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NONSEPARABLE COMPLETE ERDŐS SPACES AND SUBMEASURES ON UNCOUNTABLE CARDINALS

JAN J. DIJKSTRA, KIRSTEN I. S. VALKENBURG, AND DAVE VISSER

ABSTRACT. Among the applications of their topological characterization of complete Erdős space, $\mathfrak{E}_c$, Dijkstra and van Mill state one about copies of $\mathfrak{E}_c$ in Banach spaces ℓ^p and one concerning Polishable F_σ-ideals, that are generated by submeasures on ω. Inspired by a previous success in lifting the first result to so-called nonseparable complete Erdős spaces in ℓ^p_κ for arbitrary infinite cardinal numbers κ, we now consider the second result. For submeasures on κ and related ideals we find a theorem similar to the countable case. For a special class of submeasures, which contains the lower semicontinuous measures as a proper subclass, we establish a link between the related ideals and nonseparable complete Erdős spaces.

1. Introduction

Let A be an arbitrary set and consider the powerset $\mathcal{P}(A)$ with the symmetric difference '$\triangle$' as group structure. We equip $\mathcal{P}(A)$ with the standard product topology that comes with identification with 2^A.

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Definition 1.1. A *submeasure* φ on A is a function $\varphi : \mathcal{P}(A) \to [0, \infty]$ such that

(a) $\varphi(\emptyset) = 0$;
(b) $0 < \varphi(\{x\}) < \infty$ for any point $x \in A$;
(c) (*monotonicity*) $\varphi(X) \leq \varphi(Y)$ for all $X \subset Y \subset A$; and
(d) (*subadditivity*) $\varphi(X \cup Y) \leq \varphi(X) + \varphi(Y)$ for all $X, Y \subset A$.

We associate with a submeasure φ on A two ideals on A (see Definition 2.2):

$$\text{Exh}(\varphi) = \{X \subset A : \forall \varepsilon > 0,\ \exists F \subset A \text{ finite with } \varphi(X \setminus F) < \varepsilon\},$$

and

$$\text{Fin}(\varphi) = \{X \subset A : \varphi(X) < \infty\}.$$

Recall that for any space Z a function $f : Z \to \hat{\mathbb{R}}$ is called *lower semi-continuous* (*LSC*) if $\{z \in Z : f(z) > t\}$ is open in Z for every $t \in \mathbb{R}$, where $\hat{\mathbb{R}}$ denotes the compactification $[-\infty, \infty]$ of $\mathbb{R}$.

In [4], Dijkstra and van Mill use their characterization of complete Erdős space

$$\mathfrak{E}_{\rm c} = \{(x_n)_n \in \ell^1 : x_n \in \{0\} \cup \{1/m : m \in \mathbb{N}\},\ n \in \omega\}$$

to prove [4, Theorem 4.15]. In essence we now quote this theorem, with item (6) as an additional equivalent statement due to Dijkstra and Visser [7, Theorem 4.7]. The equivalence of items (2), (7) and (8) was already observed by Solecki [11, Corollary 3.3] and [12, Proposition 4.2]. For item (6), note that for any function $\varphi : Z \to [0, \infty]$ a subset $C \subset Z$ is called *φ-bounded* if there is an $M \in \mathbb{N}$ such that $\varphi(z) \leq M$ for all $z \in C$ and *φ-unbounded* otherwise. 2^ω denotes the Cantor set.

Theorem 1.2. *Let φ be an LSC submeasure on ω with* $\text{Exh}(\varphi) = \text{Fin}(\varphi)$. *Let τ_d be the topology on* $\text{Exh}(\varphi)$ *that is generated by the metric $d(X, Y) = \varphi(X \triangle Y)$ for $X, Y \in$* $\text{Exh}(\varphi)$ *and denote $\mathcal{I}_\omega = (\text{Exh}(\varphi), \tau_d)$. Then the following statements are equivalent:*

(1) *$\mathcal{I}_\omega$ is not homeomorphic to $\mathbb{Z}$, 2^ω, or $\mathbb{Z} \times 2^\omega$;*
(2) *there is no $B \subset \omega$ with* $\text{Exh}(\varphi) = \{X \subset \omega : X \cap B$ *is finite*$\}$*;*
(3) *for every $\varepsilon > 0$ we have $\varphi(\{n \in \omega : \varphi(\{n\}) \leq \varepsilon\}) = \infty$;*
(4) *there is a $B \subset \omega$ with $\varphi(B) = \infty$ and* $\lim_{n\to\infty} \varphi(\{n\} \cap B) = 0$*;*
(5) *$\mathcal{I}_\omega$ is homeomorphic to $\mathfrak{E}_{\rm c}$;*
(6) *every nonempty clopen subset of $\mathcal{I}_\omega$ is φ-unbounded;*

(7) $\operatorname{ind}\mathcal{I}_\omega > 0$; *and*
(8) $\mathcal{I}_\omega$ *is not locally compact.*

Our main aim is to extend this theorem to LSC submeasures on arbitrary infinite cardinal numbers. For uncountable cardinals, we arrive at the following theorem, in which we write $\lim_{\alpha\in\mu} x_\alpha = 0$ for a set μ with $|\mu| \geq \omega$ and real numbers x_α for every $\alpha \in \mu$, if for each $\varepsilon > 0$ the set $\{x_\alpha : |x_\alpha| \geq \varepsilon\}$ is finite. If κ is a cardinal number we denote by κ_{D} the cardinal κ equipped with the discrete topology.

Theorem 1.3. *Let φ be an LSC submeasure on some cardinal number $\kappa > \omega$ such that $\operatorname{Exh}(\varphi) = \operatorname{Fin}(\varphi)$. Let τ_d be the topology on $\operatorname{Exh}(\varphi)$ generated by the metric $d(X, Y) = \varphi(X \triangle Y)$ for $X, Y \in \operatorname{Exh}(\varphi)$ and denote $\mathcal{I}_\kappa = (\operatorname{Exh}(\varphi), \tau_d)$. Then the following statements are equivalent:*

(1) $\mathcal{I}_\kappa$ *is not homeomorphic to* κ_{D} *or* $\kappa_{\mathrm{D}} \times 2^\omega$;
(2) *there is no* $B \subset \kappa$ *with* $\operatorname{Exh}(\varphi) = \{X \subset \kappa : X \cap B \text{ is finite}\}$;
(3) *for every* $\varepsilon > 0$ *we have* $\varphi(\{\alpha \in \kappa : \varphi(\{\alpha\}) \leq \varepsilon\}) = \infty$;
(4) *there is a* $B \subset \kappa$ *with* $\varphi(B) = \infty$ *and* $\lim_{\alpha\in B} \varphi(\{\alpha\}) = 0$;
(5) $\mathcal{I}_\kappa \times \mathfrak{E}_{\mathrm{c}}$ *is homeomorphic to* $\mathcal{I}_\kappa$;
(6) *every nonempty clopen subset of* $\mathcal{I}_\kappa$ *is* φ*-unbounded;*
(7) $\operatorname{ind}\mathcal{I}_\kappa > 0$; *and*
(8) $\mathcal{I}_\kappa$ *is not locally compact.*

We prove this theorem in §4. Note that statement (5) above is weaker than its counterpart in the original Theorem 1.2. This is due to the nonseparability of the considered spaces $\mathcal{I}_\kappa$, whilst $\mathfrak{E}_{\mathrm{c}}$ is separable. However, in §5 we have our main result, Theorem 5.5, in which we show that for a special class of submeasures there is a stronger analogy with Theorem 1.2 in that we can replace statement (5) of Theorem 1.3 by the statement that $\mathcal{I}_\kappa$ is homeomorphic to $\mathfrak{E}_{\mathrm{c}} \times (\lambda_{\mathrm{D}})^\omega \times \kappa_{\mathrm{D}}$, where λ is the local weight of $\mathcal{I}_\kappa$. For this purpose we extend the definition of a Kadec submeasure introduced by Dijkstra and Visser [7] to uncountable cardinals. Some theory on Kadec submeasures is developed in §3, which we feel is of independent interest.

2. Preliminaries

A special class of submeasures is formed by the so-called measures.

Definition 2.1. Let A be an arbitrary set. A *measure* φ on A is a function $\varphi : \mathcal{P}(A) \to [0, \infty]$ such that

(a) $\varphi(\emptyset) = 0$;
(b) $0 < \varphi(\{x\}) < \infty$ for any point $x \in A$; and
(c) $\varphi(X \cup Y) = \varphi(X) + \varphi(Y)$ for any two disjoint subsets $X, Y \subset A$.

Before discussing some properties of the ideals $\mathrm{Exh}(\varphi)$ and $\mathrm{Fin}(\varphi)$ for arbitrary submeasures, first recall the following definition.

Definition 2.2. An *ideal* $\mathcal{I}$ on A is a subset of $\mathcal{P}(A)$ such that $\mathcal{I}$ contains the finite sets, $X \in \mathcal{I}$ whenever $X \subset Y \in \mathcal{I}$ and $X \cup Y \in \mathcal{I}$ whenever $X, Y \in \mathcal{I}$.

If $|A| < \omega$ then $\mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi) = \mathcal{P}(A)$ and if $|A| = \omega$, then the definitions of the ideals coincide with the usual definitions of $\mathrm{Exh}(\varphi)$ and $\mathrm{Fin}(\varphi)$ for a submeasure φ on a countable set. It is also easy to see that $\mathrm{Exh}(\varphi) \subset \mathrm{Fin}(\varphi)$ and that every element of $\mathrm{Exh}(\varphi)$ has a cardinality of at most ω (otherwise see Lemma 3.1). Furthermore, if φ is an LSC measure on A, then we have that $\mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$. It is an easy observation that every submeasure φ is an LSC function on $\mathrm{Exh}(\varphi)$. For $I \subset A$ the submeasure φ_I denotes the restriction of φ to $\mathcal{P}(I)$. Without loss of generality we may always assume that A is a cardinal number. Unless stated otherwise κ, λ, μ denote cardinal numbers.

