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INDUCED MAPPINGS ON SYMMETRIC PRODUCTS

GALO HIGUERA AND ALEJANDRO ILLANES

Abstract. Let X be a metric continuum. For a positive integer n, let $\mathcal{F}_n(X)$ be the hyperspace of nonempty subsets of X with at most n points. For a given mapping between continua $f: X \to Y$, we study the induced mapping $f_n: \mathcal{F}_n(X) \to \mathcal{F}_n(Y)$ given by $f_n(A) = f(A)$ (the image of A under f). Given a topological or dynamical property $\mathcal{M}$ that mappings can have, we study under which conditions the fact that f has property $\mathcal{M}$ implies that f_n has property $\mathcal{M}$, and vice versa.

1. Introduction

A *continuum* is a nonempty compact connected metric space. A continuum is said to be *nondegenerate* if it has more than one point. Given a continuum X, consider the following hyperspaces of X:

$$\begin{aligned} 2^X &= \{A \subset X : A \text{ is nonempty and closed}\}, \\ C(X) &= \{A \in 2^X : A \text{ is connected}\}, \text{ and for each } n \geq 1, \\ \mathcal{F}_n(X) &= \{A \in 2^X : A \text{ has at most } n \text{ points}\}, \\ \mathcal{C}_n(X) &= \left\{A \in 2^X : A \text{ has at most } n \text{ components}\right\}. \end{aligned}$$

All these hyperspaces are considered with the Hausdorff metric H. In this paper the word *mapping* stands for a continuous and surjective function.

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Key words and phrases. Atomic, confluent, continuum, expansive homeomorphism, homeomorphism, hyperspace, induced mapping, light, linking, mapping, mixing, MO, monotone, open, OM, refinable, semi-confluent, solenoid, symmetric product, transitive, universal, weakly confluent, weakly mixing.

Every mapping between continua $f\colon X \to Y$ induces a mapping between each of the respective hyperspaces in the following way: $2^f\colon 2^X \to 2^Y$ is defined as $2^f(A) = \{f(a) : a \in A\}$. The induced mapping to the other hyperspaces is simply the restriction of 2^f to each of such hyperspaces.

The induced mappings to other hyperspaces such as 2^X, $\mathcal{C}(X)$ and $\mathcal{C}_n(X)$ have been previously studied by several authors (see for example [3]-[7], [9]-[11], [16]-[21] and [23]). In this paper we make a systematic study of the induced mapping $f_n\colon \mathcal{F}_n(X) \to \mathcal{F}_n(Y)$, for a mapping between continua $f\colon X \to Y$. Given a topological or dynamical property $\mathcal{M}$ that mappings can have, we study under which conditions the fact that f has property $\mathcal{M}$ implies that f_n has property $\mathcal{M}$, and vice versa.

Some previous basic result have been shown in the thesis [13] and [29].

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Besides the introduction and a section of preliminaries, we divide the paper in two big sections. In the first one, we study traditional (or topological) properties defined for mappings between continua such as openness, monoteneity, confluence, etc. In the second one we study dynamical properties such as transitivity, mixing, chaos, etc.

2. Preliminaries

The symbol $\mathbb{N}$ denotes the set of positive integers. All spaces are assumed to be continua unless otherwise stated. Given a continuum X, a point $a \in X$ and $\varepsilon > 0$, the *ε-ball* around a is denoted by $B(\varepsilon, a)$. The *ε-ball* around A in $\mathcal{F}_n(X)$ is denoted by $B_H(\varepsilon, A)$. Also, for $A \subset X$, we denote the diameter of A by $\operatorname{diam}(A)$ and $N_\varepsilon(A) = \bigcup\{B(\varepsilon, a) : a \in A\}$. All the hyperspaces are considered with the Hausdorff metric (see [22, 2.1, p. 11]). It is known that this topology concides with the *Vietoris Topology* (see [22, 3.1, p. 16]) defined as follows: Given a finite collection of subsets $U_1, \ldots, U_m$ of X we define

$$\langle U_1,\ldots,U_m\rangle = \{A \in 2^X : A \subset \bigcup_{i=1}^{m} U_i \ \text{ and } \ A \cap U_i \neq \emptyset \ \text{ for each } \ i \in \{1,\ldots,m\}\}\,.$$

The family $\{\langle U_1,\ldots,U_m\rangle : m \in \mathbb{N}$ and U_i is open in X for each $i \in \{1,\ldots,m\}\}$ is a basis for the Vietoris topology. We define $\langle U_1,\ldots,U_m\rangle_n = \langle U_1,\ldots,U_m\rangle \cap \mathcal{F}_n(X)$.

Proceeding as in [8, Lemma 2.1], the following lemma can be proved.

Lemma 2.1. *Let $\mathcal{A}$ be a connected, closed subset of $\mathcal{F}_n(X)$ such that $\mathcal{A} \cap \mathcal{F}_m(X) \neq \emptyset$, for some $m \leq n$. Let $A = \bigcup\{B : B \in \mathcal{A}\}$. Then $A \in \mathcal{C}_m(X)$ and every component of A intersects every element of $\mathcal{A} \cap \mathcal{F}_m(X)$.*

We will use the following particular case of [25, Lemma 1].

Lemma 2.2. *Let $C_1,\ldots,C_m$ be pairwise disjoint subcontinua of X. Suppose that $m \leq n$. Then $\langle C_1,\ldots,C_m\rangle_n$ is a subcontinuum of $\mathcal{F}_n(X)$.*

Definition 2.3. Given $A, B \in \mathcal{C}(X)$, with $A \subsetneq B$, we say that a continuous function $\alpha\colon [0,1] \to \mathcal{C}(X)$ is an *order arc from A to B in $\mathcal{C}(X)$* if $\alpha(0) = A$, $\alpha(1) = B$ and $\alpha(s) \subsetneq \alpha(t)$ for every $0 \leq s < t \leq 1$.

The existence of order arcs is a very well known fact of the theory of hyperspaces and it is stated in the following theorem (see [22, 14.6, p. 112]).

Theorem 2.4. *Given $A, B \in \mathcal{C}(X)$ such that $A \subsetneq B$, there exists an order arc from A to B in $\mathcal{C}(X)$.*

3. Traditional Properties

3.1. **Homeomorphisms.** The following theorem is immediate.

Theorem 3.1. *The following statements for a mapping $f\colon X \to Y$ are equivalent:*

a) f is a homeomorphism,
b) f_n is a homeomorphism, for some $n \in \mathbb{N}$,
c) f_n is a homeomorphism, for every $n \in \mathbb{N}$.

3.2. **Monotone and Open Mappings.**

Definition 3.2. We say that a map $f\colon X \to Y$ is *monotone* if $f^{-1}(y)$ is a connected subset of X for every $y \in Y$.

Theorem 3.3. *The following statements for a mapping $f\colon X \to Y$ are equivalent:*

a) *f is monotone,*
b) *f_n is monotone for some $n \in \mathbb{N}$,*
c) *f_n is monotone for every $n \in \mathbb{N}$.*

Proof. a) $\Rightarrow$ c). Suppose that $f\colon X \to Y$ is monotone and let $n \in \mathbb{N}$. Let $B = \{y_1, \ldots, y_m\} \in \mathcal{F}_n(Y)$ where $m \leq n$. For every $i \in \{1, \ldots, m\}$, let $C_i = f^{-1}(y_i)$. Since f is monotone, C_i is a subcontinuum of X. In addition, the sets $C_1, \ldots, C_m$ are pairwise disjoint. It follows from Lemma 2.2 that the set $\mathcal{A} = \langle C_1, \ldots, C_m\rangle_n$ is a subcontinuum of $\mathcal{F}_n(X)$. It is easy to show that $\mathcal{A} = f_n^{-1}(B)$. This concludes the proof that $f_n^{-1}(B)$ is connected and shows that f_n is monotone.

b) $\Rightarrow$ a). Suppose that f_n is monotone for some $n \in \mathbb{N}$. Let $y \in Y$. Then $f_n^{-1}(\{y\})$ is a subcontinuum of $\mathcal{F}_n(X)$. Let $A = \bigcup\{B : B \in f_n^{-1}(\{y\})\}$. Since f is surjective, there exists $x \in X$ such that $f(x) = y$. Then $\{x\} \in f_n^{-1}(\{y\}) \cap \mathcal{F}_1(X)$. By Lemma 2.1, A is connected. Clearly, $A = f^{-1}(y)$. Thus $f^{-1}(y)$ is connected. Hence f is monotone. $\square$

Definition 3.4. A mapping $f\colon X \to Y$ is open if $f(U)$ is open in Y for every open subset U of X.

Theorem 3.5. *A mapping $f\colon X \to Y$ is open if and only if the mapping $f_2\colon \mathcal{F}_2(X) \to \mathcal{F}_2(Y)$ is open.*

Proof. (Necessity). Suppose that f is open. Consider an open subset $\mathcal{U}$ of $\mathcal{F}_2(X)$. Pick an element $f_2(A) \in f_2(\mathcal{U})$, where $A \in \mathcal{U}$. We put $A = \{p, q\}$, where possibly $p = q$. Since $\mathcal{U}$ is open, there exists $\varepsilon > 0$ such that $B_H(\varepsilon, A) \subset \mathcal{U}$. Since f is open, there exists $\delta > 0$ such that $B(\delta, f(p)) \subset f(B(\varepsilon, p))$ and $B(\delta, f(q)) \subset f(B(\varepsilon, q))$. If $f(p) \neq f(q)$, we can also ask that $B(\delta, f(p)) \cap B(\delta, f(q)) = \emptyset$.

We claim that $B_H(\delta, f_2(A)) \subset f_2(\mathcal{U})$. Let $B = \{w, z\} \in B_H(\delta, f_2(A))$. Then $H(\{w, z\}, \{f(p), f(q)\}) < \delta$. In the case that $f(p) \neq f(q)$, we may assume that $w \in B(\delta, f(p))$ and $z \in B(\delta, f(q))$.

In the case that $f(p) = f(q)$, then $w, z \in B(\delta, f(p)) = B(\delta, f(q))$. In both cases, we may assume that $w \in B(\delta, f(p)) \subset f(B(\varepsilon, p))$ and $z \in B(\delta, f(q)) \subset f(B(\varepsilon, q))$. Then there exist $u \in B(\varepsilon, p)$ and $x \in B(\varepsilon, q)$ such that $f(u) = w$ and $f(x) = z$. Notice that $H(\{u, x\}, \{p, q\}) < \varepsilon$. Hence $\{u, x\} \in \mathcal{U}$ and $f_2(\{u, x\}) = \{w, z\}$. Thus $B \in f_2(\mathcal{U})$. This shows that $B_H(\delta, f_2(A)) \subset f_2(\mathcal{U})$. Hence $f_2(\mathcal{U})$ is open.

(Sufficiency). Suppose that f_2 is open and let U be an open subset of X. Given $p \in U$, we have $\{p\} \in \langle U \rangle_2$. Since f_2 is open, $f_2(\langle U \rangle_2)$ is an open subset of $\mathcal{F}_2(Y)$ that has the element $f_2(\{p\}) = \{f(p)\}$, so there exists $\varepsilon > 0$ such that $B_H(\varepsilon, \{f(p)\}) \subset f_2(\langle U \rangle_2)$.

We claim that $B(\varepsilon, f(p)) \subset f(U)$. Let $y \in B(\varepsilon, f(p))$. Then $\{y\} \subset f_2(\langle U \rangle_2)$. So there exists $B \in \langle U \rangle_2$ such that $\{y\} = f_2(B)$. Pick a point $b \in B$. Then $b \in U$ and $f(b) = y$. This shows that $y \in f(U)$. We have shown that $B(\varepsilon, f(p)) \subset f(U)$. Hence $f(U)$ is open in Y. Therefore f is open. □

Theorem 3.6. *If $f\colon X \to Y$ is a mapping such that Y is nondegenerate and $f_n\colon \mathcal{F}_n(X) \to \mathcal{F}_n(Y)$ is open for some $n \geq 3$, then f is a homeomorphism.*

Proof. It suffices to show that f is one-to-one. Suppose, to the contrary, that there exist two points $x_1 \neq x_2$ in X such that $f(x_1) = f(x_2)$. Since Y is nondegenerate and f is surjective, there exists $x_3 \in X$ such that $f(x_3) \neq f(x_1)$. Let $\varepsilon > 0$ be such that $B(\varepsilon, f(x_1)) \cap B(\varepsilon, f(x_3)) = \emptyset$. By the continuity of f, there exists $\delta > 0$ such that the sets $B(\delta, x_1)$, $B(\delta, x_2)$ and $B(\delta, x_3)$ are pairwise disjoint, $f(B(\delta, x_3)) \subset B(\varepsilon, f(x_3))$ and $f(B(\delta, x_1)) \cup f(B(\delta, x_2)) \subset B(\varepsilon, f(x_1))$.

Since f_n is open, the set $f_n(B_H(\delta, \{x_1, x_2, x_3\}))$ is an open subset of $\mathcal{F}_n(Y)$ that has the element $\{f(x_1), f(x_3)\}$. Hence there exists $\eta > 0$ such that $B_H(\eta, \{f(x_1), f(x_3)\}) \subset f_n(B_H(\delta, \{x_1, x_2, x_3\}))$. We may assume that $\eta < \varepsilon$. Pick $n-1$ different points $y_1, \ldots, y_{n-1} \in B(\eta, f(x_3)) \setminus \{f(x_3)\}$. Let $B = \{f(x_1), y_1, \ldots, y_{n-1}\}$. Notice that $B \in B_H(\eta, \{f(x_1), f(x_3)\})$ and since this set is contained in $f_n(B_H(\delta, \{x_1, x_2, x_3\}))$, there exists $A \in B_H(\delta, \{x_1, x_2, x_3\})$ such that $f(A) = B$. Then there exist $a_1, a_2 \in A$ such that $a_1 \in B(\delta, x_1)$ and $a_2 \in B(\delta, x_2)$. Then $a_1 \neq a_2$. In addition, there exists $u_1, \ldots, u_{n-1} \in A$ such that $f(u_1) = y_1, \ldots, f(u_{n-1}) = y_{n-1}$. Since the points $y_1, \ldots, y_{n-1}$ are pairwise different, the points $u_1, \ldots, u_{n-1}$ are pairwise different. Given $i \in \{1, \ldots, n-1\}$, $f(u_i) = y_i \in B(\eta, f(x_3)) \subset B(\varepsilon, f(x_3))$. Hence $f(u_i) \notin B(\varepsilon, f(x_1))$ and $f(u_i) \notin f(B(\delta, x_1)) \cup f(B(\delta, x_2))$. This implies that $u_i \notin B(\delta, x_1) \cup B(\delta, x_2)$. Hence $u_i \neq a_1, a_2$. Then all of the points $a_1, a_2, u_1, \ldots, u_{n-1}$ are different and all of them are elements of A. This is absurd since $A \in \mathcal{F}_n(X)$. This contradiction shows that f is one-to-one. Therefore f is a homeomorphism. □

Definition 3.7. A mapping $f\colon X \to Y$ is OM (respectively, MO) if there exist a continuum Z and mappings $g\colon X \to Z$ and $h\colon Z \to Y$ such that $f = h \circ g$, g is monotone and h is open (respectively, g is open and h is monotone).

