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ON HOMOTOPICAL AND HOMOLOGICAL Z_n-SETS

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ABSTRACT. A closed subset $A \subset X$ is called a *homological* (*homotopical*) Z_n*-set* if for any $k < n+1$ and any open set $U \subset X$ the relative homology (homotopy) group $H_k(U, U \setminus A)$ ($\pi_k(U, U \setminus A)$) vanishes. A closed subset A of an LC^n-space X is a homotopical Z_n-set if and only if A is a Z_n-set in X in the sense that each map $f : [0,1]^n \to X$ can be uniformly approximated by maps into $X \setminus A$. Applying the Hurewicz isomorphism theorem, we prove that a homotopical Z_2-subset of an LC^1-space is a homotopical Z_n-set if and only if it is a homological Z_n-set in X. From the Künneth formula we derive multiplication, division, and k-root formulas for homological Z_n-sets. We prove that the set $\mathcal{Z}_n^{\mathbb{Z}}(X)$ of homological Z_n-points of a metrizable separable lc^n-space X is of type G_δ in X. We introduce and study the classes $\mathcal{Z}_n^{\mathbb{Z}}$ ($\overline{\mathcal{Z}}_n^{\mathbb{Z}}$, respectively) of topological spaces X with $\mathcal{Z}_n^{\mathbb{Z}}(X) = X$ ($\overline{\mathcal{Z}_n^{\mathbb{Z}}(X)} = X$, respectively) and prove multiplication, division, and k-root formulas for such classes. We also show that a (locally compact lc^n-)space $X \in \mathcal{Z}_n^{\mathbb{Z}}$ has Steinke dimension $\mathsf{t}(X) \geq n+1$ (has cohomological dimension $\dim_G(X) \geq n+1$ for any coefficient group G). A locally compact ANR-space $X \in \mathcal{Z}_\infty^{\mathbb{Z}}$ is not a C-space and has extension dimension e-dim$X \not\leq L$ for any non-contractible CW-complex L.

In this paper we focus on applications of homological methods to studying Z_n-sets in topological spaces. Being higher-dimensional

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counterparts of closed nowhere dense subsets, Z_n-sets are of crucial importance in infinite-dimensional and geometric topologies [6], [10], [16], [17], [31], [9] and play a role also in dimension theory [3] and theory of selections [39]. Z_n-sets were introduced by H. Toruńczyk in [35]. He defined a closed subset A of a topological space X to be a *Z_n-set* if any map $f : I^n \to X$ from the n-dimensional cube $I^n = [0,1]^n$ can be uniformly approximated by maps into $X \setminus A$. Actually, Z_n-sets work properly only in LC^n-spaces where they coincide with so-called homotopical Z_n-sets. By definition, a closed subset A of a topological space X is a *homotopical Z_n-set* if for any open cover $\mathcal{U}$ of X, every map $f : I^n \to X$ can be approximated by a map $f' : I^n \to X \setminus A$, $\mathcal{U}$-homotopic to f. In fact, homotopical Z_n-sets are nothing else but closed locally n-negligible sets in the sense of Toruńczyk [35].

In section 3, we apply the Hurewicz isomorphism theorem to characterize homotopical Z_n-sets A in Tychonoff LC^1-spaces X as homotopical $Z_{\min\{n,2\}}$-sets such that the relative homology groups $H_k(U, U \setminus A)$ vanish for all $k < n+1$ and all open sets $U \subset X$. Having in mind this characterization of homotopical Z_n-sets, we define a closed subset A of a topological space X to be a *homological Z_n-set* (more generally, a *G-homological Z_n-set* for a coefficient group G) if $H_k(U, U \setminus A) = 0$ ($H_k(U, U \setminus A; G) = 0$, respectively) for all $k < n+1$ and all open sets $U \subset X$. Therefore, a homotopical Z_2-set in a Tychonoff LC^1-space X is a homotopical Z_n-set if and only if it is a homological Z_n-set. It should be mentioned that under some restrictions on the space X this characterization of Z_n-sets has been exploited in mathematical literature [17, Proposition 4.2], [18], [28]. The homological characterization of Z_n-sets makes it possible to apply powerful tools of algebraic topology for studying Z_n-sets. Homological Z_n-sets behave like usual Z_n-sets: The union of two G-homological Z_n-sets is a G-homological Z_n-set and so is each closed subset of a G-homological Z_n-set.

In section 4, applying the technique of irreducible homological barriers, we prove that a closed subset $A \subset X$ is a homological Z_n-set in X if each point $a \in A$ is a homological Z_n-point in X and each closed subset $B \subset A$ with $|B| > 1$ can be separated by a homological Z_{n+1}-set. This characterization makes it possible to apply Steinke's separation dimension $\mathsf{t}(\cdot)$ and its transfinite extension $\mathsf{trt}(\cdot)$ to studying homological Z_n-sets. In particular, we prove that a closed subset $A \subset X$ with finite separation dimension $d = \mathsf{t}(A)$ is a G-homological Z_n-set in X if each point $a \in A$ is a G-homological Z_{n+d}-point in X. An infinite version of this result asserts that a closed subset $A \subset X$ having transfinite separation dimension $\mathsf{trt}(A)$ is a G-homological Z_∞-set in X if and only if each point $a \in A$ is a G-homological Z_∞-point in X.