A Hausdorff space Z (or a topology) is called *zero-dimensional* if the small inductive dimension (denoted by $\operatorname{ind} Z$) is 0, that is, if there is a basis consisting of clopen sets. The large inductive dimension of Z is denoted by $\operatorname{Ind} Z$. For separable metric spaces $\operatorname{ind} Z = \operatorname{Ind} Z$ and in general $\operatorname{ind} Z \le \operatorname{Ind} Z$; see [8]. For nonseperable metric spaces it is possible that Ind is greater than ind.

Definition 2.3. If $\psi : Z \to [0, \infty]$, define the following subspaces of $Z \times [0, \infty]$:

$$\begin{aligned} G_\psi^\infty &= \{(z, \psi(z)) : z \in Z,\ \psi(z) < \infty\}, \\ L_\psi^\infty &= \{(z, t) : z \in Z,\ \psi(z) \le t \le \infty\}. \end{aligned}$$

If Z is nonempty, zero-dimensional, separable, and metrizable, then an LSC function ψ is said to be an *L-Lelek function* if G_ψ^∞ is dense in L_ψ^∞.

We have the following result, see [6, Lemma 22]:

Lemma 2.4. *Let $\varepsilon > 0$ be given. If $\varphi\colon C \to [0,\infty]$ and $\psi\colon D \to [0,\infty]$ are L-Lelek functions with compact domain and if $\varphi^{-1}(0)$ and $\psi^{-1}(0)$ are singletons, then there are a homeomorphism $h\colon C \to D$ and a continuous $f\colon C \to (0,\infty)$ such that $\psi \circ h = f \cdot \varphi$ and $\sup\{|\log f(x)| : x \in C\} < \varepsilon$.*

Definition 2.5. The *weight* of a space Z is given by

$$w(Z) = \min\{|\mathcal{B}| : \mathcal{B} \text{ a basis for the topology of } Z\} + \omega$$

and the *local weight* is given by

$$lw(Z) = \min\{w(U) : U \text{ an open nonempty subset of } Z\}.$$

Proposition 2.6. *Let φ be an LSC submeasure on $\kappa \geq \omega$. Then $w(\mathcal{I}_\kappa) = \kappa$ and $lw(\mathcal{I}_\kappa) = \min_{n\in\mathbb{N}} |\{\alpha \in \kappa : \varphi(\{\alpha\}) < 1/n\}| + \omega$.*

Proof. The collection of all finite subsets of κ lies dense in $\mathcal{I}_\kappa$, by definition of $\mathrm{Exh}(\varphi)$. Therefore $w(\mathcal{I}_\kappa) \leq \kappa$. The collection of all singletons forms a discrete subset of $\mathcal{I}_\kappa$ of cardinality κ: for arbitrary $\alpha \in \kappa$ and $\beta \neq \alpha$ we have $d(\{\alpha\},\{\beta\}) = \varphi(\{\alpha,\beta\}) \geq \varphi(\{\alpha\}) > 0$ and hence $w(\mathcal{I}_\kappa) \geq \kappa$.

Since we have a topological group structure, it suffices to show for all $n \in \mathbb{N}$ that $w(\varphi^{-1}([0,1/n)) \cap \mathcal{I}_\kappa) = |L| + \omega$, where $L = \{\alpha \in \kappa : \varphi(\{\alpha\}) < 1/n\}$. The same reasoning as above yields that $w(\varphi^{-1}([0,1/n)) \cap \mathcal{I}_\kappa) \geq |L| + \omega$, so it is left to prove that we can reverse the inequality. We know that the collection $\mathcal{F}$ of finite sets in $\varphi^{-1}([0,1/n)) \cap \mathcal{I}_\kappa$ forms a dense set in $\varphi^{-1}([0,1/n)) \cap \mathcal{I}_\kappa$. Let $\mathcal{F}_L$ denote the collection of finite subsets of L. We have that $\mathcal{F} \subset \mathcal{F}_L$. This means that $w(\varphi^{-1}([0,1/n)) \cap \mathcal{I}_\kappa) \leq |\mathcal{F}| + \omega \leq |\mathcal{F}_L| + \omega = |L| + \omega$. □

Definition 2.7. Let κ be an arbitrary infinite cardinal number, and let $p \geq 1$. Recall the (possibly nonseparable) Banach space ℓ_κ^p, given by

$$\ell_\kappa^p = \left\{x = (x_\alpha)_{\alpha\in\kappa} \in \mathbb{R}^\kappa : \sum_{\alpha\in\kappa} |x_\alpha|^p < \infty\right\},$$

equipped with the topology generated by the norm $\|x\| = (\sum_{\alpha\in\kappa} |x_\alpha|^p)^{1/p}$.

Remark 2.8. The norm on $\ell^p = \ell^p_\omega$ is a Kadec norm, that is, the norm topology is the weakest topology that makes all coordinate projections and the norm function continuous. The same holds for the norm on ℓ^p_κ. Thus, the graph of the norm function when seen as a function from ℓ^p_κ with the product topology (or any other topology that lies between the product topology and the norm topology) to $\mathbb{R}$ is homeomorphic to ℓ^p_κ by the obvious map.

Definition 2.9. For $\omega \le \lambda \le \kappa$, let

$$F_\alpha = \begin{cases} \{0\} \cup \{1/n : n \in \mathbb{N}\}, & \text{if } \alpha \in \lambda; \\ \{0,1\}, & \text{if } \alpha \in \kappa \setminus \lambda. \end{cases}$$

We define $\mathfrak{E}^1_{\mathrm{c}}(\lambda,\kappa) = \{x \in \ell^1_\kappa : \forall\alpha \in \kappa, x_\alpha \in F_\alpha\}$.

By [6, Proposition 13] we have $lw(\mathfrak{E}^1_{\mathrm{c}}(\lambda,\kappa)) = \lambda$ and $w(\mathfrak{E}^1_{\mathrm{c}}(\lambda,\kappa)) = \kappa$. It is clear that $\mathfrak{E}^1_{\mathrm{c}}(\lambda,\kappa)$ is complete as a closed subset of ℓ^1_κ. For $\lambda = \kappa = \omega$, this space represents *complete Erdős space*, $\mathfrak{E}_{\mathrm{c}}$, introduced by Paul Erdős in 1940 [9]. Therefore, we will also refer to $\mathfrak{E}^1_{\mathrm{c}}(\lambda,\kappa)$ as a *nonseparable complete Erdős space* if $\kappa > \omega$. Erdős also introduced *Erdős space*, $\mathfrak{E} = \{x \in \ell^2 : x_n \in \mathbb{Q}, n \in \omega\}$ and proved that both $\mathfrak{E}$ and $\mathfrak{E}_{\mathrm{c}}$ are one-dimensional, yet totally disconnected and homeomorphic to their own squares. The spaces $\mathfrak{E}, \mathfrak{E}_{\mathrm{c}}$ and $\mathfrak{E}^\omega_{\mathrm{c}}$ were characterized by Dijkstra and van Mill [3, 5, 4] and Dijkstra [2]. Nonseparable complete Erdős spaces have analogous properties, as shown by Dijkstra, van Mill and Valkenburg [6]. That paper also concerns more general spaces defined by

$$\mathcal{E}_\mu = \{x \in \ell^p_\mu : x_\alpha \in E_\alpha,\ \alpha \in \mu\},$$

where μ is another arbitrary infinite cardinal number and the E_α are arbitrary zero-dimensional subsets of $\mathbb{R}$. Note that the values of the cardinal invariants weight and local weight can easily be determined for the space $\mathcal{E}_\mu$ whenever the sets E_α are given, see [6, Proposition 13]. The following theorem is the main result in [6].

Theorem 2.10. *The space $\mathcal{E}_\mu$ is homeomorphic to*

$$\mathfrak{E}_{\mathrm{c}} \times (\lambda_{\mathrm{D}})^\omega \times \kappa_{\mathrm{D}},$$

with $\lambda = lw(\mathcal{E}_\mu)$ and $\kappa = w(\mathcal{E}_\mu)$ if and only if $\operatorname{ind}\mathcal{E}_\mu > 0$ *and every E_α is a zero-dimensional G_δ-subset of $\mathbb{R}$.*

An easy consequence of this is [6, Theorem 41]:

Theorem 2.11. *Let $\omega \leq \lambda \leq \kappa$. Then $\mathfrak{E}^1_{\mathrm{c}}(\lambda, \kappa)$ is homeomorphic to $\mathfrak{E}_{\mathrm{c}} \times (\lambda_{\mathrm{D}})^\omega \times \kappa_{\mathrm{D}}$.*

3. Basic properties and Kadec submeasures

We start with a couple of useful relations between the theory of submeasures on ω and on an arbitrary cardinal κ. Subsequently we introduce the notion of a Kadec submeasure and we discuss some applications.