Definition 3.8. Given a sequence $\{A_m\}_{m=1}^{\infty}$ of subsets of X define $\limsup_{m\to\infty} A_m$ as the set of points $x \in X$ such that there exists a sequence of positive numbers $m_1 < m_2 < \cdots$ and there exist points $x_{m_k} \in A_{m_k}$ such that $\lim x_{m_k} = x$.

The following characterization of OM mappings was showed in [24, 2.2, p. 102, and Corollary 3.1, p. 104] by A. Lelek and D. R. Read.

Lemma 3.9. *A mapping $f\colon X \to Y$ is OM if and only if, for every sequence $\{y_m\}_{m=1}^{\infty}$ in Y that converges to a point $y \in Y$, we have that* $\limsup_{m\to\infty} f^{-1}(y_m)$ *meets every component of $f^{-1}(y)$.*

Theorem 3.10. *If $f_n\colon \mathcal{F}_n(X) \to \mathcal{F}_n(Y)$ is OM for some $n \in \mathbb{N}$, then f is OM.*

Proof. Let $\{y_m\}_{m=1}^{\infty}$ be a sequence in Y that converges to a point $y \in Y$. Let C be a component of $f^{-1}(y)$ and let $\mathcal{C}$ be the component of $f_n^{-1}(\{y\})$ that contains $\mathcal{F}_1(C)$. Then since $\mathcal{C}$ is connected and $\mathcal{F}_1(C) \subset \mathcal{C}$, it follows from Lemma 2.1 that $M = \bigcup\{E : E \in \mathcal{C}\}$ is connected. We will show that $M = C$.

Given $x \in C$, $\{x\} \in \mathcal{F}_1(C) \subset \mathcal{C}$, so $x \in M$. Hence $C \subset M$. If $x \in M$ there exists $A \in \mathcal{C}$ such that $x \in A$. Since $A \in \mathcal{C} \subset f_n^{-1}(\{y\})$, $f_n(A) = \{y\}$, so $f(x) = y$. We have that $M \subset f^{-1}(\{y\})$ and M is connected. Since $C \subset M$ and C is a component of $f^{-1}(\{y\})$, we have $C = M$.

Since f_n is OM and the sequence $\{\{y_m\}\}_{m=1}^{\infty}$ converges to $\{y\}$, there exists an element $A \in (\limsup_{m\to\infty} f_n^{-1}(\{y_m\})) \cap \mathcal{C}$. Thus there exist positive integers $m_1 < m_2 < \cdots$ and elements $A_{m_k} \in f_n^{-1}(\{y_{m_k}\})$ such that $\lim A_{m_k} = A$. Fix $x \in A$. Then there exist points $x_{m_k} \in A_{m_k}$ such that $\lim x_{m_k} = x$ ([22, 4.5, p. 25]). For each $k \in \mathbb{N}$, $f_n(A_{m_k}) = \{y_{m_k}\}$, then $f(x_{m_k}) = y_{m_k}$, so $x_{m_k} \in f^{-1}(y_{m_k})$. Since $\lim x_{m_k} = x$, $x \in \limsup_{m\to\infty} f^{-1}(y_m)$. We also know that $x \in A \in \mathcal{C}$ and that $M = C$, therefore $x \in C$. Then $x \in (\limsup_{m\to\infty} f^{-1}(y_m)) \cap C$.

We have shown that, for each component C of $f^{-1}(y)$ and each sequence $\{y_m\}_{m=1}^{\infty}$ in Y that converges to $y \in Y$, we have that $C \cap (\limsup_{m\to\infty} f^{-1}(y_m)) \neq \emptyset$. By Lemma 3.9 we conclude that f is OM. □

The next natural question is if the converse of Theorem 3.10 holds. For the case $n = 2$ we have already done all the necessary work to assure it does.

Theorem 3.11. *If f is OM (MO), then f_2 is OM (MO).*

Proof. If f is OM, then there exists a continuum Z and mappings $g\colon X \to Z$ and $h\colon Z \to Y$ such that $f = h \circ g$, g is monotone and h is open. Consider then the continuum $\mathcal{F}_2(Z)$ and the mappings $g_2\colon \mathcal{F}_2(X) \to \mathcal{F}_2(Z)$ and $h_2\colon \mathcal{F}_2(Z) \to \mathcal{F}_2(Y)$. By Theorems 3.3 and 3.5 we know that g_2 is monotone and h_2 is open. And it is clear that $f_2 = h_2 \circ g_2$, thus f_2 is OM. The proof for MO is analogous. □

Corollary 3.12. *A mapping $f\colon X \to Y$ is OM if and only if $f_2\colon \mathcal{F}_2(X) \to \mathcal{F}_2(Y)$ is OM.*

As in the case of open mappings, Corollary 3.12 cannot be extended for the case $n \geq 3$ (compare with Proposition 9 of [5]).

Example 3.13. There exist a continuum X and a mapping $f\colon X \to X$ that is OM and MO and, for each $n \geq 3$, the induced mapping $f_n\colon \mathcal{F}_n(X) \to \mathcal{F}_n(X)$ is neither OM nor MO.

Let $X = [0,1]$ and $f\colon [0,1] \to [0,1]$ be defined by:

$$f(x) = \begin{cases} 2x, & \text{if } x \in \left[0, \frac{1}{2}\right], \\ 2 - 2x, & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

The map f is known as the tent map. Since f is an open mapping and $f = f \circ \mathrm{id} = \mathrm{id} \circ f$, where $\mathrm{id}\colon [0,1] \to [0,1]$ is the identity mapping, we conclude that f is OM and MO.

Suppose that $n \geq 3$ and that f_n is OM. Then there exist a continuum Z and mappings $g\colon \mathcal{F}_n([0,1]) \to Z$ and $h\colon Z \to \mathcal{F}_n([0,1])$ such that g is monotone, h is open and $f_n = h \circ g$. Notice that $f^{-1}(y)$ has at most two points for each $y \in [0,1]$. It follows that, for every finite subset M of $[0,1]$, the set $f^{-1}(M)$ is finite, and also the set $\mathcal{P}(f^{-1}(M)) = \{W : W \subset M\}$ is finite. Given $A \in \mathcal{F}_n([0,1])$, it is clear that $f_n^{-1}(A) \subset \mathcal{P}(f^{-1}(A))$. Hence $f_n^{-1}(A)$ is finite.

Given $z \in Z$ we have $f_n(g^{-1}(z)) = (h \circ g)(g^{-1}(z)) = h(z)$, so $g^{-1}(z) \subset f_n^{-1}(h(z))$. Since g is monotone and $f_n^{-1}(h(z))$ is finite, $g^{-1}(z)$ is a one-point set. Thus g is one-to-one, then g is a homeomorphism. In particular, g is open. Since h is open, by hypothesis, we have that $f_n = h \circ g$ is open. Since $n \geq 3$, it follows from Theorem 3.6 that f is a homeomorphism, a contradiction. This shows that f is not an OM mapping.

The following theorem shows that f_n cannot be an MO mapping. $\square$

Theorem 3.14. *Let $n \geq 3$ and $f\colon X \to Y$ be a mapping such that Y is nondegenerate and f_n is MO, then f is monotone.*

Proof. Suppose to the contrary that f is not monotone. Then there exists $y_1 \in Y$ and nonempty disjoint compact subsets K and L of X such that $f^{-1}(y_1) = K \cup L$. Fix a point $y_2 \in Y - \{y_1\}$ and a sequence $\{v_m\}_{m=1}^{\infty}$ of pairwise different elements in $Y - \{y_1, y_2\}$ such that $\lim v_m = y_2$. For each $m \in \mathbb{N}$, choose a point $u_m \in X$ such that $f(u_m) = v_m$. We may assume that $\lim u_m = x_2$ for some point $x_2 \in X$. Thus $f(x_2) = y_2$.

Let Z be a continuum and let $g\colon \mathcal{F}_n(X) \to Z$, $h\colon Z \to \mathcal{F}_n(Y)$ be mappings such that g is open, h is monotone and $f_n = h \circ g$. Let $D = h^{-1}(\{y_1, y_2\})$. Then D is a subcontinuum of Z. It is easy to see that $g^{-1}(D) = \left\langle f^{-1}(y_1), f^{-1}(y_2)\right\rangle_n$. Let $E = f^{-1}(y_2)$.

Note that $\left\langle f^{-1}(y_1), f^{-1}(y_2)\right\rangle_n = \langle K \cup L, E\rangle_n = \langle K, E\rangle_n \cup \langle L, E\rangle_n \cup \langle K, L, E\rangle_n$ and $\langle K, L, E\rangle_n \neq \emptyset$ $(n \geq 3)$. Thus $\langle K, L, E\rangle_n$ is a nonempty open and closed subset of $g^{-1}(D)$ (realtive to the topology of $g^{-1}(D)$). Choose points $x_1 \in K$ and $x_3 \in L$. Let $\mathcal{C}$ be the component of $g^{-1}(D)$ containing the element $\{x_1, x_2, x_3\}$.

Since $\{x_1, x_2, x_3\} \in \langle K, L, E\rangle_n$, $\mathcal{C} \subset \langle K, L, E\rangle_n$. Let V, W be disjoint open subsets of Y such that $\{y_2\} \cup \{v_1, v_2, \dots\} \subset V$ and $y_1 \in W$. Then $E \subset f^{-1}(V)$ and $K \cup L \subset f^{-1}(W)$.

Since g is open, g is confluent ([30, Theorem (7.5), p. 148], definition of confluence is in Definition 3.17). Thus $g(\mathcal{C}) = D$. Since $\{x_1, x_2\} \in \left\langle f^{-1}(y_1), f^{-1}(y_2)\right\rangle_n = g^{-1}(D)$, $g(\{x_1, x_2\}) \in D$. Hence, there exists $C \in \mathcal{C} \subset \langle K, L, E\rangle_n$ such that $g(C) = g(\{x_1, x_2\})$. The set C is of the form $C = \{p_1, \dots, p_r, p_{r+1} \dots, p_s, p_{s+1} \dots, p_t\}$, where $t \le n$, the points $p_1, \dots, p_t$ are pairwise different, $\{p_1, \dots, p_r\} \subset K$, $\{p_{r+1} \dots, p_s\} \subset L$ and $\{p_{s+1} \dots, p_t\} \subset E$. Let $\varepsilon > 0$ be such that $B(\varepsilon, p_1), \dots, B(\varepsilon, p_t)$ are pairwise disjoint, $B(\varepsilon, p_1) \cup \dots \cup B(\varepsilon, p_s) \subset f^{-1}(W)$ and $B(\varepsilon, p_{s+1}) \cup \dots \cup B(\varepsilon, p_t) \subset f^{-1}(V)$.

Let $\mathcal{U} = \langle B(\varepsilon, p_1), \dots, B(\varepsilon, p_t)\rangle_n$. Then $\mathcal{U}$ is open in $\mathcal{F}_n(X)$, $C \in \mathcal{U}$, $g(\mathcal{U})$ is open in Z and $g(\{x_1, x_2\}) = g(C) \in g(\mathcal{U})$. Since $\lim_{m\to\infty} \{x_1, u_m, \dots, u_{m+n-2}\} = \{x_1, x_2\}$, we have

$$\lim_{m\to\infty} g(\{x_1, u_m, \dots, u_{m+n-2}\}) = g(\{x_1, x_2\}) \in g(\mathcal{U}) .$$

So, there exists $m \in \mathbb{N}$ such that $g(\{x_1, u_m, \dots, u_{m+n-2}\}) \in g(\mathcal{U})$. Thus, there exists $A \in \mathcal{U}$ such that $g(A) = g(\{x_1, u_m, \dots, u_{m+n-2}\})$. Therefore $f(A) = f_n(A) = h(g(A)) = h(g(\{x_1, u_m, \dots, u_{m+n-2}\})) = f_n(\{x_1, u_m, \dots, u_{m+n-2}\}) = \{y_1, v_m, \dots, v_{m+n-2}\}$. Let $q_1, \dots, q_{n-1} \in A$ be such that $f(q_1) = v_m, \dots, f(q_{n-1}) = v_{m+n-2}$. Notice that $\{q_1, \dots, q_{n-1}\} \subset f^{-1}(V)$.

Notice that $p_1 \in K$ and $p_{r+1} \in L$, so there exist $a_1 \in A \cap B(\varepsilon, p_1) \subset f^{-1}(W)$ and $a_2 \in A \cap B(\varepsilon, p_{r+1}) \subset f^{-1}(W)$. Thus the points $a_1, a_2, q_1, \dots, q_{n-1}$ are pairwise different and they are elements of A. This is a contradiction since A has at most n points.

Therefore f is monotone. □

Corollary 3.15. *Suppose that Y is a nondegenerate continuum. Then the following statements for a mapping $f\colon X \to Y$ are equivalent:*

a) *f is monotone,*
b) *f_n is monotone for some $n \in \mathbb{N}$,*
c) *f_n is monotone for every $n \in \mathbb{N}$,*
d) *f_n is MO for some $n \ge 3$.*

Question 3.16. Does Theorem 3.14 hold for $n = 2$?

3.3. **Confluent Mappings.**

Definition 3.17. A mapping $f\colon X \to Y$ is said to be:

1) Confluent if for every subcontinuum B of Y and every component A of $f^{-1}(B)$ we have that $f(A) = B$.
2) Weakly confluent if for every subcontinuum B of Y, there exists a component A of $f^{-1}(B)$ such that $f(A) = B$.
3) Semi-confluent if for every subcontinuum B of Y and every pair of components C and D of $f^{-1}(B)$ we have that $f(C) \subset f(D)$ or $f(D) \subset f(C)$.