In section 5, we develop the Bockstein theory for G-homological Z_n-sets. The main result is Theorem 5.5 asserting that a subset $A \subset X$ is a G-homological Z_n-set in X if and only if A is an H-homological Z_n-set for all groups $H \in \sigma(G)$, where $\sigma(G) \subset \{\mathbb{Q}, \mathbb{Z}_p, \mathbb{Q}_p, R_p : p \text{ is prime}\}$ is the Bockstein family of G.

The main result of section 6 is Theorem 6.1: For a homological (homotopical) Z_n-set A in a space X and a homological (homotopical) Z_m-set in Y the product $A \times B$ is a homological (homotopical) Z_{n+m+1}-set in $X \times Y$. For ANR's the homotopical version of this result has been proved by T. O. Banakh and Kh. R. Trushchak in [8]. It is interesting that the multiplicative theorem for homological Z_n-sets can be partly reversed, which leads to division and k-root theorems proved in section 8.

In section 9, we apply the obtained results on Z_n-sets to study Z_n-points. We show that the set of Z_n-points in a metrizable separable space X always is a G_δ-set. Moreover, if X is an LC^n-space, then the sets of homological and homotopical points also are G_δ in X.

In sections 10 and 11, we introduce and study the classes $\mathcal{Z}_n$, $\mathcal{Z}_n^{\mathbb{Z}}$ of spaces, all of whose points are homotopical (homological, respectively), Z_n-points, and classes of spaces $\overline{\mathcal{Z}}_n$ ($\overline{\mathcal{Z}}_n^{\mathbb{Z}}$, respectively) containing dense sets of homotopical (homological, respectively) Z_n-points. Applying the results from the preceding sections, we prove multiplication, division, and k-root formulas for these classes.

In sections 12 and 13, we study dimension properties of spaces from the class of $\mathcal{Z}_n^{\mathbb{Z}}$ spaces (i.e., all of whose points are homological Z_n-points). We show that for any space $X \in \mathcal{Z}_n^{\mathbb{Z}}$, the transfinite separation dimension $\mathsf{trt}(X) \geq 1 + n$. If, in addition, X is a locally compact LC^n-space, then X has cohomological dimension $\dim_G(X) \geq n+1$ for any group G. Also, the inequality e-dim$(X) \leq L$ for a CW-complex L implies that $\pi_k(L) = 0$ for all $k \leq n$. Each locally compact locally contractible space $X \in \mathcal{Z}_\infty^{\mathbb{Z}}$ is infinite-dimensional in a rather strong sense: X fails to be a C-space and has extension dimension e-dim$X \not\leq L$ for any non-contractible CW-complex L; see Theorem 13.1.

1. Preliminaries

All topological spaces considered in this paper are Tychonoff; I stands for the closed interval $[0,1]$ and n will denote a non-negative integer or infinity. In the paper we use singular homology $H_*(X;G)$ with coefficients in a non-trivial abelian group G. If $G=\mathbb{Z}$ is the group of integers, we omit the symbol $\mathbb{Z}$ and write $H_*(X)$ instead of $H_*(X;\mathbb{Z})$. By $\widetilde{H}_*(X;G)$, we denote the singular homology of X, reduced in dimension zero.

Let $\mathcal{U}$ be a cover of a space X. Two maps $f,g:Z\to X$ are called

- *$\mathcal{U}$-near* (denoted by $(f,g)\prec\mathcal{U}$) if for any $z\in Z$ there is $U\in\mathcal{U}$ with $\{f(z),g(z)\}\subset U$;
- *$\mathcal{U}$-homotopic* (denoted by $f\underset{\mathcal{U}}{\sim} g$) if there is a homotopy $h:Z\times[0,1]\to X$ such that $h(z,0)=f(z)$, $h(z,1)=g(z)$, and $h(\{z\}\times[0,1])\subset U\in\mathcal{U}$ for all $z\in Z$.

There is also a (pseudo)metric counterpart of these notions. Let ρ be a continuous pseudometric on a space X and $\varepsilon>0$ be a real number. Two maps $f,g:Z\to X$ are called

- *ε-near* if $\mathrm{dist}(f,g)<\varepsilon$ where $\mathrm{dist}(f,g)=\sup_{z\in Z}\rho(f(z),g(z))$;
- *ε-homotopic* if there is a homotopy $h:Z\times[0,1]\to X$ such that $h(z,0)=f(z)$, $h(z,1)=g(z)$, and $\mathrm{diam}_\rho h(\{z\}\times[0,1])<\varepsilon$ for all $z\in Z$.

The following easy lemma helps to reduce the "cover" version of (homotopical) nearness to the "pseudometric" one.