Lemma 3.1. *Let φ be a submeasure on $\kappa > \omega$. Then*

$$\begin{aligned} \mathrm{Exh}(\varphi) &= \{X \subset \kappa : X \cap I \in \mathrm{Exh}(\varphi_I) \textit{ for all } I \subset \kappa \textit{ with } |I| = \omega\} \\ &= \bigcup_{I \subset \kappa,\, |I| = \omega} \mathrm{Exh}(\varphi_I) \end{aligned}$$

and hence every element of $\mathrm{Exh}(\varphi)$ *is at most countable.*

Proof. Let $\mathcal{A}$ denote the family of all subsets X of κ with the property that $X \cap I \in \mathrm{Exh}(\varphi_I)$ for all $I \subset \kappa$ with $|I| = \omega$. Let $X \in \mathrm{Exh}(\varphi)$ and let $\varepsilon > 0$. We can find a finite subset F of κ such that $\varphi(X \setminus F) < \varepsilon$. Now take $I \subset \kappa$ such that $|I| = \omega$. Then $I \cap F$ is a finite subset of I and $\varphi((X \cap I) \setminus (F \cap I)) = \varphi((X \setminus F) \cap I) \leq \varphi(X \setminus F) < \varepsilon$. This means that $X \cap I \in \mathrm{Exh}(\varphi_I)$, so $\mathrm{Exh}(\varphi) \subset \mathcal{A}$.

The remaining inclusions become trivial once we show that every $X \in \mathcal{A}$ is at most countable. This is indeed the case since otherwise there exists an $\varepsilon > 0$ such that $|\{\alpha \in X : \varphi(\{\alpha\}) > \varepsilon\}| > \omega$. This would enable us to find a subset $I \subset X$ with $|I| = \omega$ and $\varphi(\{\alpha\}) > \varepsilon$ for each $\alpha \in I$, which forms a contradiction with the assumption that $X \cap I \in \mathrm{Exh}(\varphi_I)$. □

Using Lemma 3.1, the following theorem follows easily from Solecki [11], who introduced the metric below in the countable case.

Theorem 3.2. *Suppose that φ is an LSC submeasure on κ, then*

$$d_\varphi(X, Y) = \varphi(X \triangle Y)$$

defines an invariant, complete metric on $\mathrm{Exh}(\varphi)$ *and an invariant metric on* $\mathrm{Fin}(\varphi)$.

We will write d instead of d_φ whenever there is no danger of confusion. Observe that the topology τ_d generated by the metric d on $\mathrm{Exh}(\varphi)$ and $\mathrm{Fin}(\varphi)$ is stronger than the product topology τ_w that these spaces inherit from 2^κ. Also note that $(\mathrm{Exh}(\varphi), \tau_d)$ and $(\mathrm{Fin}(\varphi), \tau_d)$ are topological groups with symmetric difference '$\triangle$' as group operation.

Definition 3.3. As in Theorems 1.2 and 1.3 we write $\mathcal{I}_A = (\mathrm{Exh}(\varphi), \tau_d)$ for an LSC submeasure φ on a set A, where τ_d is the topology generated by the metric d of Theorem 3.2. For $B \subset A$ we write $\mathcal{I}_B$ for $(\mathrm{Exh}(\varphi_B), \tau_{d_{\varphi_B}})$.

Apart from the topology τ_d on $\mathrm{Exh}(\varphi)$ for a submeasure φ, one can also consider another natural topology.

Definition 3.4. Let Z and W be topological spaces and let $f\colon Z \to W$ be a function. Then τ_f is the weakest topology on Z that contains the given topology on Z and that makes f continuous.

Remark 3.5. Like in the case of the norm on ℓ^p_κ (see Remark 2.8), the graph of f when seen as a subset of $Z \times W$ is homeomorphic to (Z, τ_f) by the obvious map.

If we take f to be an LSC submeasure on κ, then $Z = 2^\kappa$ with the product topology τ_w and $W = [0, \infty]$.

Lemma 3.6. *Let φ be an LSC submeasure on κ. Then we have the following relation between the topologies $\tau_\mathrm{w}, \tau_\varphi$ and τ_d on* $\mathrm{Exh}(\varphi)$: $\tau_\mathrm{w} \subset \tau_\varphi \subset \tau_d$.

Proof. The inclusion $\tau_\mathrm{w} \subset \tau_\varphi$ follows immediately from the definition of τ_φ. Because $\varphi(\{\alpha\}) > 0$ for all $\alpha \in \kappa$ it follows that $\tau_d \supset \tau_\mathrm{w}$. Furthermore, using the monotonicity and subadditivity of φ one easily finds that $|\varphi(X) - \varphi(Y)| \le d(X, Y)$ for all $X, Y \in \mathrm{Exh}(\varphi)$, which means that φ is continuous with respect to τ_d. Since τ_φ is the weakest topology that makes φ continuous and contains τ_w, we have that $\tau_\varphi \subset \tau_d$. □

It is a natural question to ask when $\tau_\varphi = \tau_d$ on $\mathrm{Exh}(\varphi)$.

Definition 3.7. An LSC submeasure φ on κ is called a *Kadec submeasure* if we have that τ_φ equals the group topology τ_d on $\mathrm{Exh}(\varphi)$.

Proposition 3.8. *Let φ be an LSC measure on κ and let $\mathcal{I}_\kappa$ be the associated ideal. Then φ is a Kadec submeasure and if* $\operatorname{ind}\mathcal{I}_\kappa > 0$ *then $\mathcal{I}_\kappa$ is homeomorphic to $\mathfrak{E}_\mathrm{c} \times (\lambda_\mathrm{D})^\omega \times \kappa_\mathrm{D}$, where $\lambda = lw(\mathcal{I}_\kappa)$.*

Proof. The ideal $\mathcal{I}_\kappa$ (with $\mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$) can be embedded in the Banach space ℓ^1_κ. Indeed, consider the function $h\colon \mathcal{I}_\kappa \to \ell^1_\kappa$ given by

$$h(X)_\alpha = \begin{cases} \varphi(\{\alpha\}), & \text{if } \alpha \in X, \\ 0, & \text{otherwise.} \end{cases}$$

It is evident that $\varphi(X \triangle Y) = \|h(X) - h(Y)\|$, so h is an isometric embedding. Since ℓ^1 has a Kadec norm (Remark 2.8) we have that φ is a Kadec submeasure. Furthermore, we have that $h(\mathcal{I}_\kappa) = \big(\prod_{\alpha\in\kappa} E_\alpha\big) \cap \ell^1_\kappa$, with $E_\alpha = \{0, \varphi(\{\alpha\})\}$ for every $\alpha \in \kappa$. Applying Theorem 2.10, we find $\mathcal{I}_\kappa \approx \mathfrak{E}_\mathrm{c} \times (\lambda_\mathrm{D})^\omega \times \kappa_\mathrm{D}$ with $\lambda = lw(\mathcal{I}_\kappa)$ and $\kappa = w(\mathcal{I}_\kappa)$ by Proposition 2.6. □

Let for $A \subset \kappa$ the function $\zeta_A\colon 2^\kappa \to 2^\kappa$ be given by $\zeta_A(X) = X \setminus A$. We have the following characterization for Kadec submeasures.

Proposition 3.9. *Let φ be an LSC submeasure on κ. The following statements are equivalent:*

(1) *φ is a Kadec submeasure;*
(2) *$\varphi \circ \zeta_{\{\alpha\}}\colon (\mathrm{Exh}(\varphi), \tau_\varphi) \to \mathbb{R}$ is continuous for every $\alpha \in \kappa$; and*
(3) *$\zeta_{\{\alpha\}}\colon (\mathrm{Exh}(\varphi), \tau_\varphi) \to (\mathrm{Exh}(\varphi), \tau_\varphi)$ is continuous for each $\alpha \in \kappa$.*

Proof. (1) ⇒ (2). Suppose that φ is a Kadec submeasure. It is clear that for all $X, Y \in \mathrm{Exh}(\varphi)$ we have $d\big(\zeta_{\{\alpha\}}(X), \zeta_{\{\alpha\}}(Y)\big) \le d(X, Y)$, hence $\zeta_{\{\alpha\}}\colon (\mathrm{Exh}(\varphi), \tau_d) \to (\mathrm{Exh}(\varphi), \tau_d)$ is continuous for every $\alpha \in \kappa$. Since $\tau_\varphi = \tau_d$ the composition with the continuous function $\varphi : (\mathrm{Exh}(\varphi), \tau_\varphi) \to \mathbb{R}$ yields statement (2).

(2) ⇒ (3). Clearly, $\zeta_{\{\alpha\}}$ is continuous with respect to τ_w, so it suffices to show that $\zeta^{-1}_{\{\alpha\}}(\varphi^{-1}([0,t)) \cap \mathrm{Exh}(\varphi)) \in \tau_\varphi$ for $t \in (0, \infty)$. However, this set equals $(\varphi \circ \zeta_{\{\alpha\}})^{-1}([0,t)) \cap \mathrm{Exh}(\varphi)$ which is open with respect to τ_φ according to the assumption.

(3) ⇒ (1). Suppose that $\zeta_{\{\alpha\}}\colon (\mathrm{Exh}(\varphi), \tau_\varphi) \to (\mathrm{Exh}(\varphi), \tau_\varphi)$ is continuous for every $\alpha \in \kappa$. By composing these functions, one can see that $\zeta_F\colon (\mathrm{Exh}(\varphi), \tau_\varphi) \to (\mathrm{Exh}(\varphi), \tau_\varphi)$ is continuous for every finite subset $F \subset \kappa$. In view of Lemma 3.6 it suffices to show that

$\tau_d \subset \tau_\varphi$. So take a set $X \in \mathrm{Exh}(\varphi)$ and an $\varepsilon > 0$ and consider $B_d(X,\varepsilon)$, the ε-ball around X with respect to the metric d as in Theorem 3.2. We show that there is an open set O in τ_φ such that $X \in O \subset B_d(X,\varepsilon)$, which suffices to show that $\tau_d \subset \tau_\varphi$.