Clearly every confluent mapping is a weakly confluent and a semi-confluent mapping.

Example 3.18. (compare with [18, Example 5.1]). There exist continua X and Y and a confluent (and thus, weakly confluent and semi-confluent) mapping $f\colon X \to Y$ such that $f_2\colon \mathcal{F}_2(X) \to \mathcal{F}_2(Y)$ is neither a confluent, weakly conluent nor semi-confluent mapping.

Proof. Let $\mathbb{C}$ be the complex plane and $\mathbb{S}^1 \subset \mathbb{C}$ the unit circle centered at the origin. Let $X = \mathbb{S}^1 \cup I \cup J$, where I and J are two rays converging each one to one half of $\mathbb{S}^1$ as it is shown in Figure 1. Notice that the continuum X is the union of two topological copies of the $\sin(\frac{1}{x})$-continuum joined by the end points of the limit segments. Let f be the restriction of the complex function $z \to z^2$ to X. It is easy to show that f is confluent. Let $Y = f(X)$. Then $f(I)$ and $f(J)$ are two rays converging to $\mathbb{S}^1$ as it is shown in Figure 1. We will construct a subcontinuum $\mathcal{K}$ of $\mathcal{F}_2(Y)$ which will be useful to deny the three definitions of confluence stated above.

Let $\varepsilon = \frac{1}{16}$. Let $\alpha\colon [0,\infty) \to f(I)$ and $\beta\colon [0,\infty) \to f(J)$ be parametrizations of $f(I)$ and $f(J)$, respectively, by arc length. Let

$$\mathcal{A} = \{\{\alpha(t), \alpha(t+\varepsilon)\} : t \in [0,\infty)\}$$

and

$$\mathcal{B} = \{\{\beta(t), \beta(t+\varepsilon)\} : t \in [0,\infty)\} \ .$$

Notice that $\mathcal{A}$ (respectively, $\mathcal{B}$) consists of the pairs of points in $f(I)$ (respectively, $f(J)$) such that the subarc of I joining them has lenght equal to ε. Since the function $t \to \{\alpha(t), \alpha(t+\varepsilon)\}$ from $[0,\infty)$ to $\mathcal{F}_2(X)$ is continuous, we have that $\mathcal{A}$ is connected. Similarly, $\mathcal{B}$ is connected.

Let
$\mathcal{D}=\{\{e^{(-\pi+t)i},e^{(-\pi-t+\varepsilon)i}\}\in\mathcal{F}_2(\mathbb{S}^1):t\in[0,\varepsilon]\}\cup\{\{e^{(\pi-t)i},e^{(\pi+t-\varepsilon)i}\}\in\mathcal{F}_2(\mathbb{S}^1):t\in[0,\varepsilon]\}$ and $\mathcal{C}=\{\{e^{ti},e^{(t+\varepsilon)i}\}\in\mathcal{F}_2(\mathbb{S}^1):t\in[-\pi,\pi-\varepsilon]\}$.

It is easy to see that $\mathrm{cl}_{\mathcal{F}_2(Y)}(\mathcal{A})\cup\mathrm{cl}_{\mathcal{F}_2(Y)}(\mathcal{B})=\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}\cup\mathcal{D}$ and $\mathrm{cl}_{\mathcal{F}_2(Y)}(\mathcal{A})\cap\mathrm{cl}_{\mathcal{F}_2(Y)}(\mathcal{B})=\mathcal{C}\cup\mathcal{D}$. Thus, the set $\mathcal{K}=\mathrm{cl}_{\mathcal{F}_2(Y)}(\mathcal{A})\cup\mathrm{cl}_{\mathcal{F}_2(Y)}(\mathcal{B})$ is a subcontinuum of $\mathcal{F}_2(Y)$. We use $\mathcal{K}$ to deny the three definitions related to confluence.

Let $\mathbb{S}^+=\{z\in\mathbb{S}^1:\mathrm{Re}(z)\geq 0\}$ and $\mathbb{S}^-=\{z\in\mathbb{S}^1:\mathrm{Re}(z)\leq 0\}$. Let $\alpha_0:\mathbb{S}^1\to\mathbb{S}^+$ be the map defined as $\alpha_0(z)=|\mathrm{Re}(z)|+i\,\mathrm{Im}(z)$ and let $\beta_0:\mathbb{S}^1\to\mathbb{S}^-$ be the map defined as $\beta_0(z)=-|\mathrm{Re}(z)|+i\,\mathrm{Im}(z)$. For each $\theta\in\mathbb{R}$, let $\gamma(\theta)=\{\alpha_0(e^{i\theta}),\alpha_0(e^{i(\theta+\frac{\varepsilon}{2})})\}$, $\delta(\theta)=\{\beta_0(e^{i\theta}),\beta_0(e^{i(\theta+\frac{\varepsilon}{2})})\}$, $\lambda(\theta)=\{\alpha_0(e^{i\theta}),-\alpha_0(e^{i(\theta+\frac{\varepsilon}{2})})\}$ and $\eta(\theta)=\{\beta_0(e^{i\theta}),-\beta_0(e^{i(\theta+\frac{\varepsilon}{2})})\}$. Then the maps γ, δ, λ and η are defined from $\mathbb{R}$ to $\mathcal{F}_2(\mathbb{S}^1)$ and they have the following properties: $\{e^{i(\frac{\pi}{2}-\frac{\varepsilon}{4})},-e^{i(\frac{\pi}{2}-\frac{\varepsilon}{4})}\}\in\lambda(\mathbb{R})\cap\eta(\mathbb{R})$, for each $\theta\in\mathbb{R}$, $\gamma(\theta)\in\langle\mathbb{S}^+,\mathbb{S}^+-\{i,-i\}\rangle_2$, $\delta(\theta)\in\langle\mathbb{S}^-,\mathbb{S}^--\{i,-i\}\rangle_2$, $\gamma(\mathbb{R})\cap\delta(\mathbb{R})=\emptyset$, $(\gamma(\mathbb{R})\cup\delta(\mathbb{R}))\cap(\lambda(\mathbb{R})\cup\eta(\mathbb{R}))=\emptyset$ and the sets $\gamma(\mathbb{R})$, $\delta(\mathbb{R})$ and $\lambda(\mathbb{R})\cup\eta(\mathbb{R})$ are compact and connected.

Since $f|_I:I\to f(I)$ is a homeomorphism and $f^{-1}(f(I))=I$, we have that $f_2^{-1}(\mathcal{A})$ is a connected subset of $\mathcal{F}_2(I)$ that is homeomorphic to $\mathcal{A}$. Similarly, $f_2^{-1}(\mathcal{B})$ is a connected subset of $\mathcal{F}_2(J)$ that is homeomorphic to $\mathcal{B}$. It is easy to show that $f_2^{-1}(\mathcal{K})=(f_2^{-1}(\mathcal{A})\cup\gamma(\mathbb{R}))\cup(f_2^{-1}(\mathcal{B})\cup\delta(\mathbb{R}))\cup(\lambda(\mathbb{R})\cup\eta(\mathbb{R}))$ and the components of $f_2^{-1}(\mathcal{K})$ are the sets $f_2^{-1}(\mathcal{A})\cup\gamma(\mathbb{R})$, $f_2^{-1}(\mathcal{B})\cup\delta(\mathbb{R})$ and $\lambda(\mathbb{R})\cup\eta(\mathbb{R})$. Since $f_2(f_2^{-1}(\mathcal{A})\cup\gamma(\mathbb{R}))\cap f_2(f_2^{-1}(\mathcal{B})\cup\delta(\mathbb{R}))\subset\mathcal{F}_2(\mathbb{S}^1)$ and $f_2(\lambda(\mathbb{R})\cup\eta(\mathbb{R}))\subset\mathcal{F}_2(\mathbb{S}^1)$, we have that $f_2(f_2^{-1}(\mathcal{A})\cup\gamma(\mathbb{R}))\subsetneq f_2(f_2^{-1}(\mathcal{B})\cup\delta(\mathbb{R}))$, $f_2(f_2^{-1}(\mathcal{B})\cup\delta(\mathbb{R}))\subsetneq f_2(f_2^{-1}(\mathcal{A})\cup\gamma(\mathbb{R}))$ and no component $\mathcal{L}$ of $f_2^{-1}(\mathcal{K})$ has the property that $f_2(\mathcal{L})=\mathcal{K}$. Therefore, f_2 is neither confluent, weakly confluent nor semi-confluent. □

Theorem 3.19. *If $f_n\colon\mathcal{F}_n(X)\to\mathcal{F}_n(Y)$ is confluent, for some $n\in\mathbb{N}$, then $f\colon X\to Y$ is confluent.*

Proof. Let B be a subcontinuum of Y and D a component of $f^{-1}(B)$. Notice that $\mathcal{F}_1(B)$ is a subcontinuum of $\mathcal{F}_n(Y)$. Note that $\mathcal{F}_1(D)$ is a connected subset of $f_n^{-1}(\mathcal{F}_1(B))$. Let $\mathcal{C}$ be the component of $f_n^{-1}(\mathcal{F}_1(B))$ that contains $\mathcal{F}_1(D)$. By Lemma 2.1, $M=\bigcup\{E:E\in\mathcal{C}\}$ is connected. We will show that $M=D$.

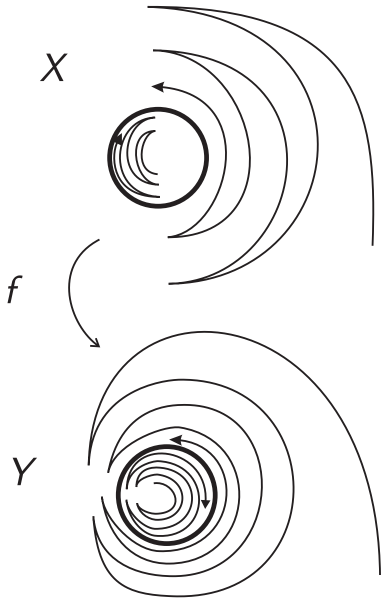

FIGURE 1

Clearly, $D \subset M$. Given $x \in M$ there exists $A \in \mathcal{C}$ such that $x \in A$. Since $A \in \mathcal{C} \subset f_n^{-1}(\mathcal{F}_1(B))$, $f_n(A) = \{b\}$ for some $b \in B$, and in particular $f(x) = b$. We have shown that $M \subset f^{-1}(B)$. Since M is connected, $D \subset M$ and D is a component of $f^{-1}(B)$, we obtain that $D = M$.

Now, we show that $f(M) = B$. Clearly, $f(M) \subset B$. Given $b \in B$, $\{b\} \in \mathcal{F}_1(B)$. Since f_n is confluent and $\mathcal{C}$ is a component of $f_n^{-1}(\mathcal{F}_1(B))$, $f_n(\mathcal{C}) = \mathcal{F}_1(B)$. Then there exists $A \in \mathcal{C}$ such that $f_n(A) = \{b\}$. Fix $x \in A$, then $f(x) = b$ and $x \in M$. Thus $f(x) \in f(M)$ and $b \in f(M)$. This shows that $B \subset f(M)$, then $f(M) = B$. Hence $f(D) = B$. Therefore f is confluent. $\square$

Theorem 3.20. *If $f_n\colon \mathcal{F}_n(X) \to \mathcal{F}_n(Y)$ is weakly confluent, for some $n \in \mathbb{N}$, then f is weakly confluent.*

Proof. Let B be a subcontinuum of Y. Since f_n is weakly confluent there exists a component $\mathcal{D}$ of $f_n^{-1}(\mathcal{F}_1(B))$ such that $f_n(\mathcal{D}) = \mathcal{F}_1(B)$.

Let $G = \bigcup\{E : E \in \mathcal{D}\}$. Clearly, $G \subset f^{-1}(B)$. Choose a component C of G. Let D be the component of $f^{-1}(B)$ that contains C. Given $y \in B$, since $f_n(\mathcal{D}) = \mathcal{F}_1(B)$, there exists $E \in \mathcal{D}$ such that $f_n(E) = \{y\}$. Since $\mathcal{D}$ is closed and connected, by Lemma 2.1, E intersects every component of G. So we can pick $z \in C \cap E$. Since $z \in E$, $f(z) = y$. In addition, since $z \in C \subset D$, $f(z) = y \in f(D)$. We have shown that $B \subset f(D)$ and then $f(D) = B$. This ends the proof that f is weakly confluent. □

Theorem 3.21. *If $f_n \colon \mathcal{F}_n(X) \to \mathcal{F}_n(Y)$ is semi-confluent, for some $n \in \mathbb{N}$, then f is semi-confluent.*

Proof. Let B be a subcontinuum of Y and let C and D be components of $f^{-1}(B)$. Notice that $\mathcal{F}_1(B)$, $\mathcal{F}_1(C)$ and $\mathcal{F}_1(D)$ are connected, $\mathcal{F}_1(C) \subset f_n^{-1}(\mathcal{F}_1(B))$ and $\mathcal{F}_1(D) \subset f_n^{-1}(\mathcal{F}_1(B))$.

Take the components $\mathcal{C}$ and $\mathcal{D}$ of $f_n^{-1}(\mathcal{F}_1(B))$ that contain $\mathcal{F}_1(C)$ and $\mathcal{F}_1(D)$, respectively. Let $M = \bigcup\{E : E \in \mathcal{C}\}$ and $N = \bigcup\{E : E \in \mathcal{D}\}$. Proceeding as in the proof of Theorem 3.19, it follows that $M = C$ and that $N = D$.

Since f_n is semi-confluent we may assume that $f_n(\mathcal{C}) \subset f_n(\mathcal{D})$. We now show that $f(C) \subset f(D)$. Let $y \in f(C)$. Let $c \in C$ be such that $f(c) = y$. Then $\{c\} \in \mathcal{C}$ and therefore $\{y\} \in f_n(\mathcal{C}) \subset f_n(\mathcal{D})$. Let $A \in \mathcal{D}$ be such that $f_n(A) = \{y\}$. Taking $x \in A$, we have that $x \in N = D$ and $f(x) = y$. Hence $y = f(x) \in f(D)$. Thus $f(C) \subset f(D)$. Therefore f is semi-confluent. □

3.4. **Light Mappings.**

Definition 3.22. A topological space X is said to be totally disconnected if every component of X is degenerate.