Lemma 1.1. *For any open cover $\mathcal{U}$ of a Tychonoff space X and any compact set $K\subset X$ there is a continuous pseudometric ρ on X such that each 1-ball $B(x,1)=\{x'\in X:\rho(x,x')<1\}$ centered at a point $x\in K$ lies in some set $U\in\mathcal{U}$.*

Proof: Embed the Tychonoff space X into a Tychonoff cube I^κ for a suitable cardinal κ. For each $x\in K$, find a finite index set $F(x)\subset\kappa$ and an open subset $W_x\subset I^{F(x)}$ whose preimage

$V_x = \mathrm{pr}^{-1}_{F(x)}(W_x)$ under the projection $\mathrm{pr}_{F(x)} : X \to I^{F(x)}$ contains the point x and lies in some $U \in \mathcal{U}$. By the compactness of K, the open cover $\{V_x : x \in K\}$ of K contains a finite subcover $\{V_{x_1}, \dots, V_{x_m}\}$. Now consider the finite set $F = \bigcup_{i=1}^m F(x_i)$ and note that each set V_{x_i} is the preimage of the some open set $W_i \subset I^F$ under the projection $\mathrm{pr}_F : X \to I^F$. Let d be any metric on the finite-dimensional cube I^F. By the compactness of $C = \mathrm{pr}_F(K) \subset \bigcup_{i=1}^m W_i$, there is $\varepsilon > 0$ such that each ε-ball centered at a point $z \in C$ lies in some W_i. Finally, define the pseudometric ρ on X letting $\rho(x, x') = \frac{1}{\varepsilon} \cdot d(\mathrm{pr}_F(x), \mathrm{pr}_F(x'))$ for $x, x' \in X$. It is easy to see that each 1-ball centered at any point $x \in K$ lies in some $U \in \mathcal{U}$. □

Let us recall that a space X is called an LC^n*-space* if for each point $x \in X$, each neighborhood U of x, and each $k < n+1$ there is a neighborhood $V \subset U$ of x such that each map $f : \partial I^k \to V$ from the boundary of the k-dimensional cube extends to a map $f : I^k \to U$.

The following homotopy approximation theorem for LC^n-spaces is well known; see [26, p. 159] or [9].

Lemma 1.2. *For any open cover $\mathcal{U}$ of a paracompact LC^n-space X and any $k < n+1$, there is an open cover $\mathcal{V}$ of X such that any two $\mathcal{V}$-near maps $f, g : K \to X$ from a simplicial complex K of dimension $\dim K \le k$ are $\mathcal{U}$-homotopic.*

This lemma has a homological counterpart. A space X is defined to be an lc^n*-space* if for each point $x \in X$, each neighborhood U of x, and each $k < n+1$, there is a neighborhood $V \subset U$ of x such that the homomorphism $i_* : \widetilde{H}_k(V) \to \widetilde{H}_k(U)$ of singular homologies induced by the inclusion map $i : V \to U$ is trivial.

It is known that each LC^n-space is an lc^n-space; see [38]. The converse is true for LC^1-spaces; see [38]. The proof of the following lemma can be found in [11, Lemma 5.4]

Lemma 1.3. *For any paracompact lc^n-space X and any $k < n+1$ there is an open cover $\mathcal{U}$ of X such that any two $\mathcal{U}$-near maps $f, g : K \to X$ defined on a simplicial complex K of dimension $\dim K \le k$ induce the same homomorphisms $f_*, g_* : H_*(K) \to H_*(X)$ on homologies.*

In the sequel, we shall need another three homological properties of lc^n-spaces. First, we recall one well-known fact from homological algebra; see [12, Lemma 16.3].

If the commutative diagram in the category of abelian groups

$$\begin{array}{ccccc} & & A_2 & \longrightarrow & A_3 \\ & & {\scriptstyle i_2}\downarrow & & {\scriptstyle i_3}\downarrow \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 \\ {\scriptstyle i_1}\downarrow & & {\scriptstyle i_4}\downarrow & & \\ C_1 & \longrightarrow & C_2 & & \end{array}$$

has exact middle row, then the finite generacy (or triviality) of the groups $i_1(B_1)$ and $i_3(A_3)$ implies the finite generacy (triviality) of the group $i_4 \circ i_2(A_2)$.

A subset A of a topological space X is called *precompact* if A has compact closure in X.

Lemma 1.4. *Any precompact set C in a Tychonoff lc^n-space X with $n < \infty$ has an open neighborhood U such that the inclusion homomorphisms $H_k(U) \to H_k(X)$ have finitely-generated image for all $k \leq n$.*

Proof: We shall prove this lemma by induction on n. For $n = 0$, the assertion of the lemma follows from the local connectedness of lc^0-spaces. Assume that for some n the lemma has been proved for all $k < n$. Consider the family $\mathcal{K}_n$ of compact subsets K of X having an open neighborhood U such that the inclusion homomorphism $H_n(U) \to H_n(X)$ has finitely generated range. We claim that for any compact subsets $A, B \in \mathcal{K}_n$, the union $A \cup B \in \mathcal{K}_n$. To show this, fix open neighborhoods O_A and O_B of the compacta A and B such that the inclusion homomorphisms $H_n(O_A) \to H_n(X)$ and $H_n(O_B) \to H_n(X)$ have finitely generated ranges. By the inductive assumption, the compact set $A \cap B$ has a neighborhood $O_{A\cap B} \subset O_A \cap O_B$ such that the inclusion homomorphism $H_{n-1}(O_{A\cap B}) \to H_{n-1}(O_A \cap O_B)$ has finitely generated range. Using the complete regularity of X, find two open neighborhoods $U_A \subset O_A$ and $U_B \subset O_B$ of the compacta A and B such that $U_A \cap U_B \subset O_{A\cap B}$. The proof will be completed as soon as we check that the inclusion homomorphism $H_n(U_A \cup U_B) \to H_n(X)$ has finitely-generated range. This follows from the diagram whose rows are Mayer-Vietoris sequences of the pairs $\{U_A, U_B\}$, $\{O_A, O_B\}$, and $\{X, X\}$:

$$\begin{array}{ccccc}
H_n(U_A) \oplus H_n(U_B) & \longrightarrow & H_n(U_A \cup U_B) & \longrightarrow & H_{n-1}(U_A \cap U_B) \\
\downarrow & & \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i_3} \\
H_n(O_A) \oplus H_n(O_B) & \longrightarrow & H_n(O_A \cup O_B) & \longrightarrow & H_{n-1}(O_A \cap O_B) \\
{\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle i_4} & & \downarrow \\
H_n(X) \oplus H_n(X) & \longrightarrow & H_n(X \cup X) & \longrightarrow & H_{n-1}(X \cap X)
\end{array}$$

Since the homomorphisms i_1 and i_3 have finitely generated ranges, so does the homomorphism $i_4 \circ i_2 : H_n(U_A \cup U_B) \to H_n(X)$. This proves the additivity of the family $\mathcal{K}_n$.

To finish the proof of the lemma, it remains to check that each compact subset C of X belongs to the family $\mathcal{K}_n$. By the lc^n-property of X, each point $x \in X$ has an open neighborhood $O_x \subset X$ such that the inclusion homomorphism $H_n(O_x) \to H_n(X)$ has finitely generated range. Take any closed neighborhood $C_x \subset O_x$ of x in the compact subset C. It is clear that $C_x \in \mathcal{K}_n$ for all $x \in C$. By the compactness of C, the cover $\{C_x : x \in C\}$ has finite subcover $C_{x_1}, \dots, C_{x_m}$. Now the additivity of the family $\mathcal{K}_n$ implies that $C = \bigcup_{i \le m} C_{x_i} \in \mathcal{K}_n$. □

Lemma 1.5. *Let X be a locally compact lc^n-space and $V \subset U$ be open subsets of X such that $\overline{V} \subset U$ and $\overline{U}$ is compact. Then for any $k \le n$ the inclusion homomorphism $H_k(X, X \setminus \bar{U}) \to H_k(X, X \setminus \bar{V})$ has finitely generated range.*

Proof: Take open sets $W_1 \subset W_2 \subset W_3 \subset X$ such that W_3 has compact closure in X and $\bar{U} \subset W_1 \subset \overline{W}_1 \subset W_2 \subset \overline{W}_2 \subset W_3$. The excision property for singular homology (see [25, Theorem 2.20]) implies that the inclusion homomorphisms $H_k(W_1, W_1 \setminus \bar{U}) \to H_k(X, X \setminus \bar{U})$ and $H_k(W_3, W_3 \setminus \bar{V}) \to H_k(X, X \setminus \bar{V})$ are isomorphisms. Thus, it suffices to check that the inclusion homomorphism $H_k(W_1, W_1 \setminus \bar{U}) \to H_k(W_3, W_3 \setminus \bar{V})$ has finitely generated range. This will be done with the help of the commutative diagram whose rows are the exact sequences of the pairs $(W_1, W_1 \setminus \bar{U})$, $(W_2, W_2 \setminus \bar{V})$, and $(W_3, W_3 \setminus \bar{V})$ and whose columns are inclusion homomorphisms in homologies:

$$\begin{array}{ccccc}
 & & H_k(W_1, W_1 \setminus \bar{U}) & \longrightarrow & H_{k-1}(W_1 \setminus \bar{U}) \\
 & & {\scriptstyle i_2}\big\downarrow & & {\scriptstyle i_3}\big\downarrow \\
H_k(W_2) & \longrightarrow & H_k(W_2, W_2 \setminus \bar{V}) & \longrightarrow & H_{k-1}(W_2 \setminus \bar{V}) \\
{\scriptstyle i_1}\big\downarrow & & {\scriptstyle i_4}\big\downarrow & & \\
H_k(W_3) & \longrightarrow & H_k(W_3, W_3 \setminus \bar{V}) & &
\end{array}$$

Lemma 1.4 implies that the homomorphisms i_1 and i_3 have finitely generated ranges (because W_2 and $W_1 \setminus \bar{U}$ have compact closures in W_3 and $W_2 \setminus \bar{V}$, respectively). Consequently, the inclusion homomorphism $i_4 \circ i_2 : H_k(W_1, W_1 \setminus \bar{U}) \to H_k(W_3, W_3 \setminus \bar{V})$ also has finitely generated range. □

Lemma 1.6. *Let X be a locally compact lc^n-space and x be a point with $H_k(X, X \setminus \{x\}) = 0$ for some $k \leq n$. Then for any neighborhood $U \subset X$ of x there is a neighborhood $V \subset U$ of x such that the inclusion homomorphism $H_k(X, X \setminus \bar{U}) \to H_k(X, X \setminus \bar{V})$ is trivial.*