Since $X \in \mathrm{Exh}(\varphi)$ there is a finite set $F \subset X$ such that $\varphi(X \backslash F) < \varepsilon/2$. Now define

$$O = \{Y \in \mathrm{Exh}(\varphi) : Y \supset F \text{ and } \varphi(Y \setminus F) < \varepsilon/2\}.$$

Clearly, $X \in O$ and

$$\{Y \in \mathrm{Exh}(\varphi) : \varphi(Y \setminus F) < \varepsilon/2\} = \zeta_F^{-1}\Big(\varphi^{-1}\big([0,\varepsilon/2)\big) \cap \mathrm{Exh}(\varphi)\Big),$$

so this set is open in τ_φ since $\zeta_F \colon (\mathrm{Exh}(\varphi), \tau_\varphi) \to (\mathrm{Exh}(\varphi), \tau_\varphi)$ is continuous. We see that O is open in τ_φ. Now take $Y \in O$. We have that

$$\begin{aligned} d(X,Y) &\leq \varphi\big((X \triangle Y) \cap F\big) + \varphi\big((X \triangle Y) \setminus F\big) \\ &\leq \varphi\big((X \cap F) \triangle (Y \cap F)\big) + \varphi(X \setminus F) + \varphi(Y \setminus F) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

This means that $O \subset B_d(X,\varepsilon)$. □

The following general result will be useful in different places in the remainder of this text.

Lemma 3.10. *Let (Z,τ) be a topological space and let $\varphi_i \colon Z \to [0,\infty]$ be LSC functions for $i \in I$, where I is an arbitrary nonempty set. Define $\varphi \colon Z \to [0,\infty]$ by $\varphi(z) = \sum_{i \in I} \varphi_i(z)$. Then φ is an LSC function and on $\varphi^{-1}\big([0,\infty)\big)$ we have that τ_{φ_i} is weaker than τ_φ for every $i \in I$.*

Proof. It is easily verified that φ is LSC, using the fact that every φ_i is nonnegative.

Now we show that $\tau_{\varphi_i} \subset \tau_\varphi$ on $\varphi^{-1}\big([0,\infty)\big)$ for every $i \in I$. Pick $i_0 \in I$. Since φ is LSC we have that

$$\mathcal{B}_\varphi = \{O \cap \varphi^{-1}\big([0,t)\big) : O \in \tau \text{ and } t \in (0,\infty)\}$$

is a basis for τ_φ on $\varphi^{-1}\big([0,\infty)\big)$. Similarly, since $\varphi^{-1}\big([0,\infty)\big) \subset \varphi_{i_0}^{-1}\big([0,\infty)\big)$, we have that

$$\mathcal{B}_{\varphi_{i_0}} = \{O \cap \varphi_{i_0}^{-1}\big([0,t)\big) : O \in \tau \text{ on } \varphi^{-1}\big([0,\infty)\big) \text{ and } t \in (0,\infty)\}$$

is a basis for $\tau_{\varphi_{i_0}}$ on $\varphi^{-1}\big([0,\infty)\big)$. Hence it is sufficient to show that

$$\varphi_{i_0}^{-1}\big([0,t)\big) \cap \varphi^{-1}\big([0,\infty)\big) \in \tau_\varphi$$

for every $t \in (0,\infty)$.

Pick $t \in (0,\infty)$. If $\varphi_{i_0}^{-1}\big([0,t)\big) \cap \varphi^{-1}\big([0,\infty)\big) = \emptyset$, there is nothing to prove, so suppose $z \in Z$ is such that $\varphi_{i_0}(z) < t$ and $\varphi(z) < \infty$. Note that $\varphi = \varphi_{i_0} + \psi$, with $\psi = \sum_{i \in I \setminus \{i_0\}} \varphi_i$. We have that ψ is just as φ an LSC function on Z and $\psi(z) \le \varphi(z) < \infty$. Put $\delta = t - \varphi_{i_0}(z) > 0$ and consider the set $V \subset Z$ given by

$$V = \psi^{-1}\big((\psi(z) - \delta/2, \infty]\big) \cap \varphi^{-1}\big([0, \varphi(z) + \delta/2)\big).$$

Clearly, $z \in V \in \mathcal{B}_\varphi$. Take $y \in V$, then $\varphi_{i_0}(y) + \psi(y) < \varphi_{i_0}(z) + \psi(z) + \delta/2$ and $\psi(y) > \psi(z) - \delta/2$. It follows that

$$\varphi_{i_0}(y) < \varphi_{i_0}(z) + \psi(z) - \psi(y) + \delta/2 < \varphi_{i_0}(z) + \delta = t.$$

This means that $z \in V \subset \varphi_{i_0}^{-1}\big([0,t)\big) \cap \varphi^{-1}\big([0,\infty)\big)$ and hence $\varphi_{i_0}^{-1}\big([0,t)\big) \cap \varphi^{-1}\big([0,\infty)\big) \in \tau_\varphi$. □

Corollary 3.11. *Let I be a nonempty set and let $\{A_i : i \in I\}$ be a collection of pairwise disjoint sets. Suppose that $\varphi_i \colon 2^{A_i} \to [0,\infty]$ is a Kadec submeasure for each $i \in I$. Then the function $\varphi \colon 2^A \to [0,\infty]$ defined by*

$$\varphi(X) = \sum_{i \in I} \varphi_i(X \cap A_i)$$

is a Kadec submeasure on A.

Proof. If $|I| = 1$ there is nothing to prove, so suppose that $|I| > 1$. We extend φ_i to an LSC function $\tilde{\varphi}_i$ on 2^A in the obvious way: we define $\tilde{\varphi}_i \colon 2^A \to [0,\infty]$ by $\tilde{\varphi}_i(X) = \varphi_i(X \cap A_i)$. Since $\varphi = \sum_{i \in I} \tilde{\varphi}_i$ it follows from Lemma 3.10 that φ is an LSC function on 2^A. Furthermore, since every φ_i is a submeasure on A_i, it is easily seen that φ is a submeasure on A.

Take $\alpha \in A$, say $\alpha \in A_j$ for some $j \in I$. We show that $\varphi \circ \zeta_{\{\alpha\}} \colon (\mathrm{Exh}(\varphi), \tau_\varphi) \to \mathbb{R}$ is continuous, which means that φ is Kadec by statement (2) of Proposition 3.9. Define $\psi \colon 2^A \to [0,\infty]$ by

$$\psi(X) = \sum_{i \in I \setminus \{j\}} \tilde{\varphi}_i(X).$$

With Lemma 3.10 we know that ψ is an LSC function and we have that $\varphi = \tilde{\varphi}_j + \psi$ and $\varphi \circ \zeta_{\{\alpha\}} = \tilde{\varphi}_j \circ \zeta_{\{\alpha\}} + \psi$. Furthermore, Lemma 3.10 tells us that $\tau_{\tilde{\varphi}_j} \subset \tau_\varphi$ and $\tau_\psi \subset \tau_\varphi$ on $\mathrm{Fin}(\varphi) \supset \mathrm{Exh}(\varphi)$. Since ψ is clearly continuous on $(\mathrm{Exh}(\varphi), \tau_\psi)$ this implies that ψ is continuous on $(\mathrm{Exh}(\varphi), \tau_\varphi)$. We are done once we show that $\tilde{\varphi}_j \circ \zeta_{\{\alpha\}} \colon (\mathrm{Exh}(\varphi), \tau_{\tilde{\varphi}_j}) \to \mathbb{R}$ is continuous. This follows from the continuity of the projection $X \mapsto X \cap A_j$ as a function from $(\mathrm{Exh}(\varphi), \tau_{\tilde{\varphi}_j})$ to $(\mathrm{Exh}(\varphi), \tau_{\varphi_j})$ together with the continuity of $\varphi_j \circ \zeta_{\{\alpha\}}$ on $(\mathrm{Exh}(\varphi_j), \tau_{\varphi_j})$, see Proposition 3.9. □

Not every LSC submeasure φ turns out to be a Kadec submeasure. An example of an LSC submeasure φ on ω with $\mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$ which is not Kadec can be found in [4, Remark 4.19]. However, for LSC submeasures on $\kappa \leq \omega$ we have the following theorem, which is a combination of [7, Theorem 3.14] and the fact that submeasures on finite sets are automatically Kadec submeasures.

Theorem 3.12. *Let φ be an LSC submeasure on $\kappa \leq \omega$. Then there exists a Kadec submeasure ψ such that $\varphi \leq \psi \leq 2\varphi$.*

Clearly, $\mathrm{Exh}(\psi) = \mathrm{Exh}(\varphi)$ and $\mathrm{Fin}(\psi) = \mathrm{Fin}(\varphi)$ and the metrics d_φ and d_ψ as in Theorem 3.2 are uniformly equivalent. This means for example that we may assume without loss of generality that the submeasure φ in Theorem 1.2 is a Kadec submeasure. Unfortunately, the proof of [7, Theorem 3.14] does not generalize to LSC submeasures on an uncountable cardinal number, so we can ask the following.

Question 3.13. Let φ be an LSC submeasure on $\kappa > \omega$. Is there a Kadec submeasure ψ on κ with $\mathrm{Exh}(\psi) = \mathrm{Exh}(\varphi)$ and such that $\tau_{d_\varphi} = \tau_\psi$ on $\mathrm{Exh}(\varphi)$?