It is well known that in compact metric spaces the components and quasi-components coincide. The next corollaries follow directly from this fact.

Corollary 3.23. *If X is a metric compact space, then X is totally disconnected if and only if for every pair of different points x and y there exists an open and closed subset U of X such that $x \in U$ and $y \notin U$.*

Corollary 3.24. *If X is a metric compact space, then X is totally disconnected if and only if for every finite subset A of X and every point $y \in X \setminus A$ there exists an open and closed subset U of X such that $A \subset U$ and $y \notin U$.*

Definition 3.25. A mapping between continua $f\colon X \to Y$ is *light* if $f^{-1}(y)$ is totally disconnected for every y in Y.

Theorem 3.26. *The following three statements for a mapping $f\colon X \to Y$ are equivalent:*

a) *f is light,*
b) *f_n is light for some $n \in \mathbb{N}$,*
c) *f_n is light for every $n \in \mathbb{N}$.*

Proof. b) $\Rightarrow$ a). Suppose that f_n is light for some $n \in \mathbb{N}$. Given $y \in Y$, $f_n^{-1}(\{y\})$ is totally disconnected. Notice that, $x \in f^{-1}(y)$ if and only if $\{x\} \in f_n^{-1}(\{y\})$. Thus $f^{-1}(y)$ is homeomorphic to $f_n^{-1}(\{y\}) \cap \mathcal{F}_1(X)$. Therefore, $f^{-1}(y)$ is totally disconnected and f is light.

a) $\Rightarrow$ c). Let $n \in \mathbb{N}$. Suppose that f is light. Let $B = \{y_1, \ldots, y_k\} \in \mathcal{F}_n(Y)$. Then $f^{-1}(y_i)$ is totally disconnected for each $i \in \{1, \ldots, k\}$.

It is easy to check that $f_n^{-1}(B) = \left\langle f^{-1}(y_1), \ldots, f^{-1}(y_k)\right\rangle_n$.

Let $A_1 = \{x_1, \ldots, x_m\}$ and $A_2 = \{u_1, \ldots, u_s\}$ be two different elements of $f_n^{-1}(B)$. We may assume that $x_1 \notin A_2$ and that $x_1 \in f^{-1}(y_1)$. By hypothesis $f^{-1}(y_1)$ is totally disconnected. Since $A_2 \cap f^{-1}(y_1)$ is finite, by Corollary 3.24, there exists an open and closed subset K of $f^{-1}(y_1)$ such that $x_1 \in K$ and $\left(A_2 \cap f^{-1}(y_1)\right) \cap K = \emptyset$. Let $L = f^{-1}(y_1) \setminus K$. Then K and L are closed in X, $K \cap L = \emptyset$, $f^{-1}(y_1) = K \cup L$, $x_1 \in K$ and $A_2 \cap f^{-1}(y_1) \subset L$.

Let $\mathcal{K} = \left\{A \in f_n^{-1}(B) : A \cap K \neq \emptyset\right\} = f_n^{-1}(B) \cap \langle K, X\rangle_n$ and let $\mathcal{L} = \left\langle L, f^{-1}(y_2), \ldots, f^{-1}(y_k)\right\rangle_n$. Clearly $\mathcal{K}$ and $\mathcal{L}$ are closed subsets of $f_n^{-1}(B)$, $A_1 \in \mathcal{K}$ and $A_2 \in \mathcal{L}$. Given $A \in f_n^{-1}(B)$, if $A \cap K \neq \emptyset$, then $A \in \mathcal{K}$, and if $A \cap K = \emptyset$, since $f_n^{-1}(B) = \left\langle f^{-1}(y_1), \ldots, f^{-1}(y_k)\right\rangle_n$, then $\emptyset \neq A \cap f^{-1}(y_1) \subset L$, so $A \in \mathcal{L}$. This shows that $f_n^{-1}(B) = \mathcal{K} \cup \mathcal{L}$.

If there is an element $A \in \mathcal{K} \cap \mathcal{L}$, then there exists $x \in A \cap K$ and $A \subset L \cup f^{-1}(y_2) \cup \cdots \cup f^{-1}(y_k)$. Since $K \subset f^{-1}(y_1)$, $x \in A \cap f^{-1}(y_1) \subset L$ and then $x \in K \cap L$, a contradiction. Therefore $\mathcal{K} \cap \mathcal{L} = \emptyset$.

This shows that $\mathcal{K}$ is an open and closed (relative to $f_n^{-1}(B)$) subset of $f_n^{-1}(B)$, $A_1 \in \mathcal{K}$ and $A_2 \notin \mathcal{K}$. Hence $f_n^{-1}(B)$ is totally disconnected. Therefore f_n is light. $\square$

3.5. **Universal Mappings.**

Definition 3.27. A mapping $f\colon X \to Y$ is universal if for every continuous function $g\colon X \to Y$ there exists $p \in X$ such that $f(p) = g(p)$.

In this definition, a particular interesting case is when f is the identity. Notice that the identity $\mathrm{id}\colon X \to X$ is universal if and only if for every continuous function $g\colon X \to X$ there exists $p \in X$ such that $g(p) = p$. In other words, the identity is universal if and only if X has the fixed point property.

Then, a first step to see if the universality of a mapping is preserved by the induced functions is to see if this happens when this mapping is the simplest of all, the identity.

Example 3.28 (J. Oledzki, [28])**.** There exists a continuum X with the fixed point property such that the hyperspace $\mathcal{F}_2(X)$ does not have the fixed point property.

Example 3.28 implies the following.

Example 3.29 ([28])**.** There exists a continuum X and the mapping $\mathrm{id}\colon X \to X$ such that id is universal, but $\mathrm{id}_2\colon \mathcal{F}_2(X) \to \mathcal{F}_2(X)$ is not universal.

The other implication does not hold either. The authors have given an example in [15].

Example 3.30 ([15])**.** There exists a continuum X such that $\mathcal{F}_2(X)$ has the fixed point property but X does not have the fixed point property.

Thus, we obtain the following.

Example 3.31. There exists a continuum X and the mapping $\mathrm{id}\colon X \to X$ such that $\mathrm{id}_2\colon \mathcal{F}_2(X) \to \mathcal{F}_2(X)$ is universal but id is not universal.

3.6. **Atomic Mappings.**

Definition 3.32. A mapping between continua $f\colon X \to Y$ is atomic if for every subcontinuum K of X we have that $f(K)$ is degenerate or $f^{-1}(f(K)) = K$.

Theorem 3.33. *If $f_n\colon \mathcal{F}_n(X) \to \mathcal{F}_n(Y)$ is atomic for some $n \geq 2$ and Y is nondegenerate then f is a homeomorphism.*

Proof. Let d be a metric for Y. It suffices to show that f is one-to-one. Suppose on the contrary that there exist $x_1 \neq x_2$ in X such that $f(x_1) = f(x_2)$. Since Y is nondegenerate and f is surjective there exists $x_3 \in X$ such that $f(x_1) \neq f(x_3)$. Let $d_0 = d(f(x_1), f(x_3))$. By Theorem 2.4 there exists an order arc α from $\{x_3\}$ to X. Let $F\colon \mathcal{C}(X) \to \mathcal{C}(Y)$ be the induced mapping by f. Then $F \circ \alpha : [0,1] \to \mathcal{C}(Y)$ is a path from $\{f(x_3)\}$ to Y.

Consider the function $g\colon \mathcal{C}(Y) \to [0,\infty)$ given by $g(K) = d(f(x_1), K) = \min\{d(f(x_1), y) : y \in K\}$. Clearly g is continuous. So the function $\beta = g \circ F \circ \alpha\colon [0,1] \to [0,\infty)$ is continuous. Thus $\beta([0,1])$ is a connected set. We have that $\beta(1) = d(f(x_1), Y) = 0$ and that $\beta(0) = d(f(x_1), \{f(x_3)\}) = d_0$. Since $\beta([0,1])$ is connected, we can choose a number $t \in \beta^{-1}(\frac{d_0}{2})$.

Notice that $F(\alpha(t))$ is a subcontinuum of Y containing $f(x_3)$ and with distance to $f(x_1)$ less than d_0, so $F(\alpha(t))$ is nondegenerate. And the distance between $F(\alpha(t))$ and $f(x_1)$ is greater than zero, so $f(x_1) \notin F(\alpha(t))$. Therefore $A = \alpha(t)$ is a subcontinuum of X that does not intersect $f^{-1}(f(x_1))$ and $f(A) = F(\alpha(t))$ is nondegenerate.

Consider $\mathcal{K} = \{\{x_1, v\} : v \in A\}$, then $\mathcal{K}$ is homeomorphic to A and, hence, is a subcontinuum of $\mathcal{F}_n(X)$. Thus $f_n(\mathcal{K}) = \{\{f(x_1), w\} : w \in f(A)\}$ is nondegenerate. Also, $f_n(\{x_2, x_3\}) = \{f(x_2), f(x_3)\} = \{f(x_1), f(x_3)\} \in f_n(\mathcal{K})$ and $\{x_2, x_3\} \notin \mathcal{K}$, so $f_n^{-1}(f_n(\mathcal{K})) \neq \mathcal{K}$. Therefore f_n is not an atomic mapping. □

3.7. **Linking Mappings.**

Definition 3.34. A mapping between continua $f\colon X \to Y$ is linking if for every subcontinuum B of Y and every pair of components C and D of $f^{-1}(B)$ we have that $f(D) \cap f(C) \neq \emptyset$.

Theorem 3.35. *If $f_n\colon \mathcal{F}_n(X) \to \mathcal{F}_n(Y)$ is a linking mapping for some $n \in \mathbb{N}$, then $f\colon X \to Y$ is a linking mapping.*

Proof. Let B be a subcontinuum of Y and let C and D be two components of $f^{-1}(B)$. Let $\mathcal{C}$ and $\mathcal{D}$ be the components of $f_n^{-1}(\mathcal{F}_1(B))$ that contain $\mathcal{F}_1(C)$ and $\mathcal{F}_1(D)$, respectively. Let $M = \bigcup\{E : E \in \mathcal{C}\}$ and $N = \bigcup\{E : E \in \mathcal{D}\}$. Proceeding as in the proof of Theorem 3.19, it follows that $C = M$ and $D = N$.

Since f_n is linking, we have that $f_n(\mathcal{C}) \cap f_n(\mathcal{D}) \neq \emptyset$. Let $A_1 \in \mathcal{C}$ and $A_2 \in \mathcal{D}$ be such that $f_n(A_1) = f_n(A_2)$. Let $a_1 \in A_1$ and $a_2 \in A_2$ be such that $f(a_1) = f(a_2)$. Since $a_1 \in A_1 \in \mathcal{C}$, we have that $a_1 \in M = C$. Analogously, $a_2 \in D$ and then $f(D) \cap f(C) \neq \emptyset$. Therefore f is linking. □

Example 3.36. There exist a continuum X and a mapping $f\colon X \to X$ such that f is linking and its induced mapping $f_2\colon \mathcal{F}_2(X) \to \mathcal{F}_2(X)$ is not linking.

Proof. Let $X = [0,1]$ and $f\colon [0,1] \to [0,1]$ be given by:

$$f(x) = \begin{cases} \frac{1}{2} + 2x, & \text{if } x \in \left[0, \frac{1}{4}\right], \\ \frac{3}{2} - 2x, & \text{if } x \in [\frac{1}{4}, \frac{3}{4}], \\ 2x - \frac{3}{2}, & \text{if } x \in \left[\frac{3}{4}, 1\right]. \end{cases}$$

Clearly f is a linking mapping.

To see that f_2 is not linking, consider the following set: $\mathcal{A} = \{\{\frac{3}{4}, y\} : y \in [\frac{1}{4}, \frac{3}{4}]\} \cup \{\{y, \frac{1}{4}\} : y \in [\frac{1}{4}, \frac{3}{4}]\}$. Notice that $\mathcal{A}$ is a subcontinuum of $\mathcal{F}_2([0,1])$ and

$$f_n^{-1}(\mathcal{A}) =$$
$$\{\{\tfrac{1}{8}, x\} : x \in [0, \tfrac{1}{8}]\} \cup \{\{\tfrac{1}{8}, x\} : x \in [\tfrac{3}{8}, \tfrac{5}{8}]\} \cup \{\{\tfrac{1}{8}, x\} : x \in [\tfrac{7}{8}, 1]\} \cup$$
$$\{\{\tfrac{3}{8}, x\} : x \in [0, \tfrac{1}{8}]\} \cup \{\{\tfrac{3}{8}, x\} : x \in [\tfrac{3}{8}, \tfrac{5}{8}]\} \cup \{\{\tfrac{3}{8}, x\} : x \in [\tfrac{7}{8}, 1]\} \cup$$
$$\{\{\tfrac{5}{8}, x\} : x \in [0, \tfrac{1}{8}]\} \cup \{\{\tfrac{5}{8}, x\} : x \in [\tfrac{3}{8}, \tfrac{5}{8}]\} \cup \{\{\tfrac{5}{8}, x\} : x \in [\tfrac{7}{8}, 1]\} \cup$$
$$\{\{\tfrac{7}{8}, x\} : x \in [0, \tfrac{1}{8}]\} \cup \{\{\tfrac{7}{8}, x\} : x \in [\tfrac{3}{8}, \tfrac{5}{8}]\} \cup \{\{\tfrac{7}{8}, x\} : x \in [\tfrac{7}{8}, 1]\}.$$

Also notice that the sets $\mathcal{D} = \{\{\frac{1}{8}, x\} : x \in [0, \frac{1}{8}]\}$ and $\mathcal{C} = \{\{\frac{7}{8}, x\} : x \in [\frac{7}{8}, 1]\}$ are closed, disjoint to each other and disjoint to all the other sets whose union is $f_n^{-1}(\mathcal{A})$. Hence $\mathcal{D}$ and $\mathcal{C}$ are components of $f_n^{-1}(\mathcal{A})$. Notice that $f_2(\mathcal{D}) = \{\{\frac{3}{4}, y\} : y \in [\frac{1}{2}, \frac{3}{4}]\}$ and $f_2(\mathcal{C}) = \{\{\frac{1}{4}, y\} : y \in [\frac{1}{4}, \frac{1}{2}]\}$ do not meet. It follows that f_2 is not a linking mapping. □

3.8. **Hereditary Properties.**

Definition 3.37. Given a property $\mathfrak{R}$ defined for mappings. We say that a maping $f\colon X \to Y$ is hereditarily $\mathfrak{R}$ if for every subcontinuum A of X we have that $f|_A\colon A \to f(A)$ has property $\mathfrak{R}$.