Proof: Without loss of generality, the neighborhood U has compact closure in X. By Lemma 1.5, there is a neighborhood $W \subset U$ of x such that the inclusion homomorphism $i_{UW} : H_k(X, X \setminus \bar{U}) \to H_k(X, X \setminus \bar{W})$ has finitely generated range $\mathrm{Im}(i_{UW})$. Pick up finitely many generators $g_1, \dots, g_m$ of the group $\mathrm{Im}(i_{UW})$. The triviality of the homology group $H_k(X, X \setminus \{x\})$ implies the triviality of the inclusion homomorphism $j : H_k(X, X \setminus \bar{W}) \to H_k(X, X \setminus \{x\})$. Then for every $i \leq m$, the image $j(g_i) = 0$ and we can find a neighborhood $V_i \subset W$ of x such that the image of g_i under the inclusion homomorphism $H_k(X, X \setminus \overline{W}) \to H_k(X, X \setminus \bar{V}_i)$ is trivial. For the neighborhood $V = \bigcap_{i \leq m} V_i$ of x, the elements $g_1, \dots, g_m$ have trivial images under the homomorphism $i_{WV} : H_k(X, X \setminus \overline{W}) \to H_k(X, X \setminus \bar{V})$. Since these elements generate the group $\mathrm{Im}(i_{UW})$, the inclusion homomorphism $i_{UV} = i_{WV} \circ i_{UW} : H_k(X, X \setminus \bar{U}) \to H_k(X, X \setminus \bar{V})$ is trivial. □

2. Characterizing locally n-negligible sets

In this section we present a homological characterization of locally n-negligible sets. Following Toruńczyk [35], we define a subset

A of a space X to be *locally n-negligible* if, given $x \in X$, $k < n+1$, and a neighborhood U of x, there is a neighborhood $V \subset U$ of x such that for each $f : (I^k, \partial I^k) \to (V, V \setminus A)$, there is a homotopy $(h_t) : (I^k, \partial I^k) \to (U, U \setminus A)$ with $h_0 = f$ and $h_1(I^k) \subset U \setminus A$. Following [33], we shall say that a pair (X, A) is *n-connected* if $\pi_i(X, A) = 0$ for all $i \le n$.

Theorem 2.1. *For a subset A of a Tychonoff space X, the following conditions are equivalent.*

(1) *A is locally n-negligible;*

(2) *given a simplicial pair (K, L) with $\dim(K) \le n$, a continuous pseudometric ρ on X and maps $\varepsilon : |K| \to (0, \infty)$ and $f : |K| \times \{0\} \cup |L| \times I \to X$ with $\rho(f(x,0), f(x,t)) < \varepsilon(x)$ and $f(x,1) \notin A$ for all $(x,t) \in |L| \times I$, there is $\bar{f} : |K| \times I \to X$ which extends f and satisfies $\rho(\bar{f}(x,t), f(x,0)) < \varepsilon(x)$ and $\bar{f}(x,1) \notin A$ for all $(x,t) \in |K| \times I$;*

(3) *for each open set $U \subset X$ and $k < n+1$, the relative homotopy group $\pi_k(U, U \setminus A)$ vanishes;*

(4) *each $x \in X$ has a basis $\mathfrak{U}_x$ of open neighborhoods with $\pi_k(U, U \setminus A) = 0$ for all $U \in \mathfrak{U}_x$ and $k < n+1$.*

If, in addition, X is an LC^1*-space and A is locally 2-negligible in X, then conditions (1)–(4) are equivalent to*

(5) *for each open $U \subset X$ and $k < n+1$, the relative homology group $H_k(U, U \setminus A)$ vanishes;*

(6) *each $x \in X$ has a basis $\mathfrak{U}_x$ of open neighborhoods with $H_k(U, U \setminus A) = 0$ for all $U \in \mathfrak{U}_x$ and $k < n+1$.*

Proof: The equivalences (1) ⇔ (2) ⇔ (3) ⇔ (4) have been proved by Toruńczyk [35, Theorem 2.3] for normal spaces X. But because of Lemma 1.1, his proof also works for any Tychonoff X.

Now assume that A is a locally 2-negligible set in an LC^1-space X.

The implication (5) ⇒ (6) is trivial while (3) ⇒ (5) easily follows from the Hurewicz isomorphism theorem [33, Ch. 7, Sec. 5, Theorem 4] (see also [25, Theorem 4.37]) because X is an LC^0-space and A is locally 1-negligible in X.

It remains to prove the implication (6) ⇒ (1). Take any point $x \in X$ and a neighborhood $U \subset X$ of x. We lose no generality assuming that U is connected.

The LC^1-property of X yields a neighborhood $V \subset U$ of x such that any map $f : \partial I^2 \to V$ extends to a map $\bar{f} : I^2 \to U$. Moreover, by (6) we can choose the neighborhood V so that $H_k(V, V \setminus A) = 0$ for all $k < n+1$. Replacing V by a connected component containing the point x, if necessary, we may assume that V is connected. Then the LC^0-property of X implies that V is path connected and the local 1-negligibility of A in X implies that $V \setminus A$ is path-connected too.