4. The small inductive dimension of $\mathcal{I}_\kappa$

We now return to the setting of general LSC submeasures. In this section we will prove Theorem 1.3, the extension of Theorem 1.2 to uncountable cardinals, and consider some consequences. Among them are the relation between the minimal weight of nonempty open subsets of $\mathcal{I}_\kappa$ and the small inductive dimension, and a statement concerning a one-point connectification of $\mathcal{I}_\kappa$.

We prove Theorem 1.3 by reduction to Theorem 1.2.

*Proof of **Theorem 1.3**.* (1) $\Rightarrow$ (2). Suppose there exists a subset B of $\kappa > \omega$ with $\mathrm{Exh}(\varphi) = \{X \subset \kappa : X \cap B \text{ is finite}\}$. Since $\kappa \setminus B$ is in the collection we have that the set is countable (Lemma 3.1) and we choose an $I \subset \kappa$ with $\kappa \setminus B \subset I$ and $|I| = \omega$. Note that $\mathrm{Fin}(\varphi_I) = \mathrm{Exh}(\varphi_I) = \{X \subset I : X \cap (I \cap B) \text{ is finite}\}$ thus Theorem 1.2 gives that $\mathcal{I}_I$ is homeomorphic to $\mathbb{Z}$, 2^ω, or $\mathbb{Z} \times 2^\omega$. If $\inf_{\alpha \in \kappa \setminus I} \varphi(\{\alpha\}) = 0$ then we can find a sequence $(\alpha_k)_{k \in \mathbb{N}}$ in $\kappa \setminus I$ for which $\varphi(\{\alpha_k\}) \leq 2^{-k}$ and hence $\{\alpha_1, \alpha_2, \ldots\} \subset \mathrm{Exh}(\varphi_{\kappa \setminus I})$. Since $\kappa \setminus I \subset B$, $\mathrm{Exh}(\varphi_{\kappa \setminus I})$ contains only the finite sets and we have a contradiction. Thus $\varepsilon = \inf_{\alpha \in \kappa \setminus I} \varphi(\{\alpha\}) > 0$ and $|\mathcal{I}_{\kappa \setminus I}| = \kappa$. Since for all sets $F \neq F' \in \mathcal{I}_{\kappa \setminus I}$ we have $d(F, F') \geq \varepsilon$, $\mathcal{I}_{\kappa \setminus I}$ is discrete and homeomorphic to κ_{D}. Since clearly $\mathcal{I}_\kappa \approx \mathcal{I}_I \times \mathcal{I}_{\kappa \setminus I}$ we have that $\mathcal{I}_\kappa$ is homeomorphic to κ_{D} or $\kappa_{\mathrm{D}} \times 2^\omega$.

(2) $\Rightarrow$ (3). We also prove this implication by contraposition. Suppose there exists an $\varepsilon > 0$ for which $\{\alpha \in \kappa : \varphi(\{\alpha\}) \leq \varepsilon\} \in \mathrm{Exh}(\varphi)$. Then for $B = \{\alpha \in \kappa : \varphi(\{\alpha\}) > \varepsilon\}$ we have $\mathrm{Exh}(\varphi) = \{X \subset \kappa : X \cap B \text{ is finite}\}$. Indeed, if $X \cap B$ is finite, then it is contained in $\mathrm{Exh}(\varphi)$, as is $\kappa \setminus B$. Hence, since $X \subset (X \cap B) \cup \kappa \setminus B$ we have $X \in \mathrm{Exh}(\varphi)$ as well. If $X \in \mathrm{Exh}(\varphi)$, then there is a finite set F with $\varphi(X \setminus F) \leq \varepsilon$. This gives $X \setminus F \subset \kappa \setminus B$ and therefore $X \cap B = X \setminus (\kappa \setminus B) \subset F$ is finite.

(3) $\Rightarrow$ (4). Assume (3). Take $m \in \mathbb{N}$ and let $X_m = \{\alpha \in \kappa : \varphi(\{\alpha\}) \leq 1/m\}$. Then we know that $X_m \notin \mathrm{Fin}(\varphi)$, so since φ is LSC it follows that there is a countable subset $I(m)$ of X_m such that $\varphi(I(m)) = \infty$. Take $I = \bigcup_{m=1}^\infty I(m)$. Then $|I| = \omega$ and for every $m \in \mathbb{N}$ we see that $\varphi(\{\alpha \in I : \varphi(\{\alpha\}) \leq 1/m\}) \geq \varphi(I(m)) = \infty$. Now look at φ_I and note that $\mathrm{Exh}(\varphi_I) = \mathrm{Fin}(\varphi_I)$. Applying Theorem 1.2 guarantees the existence of a set $B \subset I$ such that $\varphi_I(B) = \infty$ and $\lim_{\alpha \in B} \varphi(\{\alpha\}) = 0$. We see that B satisfies condition (4).

(4) $\Rightarrow$ (5). Suppose that B satisfies (4) and consider φ_B. Note that $|B| = \omega$. Since $\mathrm{Exh}(\varphi_B) = \mathrm{Fin}(\varphi_B)$ we can use Theorem 1.2 to find that $\mathcal{I}_B \approx \mathfrak{E}_{\mathrm{c}}$. Now we find that

$$\mathcal{I}_\kappa \times \mathfrak{E}_{\mathrm{c}} \approx \mathcal{I}_{\kappa \setminus B} \times \mathcal{I}_B \times \mathfrak{E}_{\mathrm{c}} \approx \mathcal{I}_{\kappa \setminus B} \times \mathfrak{E}_{\mathrm{c}}^2 \approx \mathcal{I}_{\kappa \setminus B} \times \mathfrak{E}_{\mathrm{c}} \approx \mathcal{I}_{\kappa \setminus B} \times \mathcal{I}_B \approx \mathcal{I}_\kappa.$$

(4) $\Rightarrow$ (6). Suppose that $B \subset \kappa$ satisfies condition (4) and hence $|B| = \omega$. Take a nonempty clopen subset C of $\mathcal{I}_\kappa$ and let the set X be an element of C. Put $Y = B \cup X$ and note that $|Y| = \omega$ and $\mathrm{Exh}(\varphi_Y) = \mathrm{Fin}(\varphi_Y)$. We have that $X \in C \cap \mathcal{I}_Y$, so $C \cap \mathcal{I}_Y$ is a nonempty clopen subset of $\mathcal{I}_Y$. Since $B \subset Y$ it follows

from Theorem 1.2 that $C \cap \mathcal{I}_Y$ is φ_Y-unbounded and hence C is φ-unbounded.

The implications (5) $\Rightarrow$ (7), (6) $\Rightarrow$ (7), and (8) $\Rightarrow$ (1) are trivial. For (7) $\Rightarrow$ (8) note that $\mathcal{I}_\kappa$ is totally disconnected and that a totally disconnected locally compact space is zero-dimensional. □

Remark 4.1. Note that in proving (4) $\Rightarrow$ (5) we first showed that it follows from (4) that

(9) there is a $B \subset \kappa$ with $|B| = \omega$ and $\mathcal{I}_B$ is homeomorphic to $\mathfrak{E}_\mathrm{c}$.

Then we showed that this implies statement (5). So we can add statement (9) to the list of equivalences in Theorem 1.3.

It is shown in [4, Example 4.20] that the condition $\mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$ is essential in Theorem 1.2. From that it follows easily that it is also needed in Theorem 1.3.

Proposition 4.2. *If φ is a Kadec submeasure on κ, then* $\operatorname{ind}\mathcal{I}_\kappa \leq \operatorname{Ind}\mathcal{I}_\kappa \leq 1$. *If moreover* $\mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$, *then* $\operatorname{ind}\mathcal{I}_\kappa = \operatorname{Ind}\mathcal{I}_\kappa$.

Proof. For a Kadec submeasure φ on κ one can view the space $\mathcal{I}_\kappa$ as the graph of φ when seen as a function from $(\mathrm{Exh}(\varphi), \tau_\mathrm{w})$ to $[0, \infty)$; see Remark 3.5. Define for $n \in \mathbb{Z}$ the set

$$A_n = \{\alpha \in \kappa : 2^n \leq \varphi(\{\alpha\}) < 2^{n+1}\}$$

and note that the A_n form a partition of κ. Then 2^κ can be written as $\Pi_{n\in\mathbb{Z}} 2^{A_n}$. Let τ' be the topology that $\mathrm{Exh}(\varphi)$ inherits from $\Pi_{n\in\mathbb{Z}}(2^{A_n})_\mathrm{D}$ and note that the latter space is a countable product of discrete spaces. Thus we have that $(\mathrm{Exh}(\varphi), \tau')$ is a metrizable space with Ind $= 0$; see [8, Theorems 4.1.7 and 4.1.25]. If $X, Y \in \mathrm{Exh}(\varphi)$ are such that $d(X, Y) < 2^n$, then $d(X \cap A_n, Y \cap A_n) < 2^n$ which means that $X \cap A_n = Y \cap A_n$ by the definition of A_n. Thus the projection map from $\mathcal{I}_\kappa$ to $(2^{A_n})_\mathrm{D}$ is continuous for every $n \in \mathbb{Z}$ which means that $\tau' \subset \tau_d = \tau_\varphi$. Since clearly $\tau_\mathrm{w} \subset \tau'$ we have that $\mathcal{I}_\kappa$ is also homeomorphic to the graph of φ when seen as a function from $(\mathrm{Exh}(\varphi), \tau')$ to $[0, \infty)$. By [8, Theorem 4.1.21] we now have that $\mathrm{Ind}((\mathrm{Exh}(\varphi), \tau') \times [0, \infty)) = 1$ and thus $\operatorname{Ind}\mathcal{I}_\kappa \leq 1$ by [8, Theorem 4.1.7].