Theorem 3.38. *If $f_n\colon \mathcal{F}_n(X) \to \mathcal{F}_n(Y)$ is hereditarily $\mathfrak{R}$, for some property $\mathfrak{R}$ and some $n \in \mathbb{N}$, then $f\colon X \to Y$ is hereditarily $\mathfrak{R}$.*

Proof. Notice that for every subcontinuum A of X, the mapping $f_n|_{\mathcal{F}_1(A)}\colon \mathcal{F}_1(A) \to \mathcal{F}_1(f(A))$ behaves topologically identical to $f|_A\colon A \to f(A)$. So since $\mathcal{F}_1(A)$ is a subcontinuum of $\mathcal{F}_n(X)$ and f_n is hereditarily $\mathfrak{R}$, we have that $f_n|_{\mathcal{F}_1(A)}$, and therefore $f|_A$, has the property $\mathfrak{R}$. □

Some of the most commonly studied hereditary properties are hereditarily monotone, hereditarily confluent, hereditarily weakly confluent, etc. For these properties we have the following results.

Theorem 3.39. *If $f_n\colon \mathcal{F}_n(X) \to \mathcal{F}_n(Y)$ is hereditarily monotone for some $n \geq 2$ and Y is nondegenerate, then $f\colon X \to Y$ is a homeomorphism.*

Proof. It suffices to show that f is one-to-one. Supose that there exist two different points $a, b \in X$ such that $f(a) = f(b) = y$. Since Y is nondegenerate, there exists $v \in Y \setminus \{y\}$. Fix $x_0 \in X \setminus (f^{-1}(y) \cup f^{-1}(v))$.

Let $\mathcal{A} = \{\{a, x\} : x \in X\}$, $\mathcal{C} = \{\{b, x\} : x \in X\}$ and $\mathcal{D} = \{\{x_0, x\} : x \in X\}$. Then $\mathcal{A}, \mathcal{C}$ and $\mathcal{D}$ are homeomorphic to X and therefore they are continua. Notice that $\mathcal{A} \cap \mathcal{C} = \{a, b\}$ and $\mathcal{C} \cap \mathcal{D} = \{b, x_0\}$, so $\mathcal{G} = \mathcal{A} \cup \mathcal{C} \cup \mathcal{D}$ is a subcontinuum of $\mathcal{F}_n(X)$.

We analyze the map $f_n|_{\mathcal{G}}\colon \mathcal{G} \to f_n(\mathcal{G})$. Since $f(x_0) \notin \{v, y\}$, $(f_n|_{\mathcal{G}})^{-1}(\{y, v\}) \subset \mathcal{A} \cup \mathcal{C}$, and in fact, $(f_n|_{\mathcal{G}})^{-1}(\{y, v\}) = \{\{a, x\} : x \in f^{-1}(v)\} \cup \{\{b, x\} : x \in f^{-1}(v)\}$, which are closed, disjoint and nonempty. Therefore $(f_n|_{\mathcal{G}})^{-1}(\{y, v\})$ is not connected and f_n is not hereditarily monotone. This concludes the proof of the theorem. □

Theorem 3.40. *If $f_n\colon \mathcal{F}_n(X) \to \mathcal{F}_n(Y)$ is hereditarily weakly confluent for some $n \geq 3$ (and Y is nondegenerate), then $f\colon X \to Y$ is a homeomorphism.*

Proof. Suppose that there exist two different points $a, b \in X$ such that $f(a) = f(b) = y_0$. Using order arcs we can constuct two nondegenerate and disjoint subcontinua K and L of Y which do not contain y_0. Let $e \in f^{-1}(L)$ and $c \in f^{-1}(K)$. Let $\mathcal{A} = \{\{a, b, x\} : x \in X\}$, $\mathcal{B} = \{\{a, e, x\} : x \in X\}$ and $\mathcal{C} = \{\{b, c, x\} : x \in X\}$. Then $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are subcontinua of $\mathcal{F}_n(X)$.

Since $\{a,b,e\} \in \mathcal{A} \cap \mathcal{B}$ and $\{a,b,c\} \in \mathcal{A} \cap \mathcal{C}$, $\mathcal{M} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ is a subcontinuum of $\mathcal{F}_n(X)$. Let $\mathcal{K} = \{\{y_0, f(e), w\} : w \in K\}$ and $\mathcal{L} = \{\{y_0, f(c), w\} : w \in L\}$. Since $\{y_0, f(c), f(e)\} \in \mathcal{K} \cap \mathcal{L}$, $\mathcal{R} = \mathcal{K} \cup \mathcal{L}$ is a subcontinuum of $\mathcal{F}_n(Y)$. Since $\mathcal{K} \subset f_n(\mathcal{B})$ and $\mathcal{L} \subset f_n(\mathcal{C})$, $\mathcal{R} \subset f_n(\mathcal{M})$. Since $f(e), y_0 \notin K$, $\mathcal{K} \subset \mathcal{F}_3(X) \setminus \mathcal{F}_2(X)$. Analogously, $\mathcal{L} \subset \mathcal{F}_3(X) \setminus \mathcal{F}_2(X)$ and then $\mathcal{R} \subset \mathcal{F}_3(X) \setminus \mathcal{F}_2(X)$.

We will show that $f_n|_{\mathcal{M}} \colon \mathcal{M} \to f_n(\mathcal{M})$ is not weakly confluent. Note that $(f_n|_{\mathcal{M}})^{-1}(\mathcal{R}) = (f_n^{-1}(\mathcal{R}) \cap \mathcal{A}) \cup (f_n^{-1}(\mathcal{R}) \cap \mathcal{B}) \cup (f_n^{-1}(\mathcal{R}) \cap \mathcal{C})$.

Notice that, for every $x \in X$, we have that $f_n(\{a,b,x\}) = \{y_0, f(x)\} \in \mathcal{F}_2(Y)$. Hence $f_n^{-1}(\mathcal{R}) \cap \mathcal{A} = \emptyset$ and $(f_n|_{\mathcal{M}})^{-1}(\mathcal{R}) = (f_n^{-1}(\mathcal{R}) \cap \mathcal{B}) \cup (f_n^{-1}(\mathcal{R}) \cap \mathcal{C})$.

It is easy to prove that $f_n^{-1}(\mathcal{R}) \cap \mathcal{B} = \{\{a,e,x\} : x \in f^{-1}(K)\}$ and $f_n^{-1}(\mathcal{R}) \cap \mathcal{C} = \big\{\{b,c,x\} : x \in f^{-1}(L)\big\}$.

This implies that $f_n(f_n^{-1}(\mathcal{R}) \cap \mathcal{B}) = \mathcal{K}$ and $f_n(f_n^{-1}(\mathcal{R}) \cap \mathcal{C}) = \mathcal{L}$.

Let $\mathcal{N}$ be a component of $(f_n|_{\mathcal{M}})^{-1}(\mathcal{R})$, notice that $f_n^{-1}(\mathcal{R}) \cap \mathcal{B}$ and $f_n^{-1}(\mathcal{R}) \cap \mathcal{C}$ are closed and disjoint, so $\mathcal{N} \subset f_n^{-1}(\mathcal{R}) \cap \mathcal{B}$ or $\mathcal{N} \subset f_n^{-1}(\mathcal{R}) \cap \mathcal{C}$. Since K and L are nondegenerate, $\mathcal{K} \subsetneq \mathcal{R}$ and $\mathcal{L} \subsetneq \mathcal{R}$, so $f_n(f_n^{-1}(\mathcal{R}) \cap \mathcal{B}) \subsetneq \mathcal{R}$ and $f_n(f_n^{-1}(\mathcal{R}) \cap \mathcal{C}) \subsetneq \mathcal{R}$. Thus $f_n(\mathcal{N}) \subsetneq \mathcal{R}$. Hence $f_n|_{\mathcal{M}} : \mathcal{M} \to f_n(\mathcal{M})$ is not weakly confluent, so f_n is not hereditarily confluent. □

Definition 3.41. A subcontinuum A of X is terminal in X if for every subcontinuum B of X, we have that $B \subset A$ or $A \subset B$ or $A \cap B = \emptyset$.

Theorem 3.42. *If $f_2 \colon \mathcal{F}_2(X) \to \mathcal{F}_2(Y)$ is hereditarily confluent, then $f \colon X \to Y$ is monotone and $f^{-1}(y)$ is a terminal subcontinuum of X for each $y \in Y$.*

Proof. The proof is based in the following claim.

Claim 1. If there exist $y \in Y$ and a subcontinuum A of X such that $A \cap f^{-1}(y) \neq \emptyset$, $f^{-1}(y) \nsubseteq A$ and $A \nsubseteq f^{-1}(y)$, then $f_2 : \mathcal{F}_2(X) \to \mathcal{F}_2(Y)$ is not hereditarily confluent.

Proof of Claim 1. Let $a \in A \cap f^{-1}(y)$, $b \in f^{-1}(y) \setminus A$ and $c \in A \setminus f^{-1}(y)$. Notice that $f(a), f(c) \in f(A)$ and $f(a) = y \neq f(c)$, so $f(A)$ is nondegenerate. Let $\mathcal{A} = \{\{a,x\} : x \in X\}$, $\mathcal{B} = \{\{b,x\} : x \in X\}$, $\mathcal{C} = \{\{c,x\} : x \in A\}$ and $\mathcal{M} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$. Notice that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are closed, connected, $\{a,b\} \in \mathcal{A} \cap \mathcal{B}$ and $\{a,c\} \in \mathcal{A} \cap \mathcal{C}$, hence $\mathcal{M}$ is a subcontinuum of $\mathcal{F}_2(X)$.

Let $\mathcal{K} = \{\{f(c), z\} : z \in f(A)\}$, then $\mathcal{K}$ is a nondegenerate subcontinuum of $\mathcal{F}_2(Y)$ such that $f_2(\mathcal{C}) = \mathcal{K}$, so $\mathcal{K}$ is a subcontinuum of $f_2(\mathcal{M})$. We show that $f_2|_{\mathcal{M}} \colon \mathcal{M} \to f_2(\mathcal{M})$ is not confluent.

Since $(f_2|_{\mathcal{M}})^{-1}(\mathcal{K}) = (f_2^{-1}(\mathcal{K}) \cap \mathcal{A}) \cup (f_2^{-1}(\mathcal{K}) \cap \mathcal{B}) \cup (f_2^{-1}(\mathcal{K}) \cap \mathcal{C}) = \{\{a, x\} : x \in f^{-1}(f(c))\} \cup \{\{b, x\} : x \in f^{-1}(f(c))\} \cup \mathcal{C}$ and that the sets $\{\{b, x\} : x \in f^{-1}(f(c))\}$ and $\{\{a, x\} : x \in f^{-1}(f(c))\} \cup \mathcal{C}$ are closed, disjoint and nonempty, there exists a component $\mathcal{D}$ of $(f_2|_{\mathcal{M}})^{-1}(\mathcal{K})$ that is contained in $\{\{b, x\} : x \in f^{-1}(f(c))\}$. But then $f_2(\mathcal{D}) \subset f_2\left(\{\{b, x\} : x \in f^{-1}(f(c))\}\right) = \{\{y, f(c)\}\} \subsetneq \mathcal{K}$. It follows that $f_2|_{\mathcal{M}} \colon \mathcal{M} \to f_2(\mathcal{M})$ is not confluent and therefore f_2 is not hereditarily confluent. This finishes the proof of Claim 1.

Suppose that there exists $y \in Y$ such that $f^{-1}(y)$ is not connected. Then there exist two closed, disjoint and nonempty subsets K and L of X such that $f^{-1}(y) = K \cup L$. Let C be a component of $f^{-1}(y)$ contained in K. By Theorem 2.4 there exists an order arc $\alpha \colon [0, 1] \to \mathcal{C}(X)$ from C to X. Since $\alpha(0) = C \subset K$, $\alpha(0) \subset X \setminus L$, there exists $s > 0$ such that $\alpha(s) \subset X \setminus L$ and $\alpha(s) \neq \alpha(0)$. Let $A = \alpha(s)$. Then A is a subcontinuum of X such that $\emptyset \neq L \subset f^{-1}(y) \setminus A$, $\emptyset \neq C \subset f^{-1}(y) \cap A$. If $A \subset f^{-1}(y)$, then A must be contained in a component of $f^{-1}(y)$, so $C = A$. This is absurd, so $A \not\subseteq f^{-1}(y)$. We have obtained a subcontinuum A of X such that $f^{-1}(y) \cap A \neq \emptyset$, $f^{-1}(y) \not\subseteq A$ and $A \not\subseteq f^{-1}(y)$. It follows from Claim 1 that f_2 is not hereditarily confluent; that is a contradiction. This shows that $f^{-1}(y)$ must be connected for each $y \in Y$ and therefore f is monotone.

Finally, Claim 1 also implies that $f^{-1}(y)$ is a terminal subcontinuum of X for every $y \in Y$. □

Definition 3.43. A continuum K is decomposable if it is the union of two proper subcontinua.

Theorem 3.44. *If $f_2 \colon \mathcal{F}_2(X) \to \mathcal{F}_2(Y)$ is hereditarily confluent then for every decomposable subcontinuum K of Y and for every $y \in Y \setminus K$ the set $f^{-1}(y)$ is degenerate.*

Proof. Let $K \subset Y$ be a decomposable subcontinuum. Let A and B be proper subcontinua of K such that $K = A \cup B$ and let $y \in Y \setminus K$. By Theorem 3.42, $f^{-1}(y)$, $f^{-1}(A)$ and $f^{-1}(B)$ are subcontinua of X.

Suppose that $f^{-1}(y)$ contains two points $x_1 \neq x_2$. Let $b \in f^{-1}(B) \setminus f^{-1}(A)$. Let $\mathcal{A} = \big\{\{x_1, x\} : x \in f^{-1}(K)\big\}$, $\mathcal{B} = \{\{b, x\} : x \in f^{-1}(y)\big\}$, $\mathcal{C} = \big\{\{x_2, x\} : x \in f^{-1}(B)\big\}$ and $\mathcal{M} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$. Since $\{x_1, b\} \in \mathcal{A} \cap \mathcal{B}$ and $\{b, x_2\} \in \mathcal{B} \cap \mathcal{C}$, we have $\mathcal{M}$ is a subcontinuum of $\mathcal{F}_2(X)$.