We claim that every map $f : (I^k, \partial I^k) \to (V, V \setminus A)$ with $k < n+1$ is homotopic in U to a map $\tilde{f} : (I^k, \partial I^k) \to (U \setminus A, U \setminus A)$ which will imply the local n-negligibility of A in X. This will be done by induction. For $k \le 2$, the local k-negligibility of A in X follows from the local 2-negligibility and there is nothing to prove. So we assume that the local $(k-1)$-negligibility of A has been proved for some $k > 2$. Then the implication (1) $\Rightarrow$ (3) yields $\pi_i(V, V \setminus A) = 0$ for all $i < k$. This means that the pair $(V, V \setminus A)$ is $(k-1)$-connected, which makes legal applying the relative Hurewicz isomorphism theorem [33, Ch. 7, Sec. 5, Theorem 4] to this pair.

Fix any point $* \in \partial I^k$ and let $x_0 = f(*)$. Then the map f can be considered as an element of the relative homotopy group $\pi_k(V, V \setminus A, *)$. The relative Hurewicz isomorphism theorem applied to the pair $(V, V \setminus A)$ implies that $\pi'_k(V, V \setminus A, x_0) = 0$, where $\pi'_k(V, V \setminus A, x_0)$ is the quotient group of the homotopy group $\pi_k(V, V \setminus A, x_0)$ by the normal subgroup G generated by the elements $[\gamma] \cdot [\alpha] - [\alpha]$, $[\alpha] \in \pi_k(V, V \setminus A, x_0)$, and $[\gamma] \in \pi_1(V \setminus A, x_0)$, where $\cdot : \pi_1(V \setminus A, x_0) \times \pi_k(V, V \setminus A, x_0) \to \pi_k(V, V \setminus A, x_0)$ is the left action of the fundamental group on the relative k-th homotopy group, see [33, Ch. 7, Sec. 2] or [25]. It follows from

$$0 = \pi'_k(V, V \setminus A, x_0) = \pi_k(V, V \setminus A, x_0)/G$$

that the subgroup G coincides with $\pi_k(V, V \setminus A, x_0)$. Now consider the homomorphism $i_* : \pi_k(V, V \setminus A, x_0) \to \pi_k(U, U \setminus A, x_0)$ induced by the inclusion of pairs $i : (V, V \setminus A) \to (U, U \setminus A)$. We claim that i_* is a null-homomorphism. Since the group $\pi_k(V, V \setminus A, x_0)$ is generated by the elements $[\gamma] \cdot [\alpha] - [\alpha]$, $[\alpha] \in \pi_k(V, V \setminus A, x_0)$, and $[\gamma] \in \pi_1(V \setminus A, x_0)$, it suffices to check that $i_*([\gamma] \cdot [\alpha] - [\alpha]) = 0$ for any such $[\alpha]$ and $[\gamma]$. Let $j_* : \pi_1(V \setminus A, x_0) \to \pi_1(U \setminus A, x_0)$ be the homomorphism induced by the inclusion $j : (V, x_0) \to (U, x_0)$. The local 2-negligibility of A in U and the choice of the set V implies that

$j_* = 0$ and, thus, the action of the element $j_*([\gamma]) \in \pi_1(U \setminus A, x_0)$ on $\pi_k(U, U \setminus A, x_0)$ is trivial. The naturality of the action of the fundamental group on the homotopy groups (see [33, Ch. 7, Sec. 3, Lemma 1]) implies that

$$i_*([\gamma] \cdot [\alpha] - [\alpha]) = j_*([\gamma]) \cdot i_*([\alpha]) - i_*([\alpha]) = i_*([\alpha]) - i_*([\alpha]) = 0.$$

Thus, the homomorphism $i_* : \pi_k(V, V \setminus A, x_0) \to \pi_k(U, U \setminus A, x_0)$ is trivial, which means that the map $f : (I^k, \partial I^k, *) \to (V, V \setminus A, x_0)$ is null homotopic in $(U, U \setminus A, x_0)$, and this completes the proof of the local k-negligibility of A in X. □

3. Homotopical and homological Z_n-sets

In this section, in order to study Z_n-sets, we apply the characterization of locally n-negligible sets established in the preceding section. We recall the definition of a Z_n-set and its homotopical and homological versions.

Definition 3.1. A closed subset A of a topological space X is defined to be

- a *Z_n-set* if, for any open cover $\mathcal{U}$ of X and a map $f : I^n \to X$, there is a map $f' : I^n \to X \setminus A$ which is $\mathcal{U}$-near to f;
- a *homotopical Z_n-set* in X if, for any open cover $\mathcal{U}$ of X and any map $f : I^n \to X$, there is a map $f' : I^n \to X \setminus A$ which is $\mathcal{U}$-homotopic to f;
- a *G-homological Z_n-set* in X, where G is a coefficient group, if, for any open set $U \subset X$ and any $k < n+1$, the relative homology group $H_k(U, U \setminus A; G) = 0$;
- a *homological Z_n-set* in X if it is a $\mathbb{Z}$-homological Z_n-set in X.

A point x in a space X is called a *Z_n-point* if the singleton $\{x\}$ is a Z_n-set in X. By analogy we define homotopical and homological Z_n-points.

The following theorem reveals interplay between various versions of Z_n-sets.