If $\mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$ then Theorem 1.3 applies thus if $\operatorname{ind}\mathcal{I}_\kappa = 0$ then $\mathcal{I}_\kappa \approx \kappa_\mathrm{D}$ or $\mathcal{I}_\kappa \approx \kappa_\mathrm{D} \times 2^\omega$ and hence also $\operatorname{Ind}\mathcal{I}_\kappa = 0$. □

Thus the following question is related to Question 3.13.

Question 4.3. Is in general $\operatorname{ind}\mathcal{I}_\kappa = \operatorname{Ind}\mathcal{I}_\kappa \leq 1$?

Proposition 4.4. *Suppose that φ is an LSC submeasure on $\kappa > \omega$ with $\operatorname{Exh}(\varphi) = \operatorname{Fin}(\varphi)$. If $lw(\mathcal{I}_\kappa) > \omega$ then $\operatorname{ind}\mathcal{I}_\kappa > 0$.*

Proof. According to Proposition 2.6 we have that the set $X_\varepsilon = \{\alpha \in \kappa : \varphi(\{\alpha\}) \leq \varepsilon\}$ is uncountable for every $\varepsilon > 0$ and hence it cannot be a member of $\operatorname{Exh}(\varphi)$ by Lemma 3.1. Since $\operatorname{Exh}(\varphi) = \operatorname{Fin}(\varphi)$ we have $\varphi(X_\varepsilon) = \infty$ and by Theorem 1.3 we find $\operatorname{ind}\mathcal{I}_\kappa > 0$. □

The following definition is taken from [1].

Definition 4.5. Let p be a point in a space Z. We say that p is a *fixed point of Z* if for every nonconstant continuous function $f\colon Z \to Z$ we have $f(p) = p$.

Let $\mathcal{I}_\kappa^+ = \mathcal{I}_\kappa \cup \{\Omega\}$ be a Hausdorff extension of $\mathcal{I}_\kappa$ such that for every neighbourhood U of Ω in $\mathcal{I}_\kappa^+$ we have that $\mathcal{I}_\kappa \backslash U$ is φ-bounded.

Theorem 4.6. *Suppose that φ is an LSC submeasure on $\kappa \geq \omega$ such that $\operatorname{Exh}(\varphi) = \operatorname{Fin}(\varphi)$. Then the following statements are equivalent:*

(1) *Ω is a fixed point of $\mathcal{I}_\kappa^+$;*
(2) *$\mathcal{I}_\kappa^+$ has the fixed point property;*
(3) *$\mathcal{I}_\kappa^+$ is connected; and*
(4) $\operatorname{ind}\mathcal{I}_\kappa > 0$.

Proof. The implications (1) ⇒ (2) and (2) ⇒ (3) are trivial.

(3) ⇒ (4). Assume that $\operatorname{ind}\mathcal{I}_\kappa = 0$ and select an $X \in \mathcal{I}_\kappa$. Let U and V be disjoint and open in $\mathcal{I}_\kappa^+$ such that $X \in U$ and $\Omega \in V$. Choose a clopen neighbourhood C of X in $\mathcal{I}_\kappa$ such that $C \subset U$ and note that C is also clopen in $\mathcal{I}_\kappa^+$. Thus $\mathcal{I}_\kappa^+$ is disconnected.

(4) ⇒ (1). Assume that $\operatorname{ind}\mathcal{I}_\kappa > 0$. Since τ_{w} is coarser than τ_d and zero-dimensional, the space $\mathcal{I}_\kappa$ is totally disconnected. Let U be an open neighbourhood of Ω in $\mathcal{I}_\kappa^+$ such that $V = \mathcal{I}_\kappa \setminus U \neq \emptyset$. Let C be the component of Ω in U. According to [1, Lemma 14] it suffices to show that C is not closed in the space $\mathcal{I}_\kappa^+$. By Theorem 1.3 there is a $B \subset \kappa$ with $\varphi(B) = \infty$ and $\lim_{\alpha \in B} \varphi(\{\alpha\}) = 0$. Pick a set $X \in V$ and let $E = B \cup X$. Then E is a countably infinite set with $\varphi(E) = \infty$ and $\lim_{e \in E} \varphi(\{e\}) = 0$ since $X \in \operatorname{Exh}(\varphi)$. Put $U' = \{Y \in U : Y \subset E\} \cup \{\Omega\}$, $V' = V \cap \mathcal{I}_E = \mathcal{I}_E \setminus U'$ and let C' be the component of Ω in U'. Now U' is an open neighbourhood

of Ω in $\mathcal{I}_E \cup \{\Omega\}$ such that $X \in \mathcal{I}_E \setminus U'$. According to the proof of [7, Theorem 7.5] the closure of C' intersects V'. Since C' is a subset of C we have that C is not closed in $\mathcal{I}_\kappa^+$ and the proof is complete. □

5. Submeasures and nonseparable complete Erdős spaces

The purpose of this section is to establish a link between nonseparable complete Erdős spaces and certain submeasures on κ; see also Proposition 3.8.

Proposition 5.1. *Let φ be an LSC submeasure on $\kappa > \omega$ such that $\mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$ and $\mathrm{ind}\, \mathcal{I}_\kappa > 0$. If $lw(\mathcal{I}_\kappa) = \omega$ then $\mathcal{I}_\kappa$ is homeomorphic to $\mathfrak{E}_\mathrm{c} \times \kappa_\mathrm{D}$.*

Proof. Since $\mathrm{ind}\, \mathcal{I}_\kappa > 0$ it follows from statement (9) in Remark 4.1 that there is a $B \subset \kappa$ with $|B| = \omega$ and $\mathcal{I}_B \approx \mathfrak{E}_\mathrm{c}$. Using Proposition 2.6 we define $I \subset \kappa$ as $I = B \cup \{\alpha \in \kappa : \varphi(\{\alpha\}) < 1/n\}$ with $n \in \mathbb{N}$ such that $|I| = \omega$. Note that $\mathcal{I}_I \supset \mathcal{I}_B$ so $\mathrm{ind}\, \mathcal{I}_I > 0$. It follows from Theorem 1.2 that $\mathcal{I}_I \approx \mathfrak{E}_\mathrm{c}$. Clearly, $\mathcal{I}_\kappa \approx \mathcal{I}_I \times \mathcal{I}_{\kappa \setminus I}$, so we have that $\mathcal{I}_\kappa \approx \mathfrak{E}_\mathrm{c} \times \mathcal{I}_{\kappa \setminus I}$. If $\alpha \in \kappa \setminus I$ then $\varphi(\{\alpha\}) \geq 1/n$ and this means that $\mathcal{I}_{\kappa \setminus I}$ is the collection of all finite subsets of $\kappa \setminus I$ with the discrete topology. Since $|\mathcal{I}_{\kappa \setminus I}| = \kappa$, we have that $\mathcal{I}_{\kappa \setminus I}$ is homeomorphic to κ_D. We conclude that $\mathcal{I}_\kappa \approx \mathfrak{E}_\mathrm{c} \times \kappa_\mathrm{D}$. □

In the proof of Lemma 5.3 we will use the following technical result.

Lemma 5.2. *Let (Z, τ) be a topological space and let I be a nonempty set. Suppose that $\varphi_i \colon Z \to [0, \infty]$ is an LSC function for every $i \in I$. Fix $M > 1$ and let $f_i \colon Z \to (1/M, M)$ be a continuous function for each $i \in I$. Define for $i \in I$ the function $\psi_i \colon Z \to [0, \infty]$ by $\psi_i = f_i \cdot \varphi_i$. Consider the functions $\varphi, \psi \colon Z \to [0, \infty]$ given by $\varphi = \sum_{i \in I} \varphi_i$ and $\psi = \sum_{i \in I} \psi_i$. Then $\psi^{-1}\big([0, \infty)\big) = \varphi^{-1}\big([0, \infty)\big)$ and $\tau_\varphi = \tau_\psi$.*

Proof. Clearly, $\varphi/M \leq \psi \leq M\varphi$, from which it immediately follows that $\psi^{-1}\big([0, \infty)\big) = \varphi^{-1}\big([0, \infty)\big)$.

We show that $\tau_\varphi = \tau_\psi$. We will use Lemma 3.10 to do this. It is not difficult to see that ψ_i is an LSC function for every $i \in I$. By a symmetry argument it is then enough to show that $\tau_\varphi \subset \tau_\psi$, that is, we have to prove that φ is continuous with respect to τ_ψ.

We know from Lemma 3.10 that φ is an LSC function with respect to τ, so certainly φ is an LSC function with respect to τ_ψ. It is therefore left to show that for all $t \in (0,\infty)$ the set $\varphi^{-1}\big([0,t)\big)$ is open in τ_ψ.