We will show that $f_2|_{\mathcal{M}} \colon \mathcal{M} \to f_2(\mathcal{M})$ is not confluent. Let $\mathcal{K} = \{\{y, a\} : a \in A\}$. Notice that $\mathcal{K} \subset f_2(\mathcal{A})$, so $\mathcal{K}$ is a subcontinuum of $f_2(\mathcal{M})$. Since $b \notin f^{-1}(A) \cup f^{-1}(y)$, $f_2^{-1}(\mathcal{K}) \cap \mathcal{B} = \emptyset$. Then $(f_2|_{\mathcal{M}})^{-1}(\mathcal{K}) = (f_2^{-1}(\mathcal{K}) \cap \mathcal{A}) \cup (f_2^{-1}(\mathcal{K}) \cap \mathcal{C})$. and the sets $f_2^{-1}(\mathcal{K}) \cap \mathcal{A}$ and $f_2^{-1}(\mathcal{K}) \cap \mathcal{C}$ are closed and disjoint. Clearly $f_2^{-1}(\mathcal{K}) \cap \mathcal{A} \neq \emptyset$ and since K is a continuum, $A \cap B \neq \emptyset$. This implies that $f_2^{-1}(\mathcal{K}) \cap \mathcal{C} \neq \emptyset$. Then there exists a component $\mathcal{D}$ of $(f_2|_{\mathcal{M}})^{-1}(\mathcal{K})$ such that $\mathcal{D} \subset f_2^{-1}(\mathcal{K}) \cap \mathcal{C}$. Thus

$$\begin{aligned} f_2(\mathcal{D}) &\subset f_2(f_2^{-1}(\mathcal{K}) \cap \mathcal{C}) = f_2(\mathcal{C}) \cap \mathcal{K} = \{\{y, x\} : x \in B\} \cap \mathcal{K} \\ &= \{\{y, x\} : x \in A \cap B\} \subsetneq \mathcal{K} \end{aligned}$$

and therefore $f_2|_{\mathcal{M}} : \mathcal{M} \to f_2(\mathcal{M})$ is not confluent. This contradiction proves that $f^{-1}(y)$ is a one-point set. □

Problem 3.45. Suppose that $f_2 \colon \mathcal{F}_2(X) \to \mathcal{F}_2(Y)$ is hereditarily confluent (weakly confluent) and Y is nondegenerate, must f be a homeomorphism?

3.9. **Refinable Mappings.**

Definition 3.46. Let X and Y be continua and let d be a metric for Y. Given continuous functions $f, g \colon X \to Y$, we define $dsup(f, g) = \max\{d(f(x), g(x)) : x \in X\}$.

The following lemma is immediate.

Lemma 3.47. *Let $f, g \colon X \to Y$ be a pair of continuous functions between continua, then for every $n \in \mathbb{N}$, $\mathrm{dsup}(f, g) < \varepsilon$ if and only if $\mathrm{dsup}(f_n, g_n) < \varepsilon$.*

Definition 3.48. A mapping $f \colon X \to Y$ is refinable if for every $\varepsilon > 0$, there exists a mapping $g \colon X \to Y$ such that $\mathrm{dsup}(f, g) < \varepsilon$ and $\mathrm{diam}(g^{-1}(y)) < \varepsilon$ for every $y \in Y$.

Theorem 3.49. *If $f \colon X \to Y$ is refinable, then $f_n \colon \mathcal{F}_n(X) \to \mathcal{F}_n(Y)$ is refinable for every $n \in \mathbb{N}$.*

Proof. Let $\varepsilon > 0$. Since f is refinable, there exists a mapping $g\colon X \to Y$ such that $\text{dsup}(f,g) < \varepsilon$ and $\text{diam}(g^{-1}(y)) < \varepsilon$ for every $y \in Y$. By Lemma 3.47, $\text{dsup}(f_n, g_n) < \varepsilon$. Let $A = \{y_1, \ldots, y_m\} \in \mathcal{F}_n(Y)$ and take $B, C \in g_n^{-1}(A)$. Let $c \in C$ and $i \in \{1, \ldots, m\}$ be such that $g(c) = y_i$. Since $g_n(B) = A$, there exists $b \in B$ such that $g(b) = y_i$. Since $\text{diam}(g^{-1}(y_i)) < \varepsilon$, $d(c,b) < \varepsilon$. Therefore $C \subset N_\varepsilon(B)$, analogously $B \subset N_\varepsilon(C)$, then $H(B,C) < \varepsilon$. This shows that $\text{diam}(g_n^{-1}(A)) < \varepsilon$. Therefore, f_n is refinable. □

Problem 3.50. Is it true that if $f_n\colon \mathcal{F}_n(X) \to \mathcal{F}_n(Y)$ is refinable for some $n \geq 2$, then $f\colon X \to Y$ is refinable?

4. Dynamical Properties

In this section all the functions are of the form $f\colon X \to X$ and they are continuous but not necessarily surjective, also, X always denotes a continuum. For each $k \in \{0, 1, \ldots\}$ we define, inductively, f^k as $f^0 = id_X$ and $f^k = f \circ f^{k-1}$.

4.1. **Transitivity, Mixing properties and Chaos.**

Definition 4.1. A continuous function $f\colon X \to X$ is:

(a) transitive if for every pair of nonempty open subsets U and V of X, there exists $k \in \mathbb{N}$ such that $f^k(U) \cap V \neq \emptyset$;

(b) mixing if for every pair of nonempty open subsets U and V of X there exists $N \in \mathbb{N}$ such that $f^k(U) \cap V \neq \emptyset$ for every $k \geq N$;

(c) weakly mixing if for all nonempty open subsets U_1, U_2, V_1 and V_2 of X there exists $k \in \mathbb{N}$ such that $f^k(U_i) \cap V_i \neq \emptyset$ for each $i \in \{1, 2\}$.

Lemma 4.2. *The family* $\{\langle U_1, \ldots, U_n\rangle_n \subset \mathcal{F}_n(X) : U_1, \ldots, U_n$ *are open in* $X\}$ *is a basis for the topology of* $\mathcal{F}_n(X)$.

Proof. We know that the family $\mathcal{B} = \{\langle U_1, \ldots, U_m\rangle_n \subset \mathcal{F}_n(X) :$ $m \in \mathbb{N}$ and $U_1, \ldots, U_m$ are open in $X\}$ is a basis for $\mathcal{F}_n(X)$. Then it is enough to show that for every $\langle U_1, \ldots, U_m\rangle_n$ in that family, and $A \in \langle U_1, \ldots, U_m\rangle_n$, there exist n open subsets of X, say $V_1, \ldots, V_n$, such that $A \in \langle V_1, \ldots, V_n\rangle_n \subset \langle U_1, \ldots, U_m\rangle_n$.

Case 1. $m \leq n$. In this case, we take $V_i = U_i$ for each $i \in \{1, \ldots, m\}$ and $V_i = U_m$ for each $i \in \{m+1, \ldots, n\}$. Then $\langle V_1, \ldots, V_n\rangle_n = \langle U_1, \ldots, U_m\rangle_n$ and we are done.

Case 2. $m > n$. Let $A = \{x_1, \ldots, x_k\}$. For each $i \in \{1, \ldots, k\}$, let $W_i = \bigcap\{U_j : x_i \in U_j\}$. It is easy to prove that $A \in \langle W_1, \ldots, W_k\rangle_n \subset \langle U_1, \ldots, U_m\rangle_n$. Clearly $x_i \in W_i$, for each $i \in \{1, \ldots, k\}$, so $A \in \langle W_1, \ldots, W_k\rangle_n$.

Since $A \in \mathcal{F}_n(X)$, we know that $k \leq n$ and then it follows from Case 1 that there exist open subsets $V_1, \ldots, V_n$ of X such that $A \in \langle V_1, \ldots, V_n\rangle_n \subset \langle W_1, \ldots, W_k\rangle_n \subset \langle U_1, \ldots, U_m\rangle_n$. This concludes the proof of the lemma. □

Theorem 4.3. *The following statements for a continuous function* $f\colon X \to X$ *are equivalent:*

a) *f is mixing,*
b) *f_n is mixing for some $n \in \mathbb{N}$,*
c) *f_n is mixing for every $n \in \mathbb{N}$.*

Proof. a) ⇒ c). Suppose that f is mixing and let $n \in \mathbb{N}$. Let $\langle U_1, \ldots, U_n\rangle_n$ and $\langle V_1, \ldots, V_n\rangle_n$ be two basic nonempty open subsets of $\mathcal{F}_n(X)$ (Lemma 4.2). For each $i \in \{1, \ldots, n\}$ there exists $N_i \in \mathbb{N}$ such that $f^k(U_i) \cap V_i \neq \emptyset$ for each $k \geq N_i$. Take $N = \max\{N_1, \ldots, N_n\}$, then $f^k(U_i) \cap V_i \neq \emptyset$ for every $k \geq N$ and every $i \in \{1, \ldots, n\}$. Given $k \geq N$ and $i \in \{1, \ldots, n\}$, we can pick a point $x_i \in U_i \cap f^{-k}(V_i)$. Let $A_k = \{x_1, \ldots, x_n\}$. Clearly $A_k \in \langle U_1, \ldots, U_n\rangle_n$ and $(f_n)^k(A_k) \in \langle V_1, \ldots, V_n\rangle_n$. Hence $(f_n)^k(\langle U_1, \ldots, U_n\rangle_n) \cap \langle V_1, \ldots, V_n\rangle_n \neq \emptyset$ for every $k \geq N$. Therefore f_n is mixing.

b) ⇒ a). Suppose that f_n is mixing for some $n \in \mathbb{N}$. Let U and V be two nonempty open subsets of X. Then $\langle U\rangle_n$ and $\langle V\rangle_n$ are two nonempty open subsets of $\mathcal{F}_n(X)$. Since f_n is mixing, there exists $N \in \mathbb{N}$ such that $(f_n)^k(\langle U\rangle_n) \cap \langle V\rangle_n \neq \emptyset$ for every $k \geq N$. For each $k \geq N$, let $A_k \in \langle U\rangle_n$ be such that $(f_n)^k(A_k) \in \langle V\rangle_n$ and let $x_k \in A_k \subset U$. Since $f^k(x_k) \in (f_n)^k(A_k) \in \langle V\rangle_n$, $f^k(x_k) \in V$. Therefore $f^k(U) \cap V \neq \emptyset$ for each $k \geq N$. This shows that f is mixing. □

The following theorem is well known (see for example [1]). The equivalence a) ⟺ b) is due to H. Furstenberg.

Theorem 4.4. *The following statements for a continuous function* $f\colon X \to X$ *are equivalent:*

a) *f is weakly mixing.*

b) *For each* $m \geq 2$ *and every nonempty open subsets* $U_1, \ldots, U_m$, $V_1, \ldots, V_m$ *of* X*, there exists* $k \in \mathbb{N}$ *such that* $f^k(U_i) \cap V_i \neq \emptyset$ *for each* $i \in \{1, \ldots, m\}$.

c) *For every nonempty open subsets* U, V_1, V_2 *of* X*, there exists* $k \in \mathbb{N}$ *such that* $f^k(U) \cap V_i \neq \emptyset$ *for each* $i \in \{1, 2\}$.

Theorem 4.5 (compare with Theorem 2 of [2])**.** *The following statements for a continuous function* $f\colon X \to X$ *are equivalent:*

a) f *is weakly mixing.*

b) f_n *is weakly mixing for each* $n \in \mathbb{N}$.

c) f_n *is transitive for each* $n \in \mathbb{N}$.

d) f_n *is weakly mixing for some* $n \geq 2$.

e) f_n *is transitive for some* $n \geq 2$.

Proof. a) $\Rightarrow$ b). Suppose f is weakly mixing and let $n \in \mathbb{N}$. Let $\mathcal{U}_1 = \left\langle U_1^1, \ldots, U_n^1 \right\rangle_n$, $\mathcal{U}_2 = \left\langle U_1^2, \ldots, U_n^2 \right\rangle_n$, $\mathcal{V}_1 = \left\langle V_1^1, \ldots, V_n^1 \right\rangle_n$ and $\mathcal{V}_2 = \left\langle V_1^2, \ldots, V_n^2 \right\rangle_n$ be nonempty basic open subsets of $\mathcal{F}_n(X)$ (Lema 4.2). Since f is weakly mixing, by Theorem 4.4 there exists $k \in \mathbb{N}$ such that $f^k(U_i^j) \cap V_i^j \neq \emptyset$ for every $i \in \{1, \ldots, n\}$ and $j \in \{1, 2\}$. Given $i \in \{1, \ldots, n\}$ and $j \in \{1, 2\}$ we choose a point $x_i^j \in f^{-k}(V_i^j) \cap U_i^j$. Let $A_1 = \left\{x_1^1, \ldots, x_n^1\right\}$ and $A_2 = \left\{x_1^2, \ldots, x_n^2\right\}$. Clearly $A_j \in \mathcal{U}_j$ and $(f_n)^k(A_j) \in \mathcal{V}_j$, for each $j \in \{1, 2\}$. Hence $(f_n)^k(\mathcal{U}_j) \cap \mathcal{V}_j \neq \emptyset$ for each $j \in \{1, 2\}$. This shows that f_n is weakly mixing.

The implications c) $\Rightarrow$ e) and b) $\Rightarrow$ d) are obvious; b) $\Rightarrow$ c) and d) $\Rightarrow$ e) follow directly from the definitions.

e) $\Rightarrow$ a). We use Theorem 4.4. Let U, V_1 and V_2 be nonempty open subsets of X. Let $\mathcal{U} = \langle U \rangle_n$ and $\mathcal{V} = \langle V_1, V_2 \rangle_n$. Since $n \geq 2$, $\mathcal{V} \neq \emptyset$. Since f_n is transitive, there exists $k \in \mathbb{N}$ such that $(f_n)^k(\mathcal{U}) \cap \mathcal{V} \neq \emptyset$. Then there exists $A \in \mathcal{U}$ such that $(f_n)^k(A) \in \mathcal{V}$, then $f^k(A) \cap V_1$ and $f^k(A) \cap V_1$ are both nonempty. Since $A \subset U$ it follows that $f^k(U) \cap V_i \neq \emptyset$ for each $i \in \{1, 2\}$. This shows that f is weakly mixing and finishes the proof of the theorem. □

Notation 4.6. Given a continuous function $f\colon X \to X$, $\mathrm{per}(f)$ denotes the set of periodic points of f.