Theorem 3.2. *Let G be a non-trivial Abelian group and X be a Tychonoff space.*

(1) *A subset A of X is a homotopical Z_n-set in X if and only if A is closed and locally n-negligible set in X.*

(2) *Each homotopical Z_n-set in X is a Z_n-set in X.*
(3) *If X is an* LC^n*-space, then each Z_n-set in X is a homotopical Z_n-set.*
(4) *Each homotopical Z_n-set in X is a G-homological Z_n-set.*
(5) *Each G-homological Z_0-set in X is a homotopical Z_0-set.*
(6) *Each G-homological Z_1-set in X is a Z_1-set.*

Proof: The first item follows immediately from Theorem 2.1 while the second is trivial. The third item was proved in [35, Corollary 3.3] and follows from Lemma 1.2. The fourth item can be easily derived from [13, Corollary 10.6] (of the relative Hurewicz theorem). The fifth item follows from the fact that a relative homology group $H_0(U, U \setminus A; G)$ vanishes if and only if each path-connected component of U meets the set $U \setminus A$.

To prove the sixth item, assume that A is a G-homological Z_1-set in X. Being a G-homological Z_0-set, A is a homotopical Z_0-set in X. To show that A is a Z_1-set in X, fix an open cover $\mathcal{U}$ of X and a map $f : I \to X$. Consider the open cover $f^{-1}(\mathcal{U}) = \{f^{-1}(U) : U \in \mathcal{U}\}$ of the interval $I = [0,1]$ and find a sequence $0 = t_0 < t_1 < \cdots < t_m = 1$ such that for each $i \le m$, the interval $[t_{i-1}, t_i]$ lies in $f^{-1}(U_i)$ for some set $U_i \in \mathcal{U}$. Let $W_m = U_m$ and $W_i = U_i \cap U_{i+1}$ for $i < m$.

Since $H_0(W_i, W_i \setminus A; G) = 0$, the path-connected component of W_i containing the point $f(t_i)$ meets the set $W_i \setminus A$ at some point x_i. We claim that the points x_{i-1} and x_i lie in the same path-connected component of $U_i \setminus A$. Assuming the converse, we would get a nontrivial cycle $\alpha = g \cdot x_{i-1} - g \cdot x_i$ in $H_0(U_i \setminus A; G)$ with $g \in G$ being any non-zero element. On the other hand, this cycle is the boundary of an obvious 1-chain β in U_i and thus vanishes in the homology group $H_0(U_i; G)$. But this contradicts the exact sequence $0 = H_1(U_i, U_i \setminus A; G) \to H_0(U_i \setminus A; G) \to H_0(U_i; G)$ for the pair $(U_i, U_i \setminus A)$.

Therefore x_{i-1} and x_i lie in the same path-connected component of $U_i \setminus A$, ensuring the existence of a continuous map $g_i : [t_{i-1}, t_i] \to U_i \setminus A$ with $g_i(t_{i-1}) = x_{i-1}$ and $g_i(t_i) = x_i$. The maps g_i and $i \le m$ compose a single continuous map $g : [0,1] \to X \setminus A$ which is $\mathcal{U}$-near to f, witnessing the Z_1-set property of A. □

Combining Theorem 3.2(1) with Theorem 2.1, we get the following important characterization of homotopical Z_n-sets.

Theorem 3.3. *A homotopical Z_2-set A in a Tychonoff* LC^1*-space X is a homotopical Z_n-set in X if and only if A is a homological Z_n-set in X.*

Remark 3.4. Theorem 3.3 has been known as folklore, and its particular cases have appeared in literature; see e.g., [28], [18], [17]. It should also be mentioned that a substantial part of [17] is devoted to closed sets of infinite codimension (coinciding with our homological Z_∞-sets).

Next, we establish some elementary properties of G-homological Z_n-sets. From now on, G is a non-trivial abelian group.

Proposition 3.5. *Let A and B be G-homological Z_n-sets in a topological space X.*

1. *Any closed subset $F \subset A$ is a G-homological Z_n-set in X.*
2. *The union $A \cup B$ is a G-homological Z_n-set in X.*

Proof: (1) We should check that $H_k(U, U \setminus F; G) = 0$ for all open sets $U \subset X$ and all $k < n+1$. The G-homology Z_n-set property of A yields $H_k(U, U \setminus A; G) = 0$. Since $(U \setminus F, U \setminus A) = (U \setminus F, (U \setminus F) \setminus A)$, we get also $H_{k-1}(U \setminus F, U \setminus A; G) = 0$. Writing the exact sequence of the triple $(U, U \setminus F, U \setminus A)$ gives us $H_k(U, U \setminus F; G) = 0$.

(2) Given any $k < n+1$ and any open set $U \subset X$ consider the exact sequence of the triple $(U, U \setminus A, U \setminus (A \cup B))$:

$$0 = H_k(U \setminus A, U \setminus (A \cup B); G) \to H_k(U, U \setminus (A \cup B); G) \to H_k(U, U \setminus A; G) = 0$$

and conclude that $H_k(U, U \setminus (A \cup B); G) = 0$. □

Next, we show that in the definition of a G-homological Z_n-set we can require that U runs over some base for X.