Take $t \in (0,\infty)$. Since $\psi^{-1}\big([0,\infty)\big) = \varphi^{-1}\big([0,\infty)\big)$ we know that $\varphi^{-1}\big([0,t)\big) \subset \psi^{-1}\big([0,\infty)\big)$. Clearly, φ_i is continuous with respect to τ_{ψ_i} for every $i \in I$. With Lemma 3.10 we know that $\tau_{\psi_i} \subset \tau_\psi$ on $\psi^{-1}\big([0,\infty)\big)$, so we have that every φ_i is continuous with respect to τ_ψ on $\psi^{-1}\big([0,\infty)\big)$. Take $z \in \varphi^{-1}\big([0,t)\big)$ and put $\varepsilon = t - \varphi(z)$. We know that $\psi(z) < \infty$ so we can find a finite set $F \subset I$ such that $\sum_{i\in I\setminus F} \psi_i(z) < \varepsilon/(2M)$. Using that $\sum_{i\in F}\varphi_i$ is continuous with respect to τ_ψ on $\psi^{-1}\big([0,\infty)\big)$ and that $\psi^{-1}\big([0,\infty)\big) \in \tau_\psi$, we find an open set $U \subset \psi^{-1}\big([0,\infty)\big)$ in τ_ψ such that $z \in U$ and for all $z' \in U$ we have that $\sum_{i\in F}\varphi_i(z') < \varphi(z) + \varepsilon/2$. Let $\tilde\psi = \sum_{i\in I\setminus F}\psi_i$, then we have that $\psi = \sum_{i\in F}\psi_i + \tilde\psi$. Lemma 3.10 tells us that $\sum_{i\in F}\psi_i$ and $\tilde\psi$ are LSC functions and that $\tau_{\tilde\psi} \subset \tau_\psi$ on $\psi^{-1}\big([0,\infty)\big)$. Define the set $V \subset \psi^{-1}\big([0,\infty)\big)$ by $V = U \cap \tilde\psi^{-1}\left(\left[0, \frac{\varepsilon}{2M}\right)\right)$. It is clear that $z \in V \in \tau_\psi$. Moreover, for every $z' \in V$ we have that

$$\begin{aligned}\varphi(z') = \sum_{i\in F}\varphi_i(z') + \sum_{i\in I\setminus F}\varphi_i(z') &\le \sum_{i\in F}\varphi_i(z') + M\,\tilde\psi(z') \\ &< \varphi(z) + \frac{\varepsilon}{2} + M\frac{\varepsilon}{2M} = t.\end{aligned}$$

We showed that $z \in V \subset \varphi^{-1}\big([0,t)\big)$ with $V \in \tau_\psi$, which means that $\varphi^{-1}\big([0,t)\big)$ is open in τ_ψ. □

In the next lemma we look at submeasures that behave a bit like measures.

Lemma 5.3. *Let φ be a Kadec submeasure on $\kappa > \omega$ such that* $\mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$. *Suppose there exists a partition $\{A_\beta : \beta \in \kappa\}$ of κ with $|A_\beta| \le \omega$ for every $\beta \in \kappa$, $|\{\beta \in \kappa : \mathrm{ind}\,\mathcal{I}_{A_\beta} > 0\}| = \kappa$ and with the property that there exists an $M \in \mathbb{N}$ such that for every $X \in \mathrm{Exh}(\varphi)$ we have*

$$(*) \qquad \sum_{\beta\in\kappa}\varphi(X \cap A_\beta) \le M\varphi(X).$$

Then $\mathcal{I}_\kappa$ is homeomorphic to $\mathfrak{E}_{\mathrm{c}} \times (\kappa_{\mathrm{D}})^\omega$.

Proof. Note that $(*)$ is trivially true for $X \notin \text{Exh}(\varphi) = \text{Fin}(\varphi)$. By taking appropriate unions of different sets A_β if necessary, and using the subadditivity of φ, one sees that we may, and we will from now on, assume that $\{A_\beta : \beta \in \kappa\}$ is a partition of κ such that $\text{ind}\,\mathcal{I}_{A_\beta} > 0$ and $|A_\beta| = \omega$ for all $\beta \in \kappa$.

Claim 5.4. *For all $\beta \in \kappa$ we have that φ_{A_β} is an L-Lelek function with compact domain for which $\varphi_{A_\beta}^{-1}(0) = \{\emptyset\}$.*

Proof. Pick a $\beta \in \kappa$. We only have to prove that φ_{A_β} is an L-Lelek function on 2^{A_β}, that is, that $G^\infty_{\varphi_{A_\beta}}$ is dense in $L^\infty_{\varphi_{A_\beta}}$ (see Definition 2.3). Let $X \subset A_\beta$ be arbitrary and consider a standard neighbourhood U of X in 2^{A_β}. So we have that $U = \{Y \subset A_\beta : Y \cap F = X \cap F\}$ for some finite set $F \subset A_\beta$. Note that $\mathcal{I}_{A_\beta} \cap U$ is a clopen subspace of $\mathcal{I}_{A_\beta}$, which contains $X \cap F$. Let t be such that $\varphi_{A_\beta}(X \cap F) < t < \infty$. It now suffices to show the existence of some $Y \in \mathcal{I}_{A_\beta} \cap U$ with $\varphi_{A_\beta}(Y) = t$. We know that φ_{A_β} is a continuous function on $\mathcal{I}_{A_\beta}$. If there were no $Y \in \mathcal{I}_{A_\beta} \cap U$ with $\varphi_{A_\beta}(Y) = t$ then the nonempty clopen subset $\{Y \in \mathcal{I}_{A_\beta} \cap U : \varphi_{A_\beta}(Y) < t\} \subset \mathcal{I}_{A_\beta}$ would be φ-bounded. This violates Theorem 1.2 and thereby the claim is proved. □

Put $T = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ and define the function $\psi_{A_\beta} : T^{A_\beta} \to [0,\infty]$ by $\psi_{A_\beta}(x) = \sum_{\alpha \in A_\beta} x_\alpha$. Notice that ψ_{A_β} coincides with the $\ell^1_{A_\beta}$-norm when both are restricted to the set $\{x \in T^{A_\beta} : \sum_{\alpha\in A_\beta} x_\alpha < \infty\}$. We can now apply Lemma 2.4 to $\varphi_{A_\beta} : 2^{A_\beta} \to [0,\infty]$ and ψ_{A_β} for every $\beta \in \kappa$, to find a homeomorphism $h_\beta : 2^{A_\beta} \to T^{A_\beta}$ and a continuous function $f_\beta : 2^{A_\beta} \to (1/2, 2)$ such that $\psi_{A_\beta}(h_\beta(X)) = f_\beta(X)\varphi_{A_\beta}(X)$ for every $X \in 2^{A_\beta}$. Define $\psi : 2^\kappa \to [0,\infty]$ by

$$\psi(X) = \sum_{\beta\in\kappa} \psi_{A_\beta}(h_\beta(X \cap A_\beta)),$$

which implies that

$$\psi(X) = \sum_{\beta\in\kappa} f_\beta(X \cap A_\beta)\varphi_{A_\beta}(X \cap A_\beta) = \sum_{\beta\in\kappa} \tilde{f}_\beta(X)\varphi_\beta(X),$$

where $\tilde{f}_\beta : 2^\kappa \to (1/2, 2)$ is given by $\tilde{f}_\beta(X) = f_\beta(X \cap A_\beta)$ and $\varphi_\beta : 2^\kappa \to [0, \infty]$ is given by $\varphi_\beta(X) = \varphi_{A_\beta}(X \cap A_\beta)$. Since the function $X \mapsto X \cap A_\beta$ from 2^κ to 2^{A_β} is continuous it follows that $\tilde{f}_\beta$ is continuous for all $\beta \in \kappa$ and φ_β is an LSC function for all $\beta \in \kappa$. Observe that ψ is not in general a submeasure on κ.

Now consider the function $\tilde{\varphi}\colon 2^\kappa \to [0, \infty]$ given by $\tilde{\varphi} = \sum_{\beta\in\kappa} \varphi_\beta$. Since φ is a Kadec submeasure it is clear that every φ_{A_β} is a Kadec submeasure and we can apply Corollary 3.11 to see that $\tilde{\varphi}$ is a Kadec submeasure as well. Using that φ is an LSC submeasure it is not difficult to show that $\varphi \le \tilde{\varphi}$. By assumption we have that $\tilde{\varphi} \le M\varphi$ and these inequalities, together with the equality $\mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$, imply that $\mathrm{Exh}(\tilde{\varphi}) = \mathrm{Fin}(\tilde{\varphi})$ with $\mathrm{Exh}(\tilde{\varphi}) = \mathrm{Exh}(\varphi)$. We also see that the metrics d_φ and $d_{\tilde{\varphi}}$ on $\mathrm{Exh}(\varphi)$ as defined in Theorem 3.2 are uniformly equivalent. We can now write

$$\mathcal{I}_\kappa = (\mathrm{Exh}(\varphi), \tau_{d_\varphi}) = (\mathrm{Exh}(\tilde{\varphi}), \tau_{d_{\tilde{\varphi}}}) = (\mathrm{Exh}(\tilde{\varphi}), \tau_{\tilde{\varphi}}).$$

Since $\psi = \sum_{\beta\in\kappa} \tilde{f}_\beta \cdot \varphi_\beta$ it follows from Lemma 5.2 that $\mathrm{Fin}(\tilde{\varphi}) = \psi^{-1}\big([0, \infty)\big)$ and $\tau_{\tilde{\varphi}} = \tau_\psi$. Using that $\mathrm{Exh}(\tilde{\varphi}) = \mathrm{Fin}(\tilde{\varphi})$ we get with the previous equalities that $\mathcal{I}_\kappa = (\psi^{-1}\big([0, \infty)\big), \tau_\psi)$.