Lemma 4.7 (compare with Lemma 1 of [2])**.** *For each continuous function* $f\colon X \to X$ *and each* $n \in \mathbb{N}$*, the set* $\mathrm{per}(f_n)$ *is dense in* $\mathcal{F}_n(X)$ *if and only if* $\mathrm{per}(f)$ *is dense in* X.

Proof. (Necessity). Let $n \in \mathbb{N}$. Suppose that $\mathrm{per}(f_n)$ is dense in $\mathcal{F}_n(X)$. Let $x \in X$ and $\varepsilon > 0$. Since $\mathrm{per}(f_n)$ is dense in $\mathcal{F}_n(X)$, there exists $A \in \mathrm{per}(f_n) \cap B_H(\varepsilon, \{x\})$. Since $A \in \mathrm{per}(f_n)$, there exists $k \in \mathbb{N}$ such that $(f_n)^k(A) = A$. This means that $f^k|_A\colon A \to A$ is a permutation of the elements of A. Since A is finite and permutations of finite sets have finite order, there exists $r \in \mathbb{N}$ such that $\left(f^k\right)^r|_A = id_A$. Then $A \subset \mathrm{per}(f)$. Since $A \in B_H(\varepsilon, \{x\})$, we have that $A \subset B(\varepsilon, x)$, this shows that $B(\varepsilon, x) \cap \mathrm{per}(f) \neq \emptyset$ and $\mathrm{per}(f)$ is dense in X.

(Sufficiency). Suppose $\mathrm{per}(f)$ is dense in X. Let $n \in \mathbb{N}$, $A = \{x_1, \ldots, x_m\} \in \mathcal{F}_n(X)$ and $\varepsilon > 0$. Since $\mathrm{per}(f)$ is dense, for each $i \in \{1, \ldots, m\}$, there exist $p_i \in \mathrm{per}(f) \cap B(\varepsilon, x_i)$ and $n_i \in \mathbb{N}$ such that $f^{n_i}(p_i) = p_i$. Take $k = n_1 \cdots n_m$, then $f^k(p_i) = p_i$ for each $i \in \{1, \ldots, m\}$. Let $B = \{p_1, \ldots, p_m\}$. By construction, $(f_n)^k(B) = \left\{f^k(p_1), \ldots, f^k(p_m)\right\} = B$ so $B \in \mathrm{per}(f_n)$. Since for every $i \in \{1, \ldots, m\}$, $p_i \in B(\varepsilon, x_i)$, we have $B \in B_H(\varepsilon, A)$. This shows that $B \in \mathrm{per}(f_n) \cap B_H(\varepsilon, A)$. Therefore, $\mathrm{per}(f_n)$ is dense in $\mathcal{F}_n(X)$. □

Definition 4.8. A continuous function $f\colon X \to X$ is chaotic if it is transitive and $\mathrm{per}(f)$ is dense in X.

Theorem 4.9. *The following statements for a continuous function $f\colon X \to X$ are equivalent:*

a) *f is chaotic and weakly mixing.*
b) *f_n is chaotic for some $n \geq 2$.*
c) *f_n is chaotic for each $n \geq 2$.*

Proof. It follows directly from Lemma 4.7 and Theorem 4.5. □

Example 4.10. There exists a continuum X and a continuous chaotic function $f\colon X \to X$ such that $f_n\colon \mathcal{F}_n(X) \to \mathcal{F}_n(X)$ is not chaotic for any $n \geq 2$.

Let $X = [0, 1]$ and $f\colon [0, 1] \to [0, 1]$ be defined as follows:

$$f(x) = \begin{cases} 2x + \frac{1}{2}, & \text{if } x \in \left[0, \frac{1}{4}\right], \\ \frac{3}{2} - 2x, & \text{if } x \in \left[\frac{1}{4}, \frac{1}{2}\right], \\ 1 - x, & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

By Theorem 4.9 it is enough to show that f is chaotic but not weakly mixing.

Claim 1. f is not weakly mixing.

In order to prove Claim 1, let $U_1 = U_2 = V_1 = (0, \frac{1}{2})$ and $V_2 = (\frac{1}{2}, 1)$. Let $W_1 = [0, \frac{1}{2}]$ and $W_2 = [\frac{1}{2}, 1]$. Clearly, $f(W_1) = W_2$ and $f(W_2) = W_1$. Thus

$$f^k(W_1) = \begin{cases} W_1, & \text{if } k \text{ is even,} \\ W_2, & \text{if } k \text{ is odd.} \end{cases}$$

Suppose that there exists $k \in \mathbb{N}$ such that $f^k(U_1) \cap V_1 \neq \emptyset$ and $f^k(U_2) \cap V_2 \neq \emptyset$. Take points $x \in U_1$ and $y \in U_2$ such that $f^k(x) \in V_1$ and $f^k(y) \in V_2$. If k is odd, then $f^k(x) \in f^k(W_1) = W_2$, so $f^k(x) \in W_2 \cap V_1 = \emptyset$, a contradiction. If k is even, $f^k(y) \in f^k(W_1) = W_1$, so $f^k(y) \in W_1 \cap V_2 = \emptyset$, a contradiction. This proves that f is not weakly mixing.

Claim 2. f is chaotic.

In order to show Claim 2, lets take a look at f^2

$$f^2(x) = \begin{cases} \frac{1}{2} - 2x, & \text{if } x \in \left[0, \frac{1}{4}\right], \\ 2x - \frac{1}{2}, & \text{if } x \in \left[\frac{1}{4}, \frac{3}{4}\right], \\ \frac{5}{2} - 2x, & \text{if } x \in \left[\frac{3}{4}, 1\right]. \end{cases}$$

Note that we can consider $f^2|_{[0,\frac{1}{2}]} \colon \left[0, \frac{1}{2}\right] \to \left[0, \frac{1}{2}\right]$ and $f^2|_{[\frac{1}{2},1]} \colon \left[\frac{1}{2}, 1\right] \to \left[\frac{1}{2}, 1\right]$ as independent dynamical systems. It is easy to see that both of this mappings are topologically conjugate ([12]) to the map known as the"tent map" which was described in Example 3.13 in the previous section. It is known that this mapping is chaotic ([12]). It easily follows from this that f is chaotic.

4.2. **Specification and the P property.**

Definition 4.11. A continuous function $f\colon X \to X$ has specification if for every $\varepsilon > 0$ there exists M_ε such that for each $k \geq 2$, for every $x_1, \dots, x_k \in X$ and for every nonnegative integers $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$ such that $a_i - b_{i-1} \geq M_\varepsilon$, there exists $z \in X$ such that for every $i \in \{1, \dots, k\}$ and every $m \in \{a_i, \dots, b_i\}$ we have that $d(f^m(z), f^m(x_i)) < \varepsilon$.

Definition 4.12. A continuous function $f\colon X \to X$ has the P property if for every pair of nonempy open subsets U_0, U_1 of X there is an $N \in \mathbb{N}$ such that for each $k \geq 2$ and each $s = (s(1), \dots, s(k)) \in \{0,1\}^k$ there exists $x \in X$ such that $x \in U_{s(1)}, f^N(x) \in U_{s(2)}, f^{2N}(x) \in U_{s(3)}, \dots, f^{(k-1)N}(x) \in U_{s(k)}$.

The relation between a mapping having one of these properties and the induced mapping to some hyperspaces having it was studied by Rongbao Gu and Wenjing Guo in [14]. The proofs of Theorems 4.1 and 4.2 of [14] can be very easily adapted to prove the following theorems.

Theorem 4.13. *The following statements for a continuous function $f\colon X \to X$ are equivalent:*

a) *f has specification,*
b) *f_n has specification for some $n \geq 2$,*
c) *f_n has specification for each $n \geq 2$.*

Theorem 4.14. *The following statements for a continuous function $f\colon X \to X$ are equivalent*

a) *f has the P property,*
b) *f_n has the P property for some $n \geq 2$,*
c) *f_n has the P property for each $n \geq 2$.*

4.3. **Expansive Homeomorphisms.** Given a homeomorphism $f\colon X \to X$ we can define its negative iterations as $f^{-n} = \left(f^{-1}\right)^n$ for each $n \in \mathbb{N}$, where f^{-1} is the inverse of f.

Definition 4.15. A homeomorphism $f\colon X \to X$ is expansive if there exists $c > 0$ such that for every two differents points $x, y \in X$ there is $k \in \mathbb{Z}$ such that $d(f^k(x), f^k(y)) > c$. In such case we call c an expansion constant for f.

Theorem 4.16. *If $f_n\colon \mathcal{F}_n(X) \to \mathcal{F}_n(X)$ is an expansive homeomorphism for some $n \in \mathbb{N}$, then $f\colon X \to X$ is an expansive homeomorphism.*

Proof. Let $n \in \mathbb{N}$ be such that f_n is an expansive homeomorphism. By Theorem 3.1, f is a homeomorphism. Let $c > 0$ be an expansion constant for f_n. We will show that c is also an expansion constant for f.

Given $x \neq y \in X$ we know that $\{x\} \neq \{y\} \in \mathcal{F}_n(X)$ and then there exists a $k \in \mathbb{Z}$ such that

$$c < H(\left(f_n\right)^k(\{x\}), \left(f_n\right)^k(\{y\})) = d(f^k(x), f^k(y)).$$

Thus c is an expansion constant for f. Therefore f is an expansive homeomorphism. □

4.3.1. *Inverse Limits and Shift Homeomorphisms.*

Definition 4.17. An inverse sequence is a sequence of pairs $\{(X_k, f_k)\}_{k=1}^{\infty}$, where for every $k \in \mathbb{N}$, X_k is a continuum and $f_k\colon X_{k+1} \to X_k$ is a continuous function.

Definition 4.18. Given an inverse sequence $\{(X_k, f_k)\}_{k=1}^{\infty}$ we define its inverse limit as:

$$\varprojlim \{(X_k, f_k)\} = \{(x_k)_{k=1}^{\infty} \in \prod_{k=1}^{\infty} X_k : f_k(x_{k+1}) = x_k \text{ for every } k \in \mathbb{N}\}$$

The metric in $\prod\limits_{k=1}^{\infty} X_k$ is given by $\rho((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = \sum\limits_{k=1}^{\infty} \frac{d_k(x_k, y_k)}{2^k}$, where d_k is a metric for X_k bounded by a number M. It is known that the inverse limit of each inverse sequence $\{(X_k, f_k)\}_{k=1}^{\infty}$, where each X_k is a continuum, is a continuum.

Notation 4.19. Given a continuous function $f\colon X \to X$ one can consider the inverse sequence $\{X_k, f_k\}_{k=1}^{\infty}$, where $X_k = X$ and $f_k = f$ for every $k \in \mathbb{N}$. The inverse limit of this particular inverse sequence is simply denoted by $\varprojlim(X, f)$.

Definition 4.20. Given $Y = \varprojlim(X, f)$ we define a function $\widehat{f}\colon Y \to Y$ as $\widehat{f}((x_1, x_2, \ldots)) = (f(x_1), f(x_2), \ldots) = (f(x_1), x_1, x_2, \ldots)$.

Notice that $(f(x_1), x_1, x_2, \ldots)$ belongs to Y, so $\widehat{f}$ is well defined.

The following theorem is immediate.

Theorem 4.21. *Given a continuous function $f\colon X \to X$, if $Y = \varprojlim(X, f)$, then the function $\widehat{f}\colon Y \to Y$ is a homeomorphism, called the shift homeomorphism, and the inverse of $\widehat{f}$ is given by $(\widehat{f})^{-1}$ $((x_1, x_2, \ldots)) = ((x_2, x_3, \ldots))$.*

Example 4.22. There exists a continuum X and an expansive homeomorphism $g\colon X \to X$ such that $g_n\colon \mathcal{F}_n(X) \to \mathcal{F}_n(X)$ is not an expansive homeomorphism for any $n \geq 2$.

Let $n \geq 2$. Consider the unit circle $\mathbb{S}^1$, centered at the origin in the complex plane $\mathbb{C}$. We give to $\mathbb{S}^1$ the metric of the shorter arc, that is, given $z, w \in \mathbb{S}^1$ we define their distance to be $d(z, w) =$ the length of the shortest arc containing z and w.

This is clearly a metric which is compatible with the norm metric inherited from $\mathbb{C}$. Let $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ be defined as $f(z) = z^2$.

Let $X = \varprojlim(\mathbb{S}^1, f)$. It is known that $\widehat{f}\colon X \to X$ is an expansive homeomorphism (see for example [31]). We show that $\widehat{f_n}\colon \mathcal{F}_n(X) \to \mathcal{F}_n(X)$ is not an expansive homeomorphism for any $n \geq 2$.

Claim 1. Given $z, w \in \mathbb{S}^1$ there exist $u, v \in \mathbb{S}^1$ such that $f(u) = z$, $f(v) = w$ and $d(u, v) = \frac{d(z,w)}{2}$

To prove Claim 1, we consider two cases.

Case 1. The complex number 1 is not in the interior of the shortest arc of $\mathbb{S}^1$ joining z and w.

In this case, we write $z = e^{i\alpha}$ and $w = e^{i\beta}$ with $\alpha, \beta \in [0, 2\pi)$. Then $d(z, w) = |\alpha - \beta|$. In this case, define $u = e^{i\frac{\alpha}{2}}$ and $v = e^{i\frac{\beta}{2}}$.

Case 2. The complex number 1 is in the interior of the shortest arc of $\mathbb{S}^1$ joining z and w.

In this case, the complex number -1 is not in the shortest arc joining z and w. Then we write $z = e^{i\alpha}$ and $w = e^{i\beta}$ with $\alpha, \beta \in (-\pi, \pi)$. In this case, let $u = e^{i\frac{\alpha}{2}}$ and $v = e^{i\frac{\beta}{2}}$.

The following claim is easy to prove.

Claim 2. Let $z, w \in \mathbb{S}^1$ be such that $d(z, w) < \frac{\pi}{2}$ then $d(f(z), f(w)) = 2d(z, w)$.

Claim 3. The homeomorphism $\widehat{f_n} : \mathcal{F}_n(X) \to \mathcal{F}_n(X)$ is not expansive.

Proof of Claim 3. Let $0 < c < \frac{\pi}{4}$. We will show that c is not an expansion constant for f_n. Let $p \in \mathbb{N}$ and $z_1, w_1 \in \mathbb{S}^1$ be such that $\frac{\pi}{2^p} < c$ and $0 < d(z_1, w_1) < \frac{c}{2^p}$. Applying Claim 1, succesively, it is possible to construct two sequences $\{z_k\}_{k=1}^{\infty}$ and $\{w_k\}_{k=1}^{\infty}$ in $\mathbb{S}^1$ such that $f(z_{k+1}) = z_k$, $f(w_{k+1}) = w_k$ and $d(z_{k+1}, w_{k+1}) = \frac{d(z_1,w_1)}{2^k}$ for every $k \in \mathbb{N}$.

Notice that the points $z = (z_1, z_2, \ldots)$ and $w = (w_1, w_2, \ldots)$ belong to X.

First, we will show that $\rho(\widehat{f}^m(z), \widehat{f}^m(w)) < c$ for every $m \in \{\ldots, p-1, p\}$.

By the definition of $(\widehat{f})^{-1}$ we have that, for each $m \in \{0, 1, \ldots\}$,

$$\begin{aligned}\rho(\widehat{f}^{-m}(z), \widehat{f}^{-m}(w)) &= \rho((z_{m+1}, z_{m+2}, \ldots), (w_{m+1}, w_{m+2}, \ldots)) \\ &= \sum_{k=1}^{\infty} \frac{d(z_{m+k}, w_{m+k})}{2^k} = \sum_{k=1}^{\infty} \frac{\frac{d(z_1,w_1)}{2^{m+k}}}{2^k} \\ &= \sum_{k=1}^{\infty} \frac{d(z_1, w_1)}{2^{2k+m}} \leq \sum_{k=1}^{\infty} \frac{d(z_1, w_1)}{2^k} \\ &= d(z_1, w_1) < \frac{c}{2^p} < c\,.\end{aligned}$$

Hence, we have shown that $\rho(\widehat{f}^m(z), \widehat{f}^m(w)) < c$ for each $m \in \{\ldots, -2, -1, 0\}$.

Next, by induction, we will show that for each $m \in \{0, 1, \ldots, p\}$, $d(f^m(z_1), f^m(w_1)) = 2^m d(z_1, w_1)$. The case $m = 0$ is immediate. Suppose that $m \in \{0, 1, \ldots, p-1\}$ is such that $d(f^m(z_1), f^m(w_1)) = 2^m d(z_1, w_1)$. Since $m < p$ and $2^p d(z_1, w_1) < c < \frac{\pi}{4}$, $d(f^m(z_1), f^m(w_1)) < \frac{\pi}{4}$. Claim 2 implies that $d(f^{m+1}(z_1), f^{m+1}(w_1)) = 2d(f^m(z_1), f^m(w_1)) = 2^{m+1} d(z_1, w_1)$. This concludes the induction.

Finally, again by induction, we will prove that $\rho(\widehat{f}^m(z), \widehat{f}^m(w)) \leq 2^m d(z_1, w_1)$ for each $m \in \{0, 1, \ldots, p\}$. The case $m = 0$ was showed in the chain of inequalities above. Therefore, suppose that if $m \in \{1, \ldots, p\}$, then $\rho(\widehat{f}^{m-1}(z), \widehat{f}^{m-1}(w)) \leq 2^{m-1} d(z_1, w_1)$. By the definitions of ρ and $\widehat{f}$ we have

$$\begin{gathered}\rho(\widehat{f}^m(z), \widehat{f}^m(w)) = \\ \rho((f^m(z_1), \ldots, f^0(z_1), z_2, \ldots), (f^m(w_1), \ldots f^0(w_1), w_2, \ldots)) \\ = \sum_{k=0}^{m-1} \tfrac{d(f^{m-k}(z_1), f^{m-k}(w_1))}{2^{k+1}} + \sum_{k=1}^{\infty} \tfrac{d(z_k, w_k)}{2^{m+k}} \\ = \tfrac{d(f^m(z_1), f^m(w_1))}{2} + \sum_{k=1}^{m-1} \tfrac{d(f^{m-k}(z_1), f^{m-k}(w_1))}{2^{k+1}} + \sum_{k=1}^{\infty} \tfrac{d(z_k, w_k)}{2^{m+k}} \\ = \tfrac{d(f^m(z_1), f^m(w_1))}{2} + \tfrac{1}{2} \sum_{k=0}^{m-2} \tfrac{d(f^{(m-1)-k}(z_1), f^{(m-1)-k}(w_1))}{2^{k+1}} + \tfrac{1}{2} \sum_{k=1}^{\infty} \tfrac{d(z_k, w_k)}{2^{(m-1)+k}} \\ = \tfrac{d(f^m(z_1), f^m(w_1))}{2} + \tfrac{\rho(\widehat{f}^{m-1}(z), \widehat{f}^{m-1}(w))}{2} \leq \tfrac{2^m d(z_1, w_1)}{2} + \tfrac{2^{m-1} d(z_1, w_1)}{2} \\ \leq 2^m d(z_1, w_1).\end{gathered}$$

We have shown that $\rho(\widehat{f}^m(z),\widehat{f}^m(w)) \leq 2^m d(z_1,w_1)$. This concludes the induction and shows that $\rho(\widehat{f}^m(z),\widehat{f}^m(w)) \leq 2^m d(z_1,w_1)$ for all $m \in \{0,1,\ldots,p\}$. By the choice of p, $2^p d(z_1,w_1) < c$, it follows that $\rho(\widehat{f}^m(z),\widehat{f}^m(w)) < c$ for each $m \in \{0,1,\ldots,p\}$.

This completes the proof that $\rho(\widehat{f}^m(z),\widehat{f}^m(w)) < c$ for all $m \in \{\ldots,p-1,p\}$.

Let $z'=(e^{\pi i}z_1, e^{\frac{\pi}{2}i}z_2, e^{\frac{\pi}{4}i}z_3,\ldots)$ and $w'=(e^{\pi i}w_1, e^{\frac{\pi}{2}i}w_2, e^{\frac{\pi}{4}i}w_3,\ldots)$. Clearly $z',w' \in X$. It is clear that rotations preserve the given metric on $\mathbb{S}^1$. Therefore, $\rho(\widehat{f}^m(z'),\widehat{f}^m(w')) < c$ for each $m \in \{\ldots,-1,0,1,\ldots,p\}$.

Let $A = \{z,w'\}, B = \{w,z'\} \in \mathcal{F}_n(Y)$. We are going to show that, for each $m \in \mathbb{Z}$, $H((\widehat{f_n})^m(A),(\widehat{f_n})^m(B)) < c$. We have seen that $\rho(\widehat{f}^m(z),\widehat{f}^m(w))$ and $\rho(\widehat{f}^m(z'),\widehat{f}^m(w')) < c$, for each $m \in \{\ldots,p-1,p\}$. This implies that $H((\widehat{f_n})^m(A),(\widehat{f_n})^m(B)) < c$, for each $m \in \{\ldots,p-1,p\}$.

Let $m \in \{p+1,p+2,\ldots\}$. We will show that $H((\widehat{f_n})^m(A),(\widehat{f_n})^m(B)) < c$. Since $f(e^{\pi i}z_1) = e^{2\pi i}(z_1)^2 = (z_1)^2 = f(z_1)$, we have that the first m coordinates of $\widehat{f}^m(z)$ coincide with the first m coordinates of $\widehat{f}^m(z')$. By definition, the metric d is bounded by π and by the choice of p, $\frac{\pi}{2^p} < c$. So

$$\begin{aligned}\rho(\widehat{f}^m(z),\widehat{f}^m(z')) &= \sum_{k=m+1}^{\infty} \frac{d(z_{k-m}, e^{\frac{\pi}{2^{k-m-1}}i}z_{k-m})}{2^k} \\ &\leq \sum_{k=m+1}^{\infty} \frac{\pi}{2^k} = \frac{\pi}{2^m} < \frac{\pi}{2^p} < c.\end{aligned}$$

Similarly, $\rho(\widehat{f}^m(w),\widehat{f}^m(w')) < c$. Therefore, $H((\widehat{f_n})^m(A),(\widehat{f_n})^m(B)) < c$.

This completes the proof that $H((\widehat{f_n})^m(A),(\widehat{f_n})^m(B)) < c$, for each $m \in \mathbb{Z}$ and that c is not an expansion constant for $\widehat{f_n}$. Since c was chosen arbitrarily, we have shown that $\widehat{f_n}$ has no expansion constant. Therefore, $\widehat{f_n}$ is not an expansive homeomorphism. $\square$

Definition 4.23. A homeomorphism $f\colon X \to X$ is continuum-wise expansive if there exists a constant $c > 0$ such that for each nondegenerate subcontinuum Y of X, there exists $k \in \mathbb{Z}$ such that $\operatorname{diam}(f^k(Y)) > c$. In such case c is a continuum-wise expansion constant for f.

Lemma 4.24. *Let $\mathcal{K}$ be a nondegenerate subcontinuum of $\mathcal{F}_n(X)$ and let D be a component of the set $\bigcup\{E : E \in \mathcal{K}\}$. Then* $\operatorname{diam}(\mathcal{K}) \geq \frac{\operatorname{diam}(D)}{2n}$.

Proof. Fix $A = \{a_1, \ldots, a_m\} \in \mathcal{K}$ and let $r < \frac{\operatorname{diam}(D)}{2n}$. For every $i \in \{1, \ldots, m\}$ let $U_i = B(r, a_i) \cap D$. Then $U_i = \emptyset$ or $\operatorname{diam}(U_i) \leq 2r$.

We show that $U_1 \cup \cdots \cup U_m \subsetneq D$. Suppose, to the contrary, that $D = U_1 \cup \cdots \cup U_m$. Let $x, y \in D$ be such that $d(x, y) = \operatorname{diam}(D)$. Let $i_0, j_0 \in \{1, \ldots, m\}$ be such that $x \in U_{i_0}$ and $y \in U_{j_0}$. Since D is connected and every U_i is open in D, there exists $\{i_0, i_1, \ldots, i_k\} \subset \{1, \ldots, m\}$ such that $i_k = j_0$, $i_p \neq i_q$ for every $p \neq q$ and $U_{i_p} \cap U_{i_{p+1}} \neq \emptyset$ for each $p \in \{0, 1, \ldots, k-1\}$. This is a chain of $k+1$ open subsets from x to y and every link has diameter at most $2r$. Then, $d(x, y) \leq 2r(k+1) \leq 2rm \leq 2rn < \operatorname{diam}(D)$; this is a contradiction. Therefore $U_1 \cup \cdots \cup U_m \subsetneq D$.

Let $b \in D \setminus (U_1 \cup \cdots \cup U_m) = D \setminus \bigcup\{B(r, a_i) : i \in \{1, \ldots, m\}\}$. This implies that $a_i \notin B(r, b)$, for any $i \in \{1, \ldots, m\}$. Let $B \in \mathcal{K}$ be such that $b \in B$. Notice that $H(A, B) \geq r$. Then $\operatorname{diam}(\mathcal{K}) \geq r$. Since r was an arbitrary number less than $\frac{\operatorname{diam}(D)}{2n}$, we conclude that $\operatorname{diam}(\mathcal{K}) \geq \frac{\operatorname{diam}(D)}{2n}$. □

Theorem 4.25. *The following statements for a continuous function $f\colon X \to X$ are equivalent:*

a) *f is a continuum-wise expansive homeomorphism,*
b) *f_n is a continuum-wise expansive homeomorphism for some $n \geq 2$,*
c) *f_n is a continuum-wise expansive homeomorphism for each $n \geq 2$.*

Proof. b) ⇒ a). Suppose that f_n is a continuum-wise expansive homeomorphism and let $c > 0$ be a continuum-wise expansion constant for f_n. We claim that c is an continuum-wise expansion constant for f. Let K be a nondegenerate subcontinuum of X.

Then $\mathcal{F}_1(K)$ is a nondegenerate subcontinuum of $\mathcal{F}_n(X)$. Therefore there exists $k \in \mathbb{Z}$ such that $\text{diam}((f_n)^k (\mathcal{F}_1(K))) > c$. Clearly $\text{diam}(f^k(K)) = \text{diam}((f_n)^k (\mathcal{F}_1(K)))$. Therefore c is a continuum-wise expansion constant for f.

a) $\Rightarrow$ c). Let $n \in \mathbb{N}$. Suppose that f is a continuum-wise expansive homeomorphism and let $c > 0$ be a continuum-wise expansion constant for f. We claim that $\frac{c}{2n}$ is a continuum-wise expansion constant for f_n. Let $\mathcal{K}$ be a nondegenerate subcontinuum of $\mathcal{F}_n(X)$. Lemma 2.1 implies that $\bigcup \{E : E \in \mathcal{K}\}$ has at most n components, but $\mathcal{K}$ has infinitely many elements, so one of those components must be nondegenerate. Let D be one nondegenerate component of $\bigcup \{E : E \in \mathcal{K}\}$. Since c is a continuum-wise expansion constant for f, there exists $k \in \mathbb{Z}$ such that $\text{diam}(f^k(D)) > c$. Given $a \in f^k(D)$ there exists $x \in D$ such that $f^k(x) = a$ and since $D \subset \bigcup \{E : E \in \mathcal{K}\}$ there is $A \in \mathcal{K}$ such that $x \in A$. Then $a = f^k(x) \in (f_n)^k (A) \in (f_n)^k (\mathcal{K})$. This shows that $f^k(D) \subset \bigcup \left\{E : E \in (f_n)^k (\mathcal{K})\right\}$. Let E_0 be the component of $\bigcup \left\{E : E \in (f_n)^k (\mathcal{K})\right\}$ that contains $f^k(D)$. Then $\text{diam}(E_0) \geq \text{diam}(f^k(D)) > c$. Lemma 4.24 implies that $\text{diam}((f_n)^k (\mathcal{K})) \geq \frac{\text{diam}(E_0)}{2n}$ and therefore $\text{diam}((f_n)^k (\mathcal{K})) > \frac{c}{2n}$. This shows that $\frac{c}{2n}$ is a continuum-wise expansion constant for f_n. Therefore, f_n is a continuum-wise expansive homeomorphism. $\square$

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Instituto de Matemáticas, Universidad Nacional Autónoma de México, Circuito Exterior, Cd. Universitaria, México 04510, D.F. México

E-mail address: illanes@matem.unam.mx
E-mail address: galoolag@gmail.com