Next, define $H : 2^\kappa \to T^\kappa$ coordinatewise by $H(X)_\alpha = h_\beta(X \cap A_\beta)_\alpha$, whenever $\alpha \in A_\beta$. This is obviously a homeomorphism with respect to the underlying product topologies. We now find

$$\psi(X) = \sum_{\alpha\in\kappa} H(X)_\alpha,$$

hence $X \mapsto \sum_{\alpha\in\kappa} H(X)_\alpha$ is continuous on $\mathcal{I}_\kappa$. Notice that whenever $\psi(X) < \infty$, this value equals the ℓ^1_κ-norm of $H(X)$. Using Remark 2.8 we find that $H : \mathcal{I}_\kappa \to T^\kappa \cap \ell^1_\kappa$ is continuous. A symmetric argument yields continuity of its inverse and combining this with the equality $T^\kappa \cap \ell^1_\kappa = \mathfrak{E}^1_\mathrm{c}(\kappa, \kappa)$ and Theorem 2.11 we conclude that $\mathcal{I}_\kappa \approx \mathfrak{E}_\mathrm{c} \times (\kappa_D)^\omega$. □

Theorem 5.5. *Let φ be an LSC submeasure on $\kappa > \omega$ such that $\mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$ and $lw(\mathcal{I}_\kappa) = \lambda > \omega$. Suppose there exists a partition $\{A_\beta : \beta \in \kappa\}$ of κ with $|A_\beta| \le \omega$ for every $\beta \in \kappa$ and with the property that there exists an $M \in \mathbb{N}$ such that for every $X \in \mathrm{Exh}(\varphi)$ we have*

$$\sum_{\beta\in\kappa} \varphi(X \cap A_\beta) \le M\varphi(X).$$

Then $\mathcal{I}_\kappa$ is homeomorphic to $\mathfrak{E}_\mathrm{c} \times (\lambda_\mathrm{D})^\omega \times \kappa_\mathrm{D}$.

Proof. We start with the observation that we may assume without loss of generality that φ is a Kadec submeasure on κ. We can see this as follows. Apply Theorem 3.12 to find a Kadec submeasure ψ_β on A_β such that $\varphi_{A_\beta} \le \psi_\beta \le 2\varphi_{A_\beta}$. Define $\psi\colon 2^\kappa \to [0,\infty]$ by $\psi(X) = \sum_{\beta\in\kappa} \psi_\beta(X \cap A_\beta)$. By Corollary 3.11 ψ is a Kadec submeasure on κ. Using that φ is an LSC submeasure and $\varphi_{A_\beta} \le \psi_\beta$ for all $\beta \in \kappa$ it follows easily that $\varphi \le \psi$. Furthermore, since $\psi_\beta \le 2\varphi_{A_\beta}$ for all $\beta \in \kappa$ we also have that $\psi \le 2M\varphi$ by assumption. Together with the fact that $\mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$ this implies that $\mathrm{Exh}(\psi) = \mathrm{Fin}(\psi)$, with $\mathrm{Exh}(\psi) = \mathrm{Exh}(\varphi)$, and also that the metrics d_φ and d_ψ on $\mathrm{Exh}(\varphi)$ as defined in Theorem 3.2 are uniformly equivalent. So ψ produces the same space $\mathcal{I}_\kappa$ as φ. In addition, it is clear that $\sum_{\beta\in\kappa} \psi(X \cap A_\beta) = \psi(X)$ for all $X \subset \kappa$ so ψ satisfies the conditions of this theorem. We see that we can replace φ by ψ and therefore we will continue the proof assuming that φ is a Kadec submeasure on κ.

Using Proposition 2.6 we take an $n_0 \in \mathbb{N}$ such that $|\{\alpha \in \kappa : \varphi(\{\alpha\}) < 1/n_0\}| = \lambda$. Put $C = \{\alpha \in \kappa : \varphi(\{\alpha\}) < 1/n_0\}$. Using transfinite recursion we construct a collection $\{A'_\beta : \beta \in \lambda\}$ of pairwise disjoint countable subsets of κ where each set A'_β is a union of sets A_γ with $\gamma \in \kappa$, such that $\operatorname{ind}\mathcal{I}_{A'_\beta} > 0$ for all $\beta \in \lambda$. Assume that $\alpha \in \lambda$ is such that A'_β has been found for $\beta < \alpha$. Put $B = \bigcup_{\beta<\alpha} A'_\beta$ and note that $|B| < \lambda$. It follows from Proposition 2.6 that $lw(\mathcal{I}_{C\setminus B}) = \lambda$ and since clearly $\mathrm{Exh}(\varphi_{C\setminus B}) = \mathrm{Fin}(\varphi_{C\setminus B})$ we get from Proposition 4.4 and Remark 4.1 that there is a countable set $C_\alpha \subset C \setminus B$ such that $\operatorname{ind}\mathcal{I}_{C_\alpha} > 0$. Put $A'_\alpha = \bigcup\{A_\beta : A_\beta \cap C_\alpha \neq \emptyset\}$ and note that it is countable and disjoint from A'_β for $\beta < \alpha$. In this way we get the desired sets A'_β for every $\beta \in \lambda$.

If $\lambda = \kappa$, then put $A' = \bigcup_{\beta\in\lambda} A'_\beta$ and define the collection $\mathcal{A}$ of subsets of κ as

$$\mathcal{A} = \{A'_\beta : \beta \in \lambda\} \cup \{A_\beta : A_\beta \cap A' = \emptyset\}.$$

With the subadditivity of φ and the fact that every A'_β is a union of sets A_γ with $\gamma \in \kappa$, it follows easily that $\mathcal{A}$ is a partition of κ that satisfies the conditions of Lemma 5.3, which means that $\mathcal{I}_\kappa \approx \mathfrak{E}_\mathrm{c} \times (\kappa_\mathrm{D})^\omega$.

Now suppose that $\lambda < \kappa$. Again, put $A' = \bigcup_{\beta\in\lambda} A'_\beta$ and now define the collection $\mathcal{A}$ of subsets of κ as

$$\mathcal{A} = \{A'_\beta : \beta \in \lambda\} \cup \{A_\beta : A_\beta \cap (C \setminus A') \neq \emptyset\}.$$

We have that $|\mathcal{A}| = \lambda$, so we can write $\mathcal{A} = \{B_\beta : \beta \in \lambda\}$. We define $A = \bigcup \mathcal{A}$, so $\mathcal{A}$ is a partition of A. Since every element of $\mathcal{A}$ is at most countable we have that $|A| = \lambda$. We know that $\operatorname{ind} \mathcal{I}_{A'_\beta} > 0$ for all $\beta \in \lambda$, which means that $|\{\beta \in \lambda : \operatorname{ind} \mathcal{I}_{B_\beta} > 0\}| = \lambda$. Furthermore, since φ is subadditive and every B_β is a union of sets A_γ with $\gamma \in \kappa$, we have for every $X \subset A$ that

$$\sum_{\beta\in\lambda} \varphi_A(X \cap B_\beta) \leq \sum_{\beta\in\kappa} \varphi(X \cap A_\beta) \leq M\varphi(X) = M\varphi_A(X).$$

Note that $\operatorname{Exh}(\varphi_A) = \operatorname{Fin}(\varphi_A)$ and since it is clear that φ_A is a Kadec submeasure on A we may apply Lemma 5.3 which says that $\mathcal{I}_A$ is homeomorphic to $\mathfrak{E}_c \times (\lambda_D)^\omega$.

Note that $A \supset C$, so if $\alpha \in \kappa \setminus A$ then $\varphi(\{\alpha\}) \geq 1/n_0$ and this means that $\mathcal{I}_{\kappa\setminus A}$ is the collection of all finite subsets of $\kappa \setminus A$ with the discrete topology. Since $|\mathcal{I}_{\kappa\setminus A}| = \kappa$, we have that $\mathcal{I}_{\kappa\setminus A}$ is homeomorphic to κ_D. Clearly, $\mathcal{I}_\kappa$ is homeomorphic to $\mathcal{I}_A \times \mathcal{I}_{\kappa\setminus A}$, which gives the desired result. □

Example 5.6. Until Lemma 5.3, we were only able to classify (up to homeomorphism) the zero-dimensional ideals of Theorem 1.3, the ideals mentioned in Proposition 5.1, and the ideals generated by LSC measures mentioned in Proposition 3.8. Let us now consider the submeasure φ on $\mathbb{N} \times \kappa$ for some $\kappa > \omega$ given by

$$\varphi(X) = \sum_{\alpha\in\kappa} \left(|\pi_\alpha(X) \cap I_0| + \sum_{k=1}^{\infty} \frac{\min(k, |\pi_\alpha(X) \cap I_k|)}{k^2} \right),$$

where $\pi_\alpha(X) = \{n : (n, \alpha) \in X\}$ and $I_k = [2^k, 2^{k+1}) \cap \mathbb{N}$, for $k \in \omega$. This submeasure restricted to $\mathbb{N} \times \{\alpha_0\}$ for any $\alpha_0 \in \kappa$ is in essence the submeasure studied in [10, Example 1.11.1]. In this example, Farah proves that there does not exist a measure on $\mathbb{N}$ generating the same ideal as this submeasure. Note that Theorem 5.5 applies to φ and if $\mathcal{I}_\kappa$ were generated by a measure on $\mathbb{N} \times \kappa$, this would imply that the ideal in Farah's example is generated by a measure as well. Hence, we need Theorem 5.5 to conclude that $\mathcal{I}_\kappa \approx \mathfrak{E}_c \times (\kappa_D)^\omega$.

Question 5.7. Does there exist an LSC submeasure φ on some cardinal number $\kappa > \omega$ such that $\operatorname{Exh}(\varphi) = \operatorname{Fin}(\varphi)$ and $\operatorname{ind} \mathcal{I}_\kappa > 0$, where $\mathcal{I}_\kappa$ is not homeomorphic to a space of the form $\mathfrak{E}_c \times (\lambda_D)^\omega \times \kappa_D$?

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