Proposition 3.6. *Let $\mathcal{B}$ be a base of the topology of a topological space X. A closed set $A \subset X$ is a G-homological Z_n-set in X if and only if $H_k(U, U \setminus A; G) = 0$ for all $k < n+1$ and $U \in \mathcal{B}$.*

Proof: The "only if" part of the theorem is trivial. To prove the "if" part, assume that $H_k(U, U \setminus A; G) = 0$ for all $k < n+1$ and all sets $U \in \mathcal{B}$. Let $\mathcal{U}_k$ be the family of all open subsets $U \subset X$ such that $H_k(U, U \setminus A; G) = 0$. By induction on $k < n+1$, we shall show that $\mathcal{U}_k$ consists of all open sets in X.

First, we verify the case $k = 0$. Take any non-empty open set $U \subset X$. The equality $H_0(U, U \setminus A; G) = 0$ will follow as soon as we show that each path-connected component C of U intersects the set $U\setminus A$. Fix any point $c \in C$ and find a neighborhood $V \in \mathcal{B}$ of x lying in U. The equality $H_0(V, V \setminus A; G) = 0$ implies that $V \setminus A$ meets each path-connected component of the set V, in particular, the path-connected component C' of the point c in V. Since $C' \subset C$, we conclude that $U \setminus A \supset V \setminus A$ meets the component C, which completes the proof of the case $k = 0$.

Assuming that for some positive $k < n + 1$ the family $\mathcal{U}_{k-1}$ consists of all open subsets of X, we first show that the family $\mathcal{U}_k$ is closed under finite unions. Indeed, given any two sets $U, V \in \mathcal{U}_k$ we can write down the piece of the relative Mayer-Vietoris exact sequence

$$H_k(U, U \setminus A; G) \oplus H_k(V, V \setminus A; G) \to H_k(U \cup V, (U \cup V) \setminus A; G) \to H_{k-1}(U \cap V, U \cap V \setminus A; G)$$

and conclude that $U \cup V \in \mathcal{U}_k$ because $U \cap V \in \mathcal{U}_{k-1}$. Since $\mathcal{U}_k$ contains the base $\mathcal{B}$, it contains all possible finite unions of sets of the base. Because the singular homology theory has compact support, $\mathcal{U}$ contains all possible unions of sets from the base $\mathcal{B}$ and consequently, $\mathcal{U}_k$ consists of all open sets in X. □

Corollary 3.7. *A subset A of a topological space X is a G-homological Z_n-set in X if and only if there is an open cover $\mathcal{U}$ of X such that for every $U \in \mathcal{U}$ the intersection $U \cap A$ is a G-homological Z_n-set in U.*

4. Detecting G-homological Z_n-sets with help of partitions

In this section we apply the technique of irreducible homological barriers to detect G-homological Z_n-sets with help of their partitions.

A closed subset B of a topological space X is called an *irreducible barrier* for a non-zero homology element $\alpha \in H_n(X, X \setminus B; G)$ if for every closed subset $A \subset B$ with $A \neq B$ the image $i_A^B(\alpha)$ under the inclusion homomorphism $i_A^B : H_n(X, X \setminus B; G) \to H_n(X, X \setminus A; G)$ is trivial. We shall say that a subset A of a topological space X is *separated* by a subset $B \subset X$ if the complement $A \setminus B$ is disconnected. In dimension theory, closed separating sets are also referred to as *partitions*; see [23, 1.1.3].

We shall need two elementary properties of irreducible barriers, which were also exploited in [14] and [4].

Lemma 4.1. *Let X be a topological space and G be a non-trivial abelian group.*

(1) *Each closed subset A of X with $H_n(X, X \setminus A; G) \neq 0$ for some $n \geq 0$ contains an irreducible barrier $B \subset A$ for some element $\alpha \in H_n(X, X \setminus B; G)$.*
(2) *If A is an irreducible barrier for some element $\alpha \in H_n(X, X \setminus A; G)$, then $H_{n+1}(X, X \setminus B; G) \neq 0$ for any closed subset $B \subset A$ separating A.*

Proof: (1) Given a closed subset $A \subset X$ and a non-zero element $\alpha \in H_n(X, X \setminus A; G) \neq 0$, consider the family $\mathcal{B}$ of closed subsets of A such that for every $B \in \mathcal{B}$ the image $i^A_B(\alpha)$ under the inclusion homomorphism $i^A_B : H_n(X, X \setminus A; G) \to H_n(X, X \setminus B; G)$ is not trivial. We claim that $\mathcal{B}$ contains a minimal element. This will follow from Zorn's Lemma as soon as we prove that the intersection $\cap\mathcal{C}$ of any linearly ordered subfamily $\mathcal{C} \subset \mathcal{B}$ belongs to $\mathcal{B}$. Since singular homology has compact support, the homology group $H_n(X, X \setminus \cap\mathcal{C}; G)$ is the direct limit of the groups $H_n(X, X \setminus C; G)$, $C \in \mathcal{C}$; see [33, Ch. 4, Sec. 8, Theorem 13]. Since all the elements $i^A_C(\alpha)$, $C \in \mathcal{C}$, are not trivial, so neither is the element $i^A_{\cap C}(\alpha)$, which means that $\cap\, \mathcal{C} \in \mathcal{B}$. Now, Zorn's Lemma yields a minimal element B in $\mathcal{B}$. Let $\beta = i^A_B(\alpha) \neq 0$. The minimality of B implies that for any closed subset $C \subset B$ with $C \neq B$, we get $i^A_C(\alpha) = i^B_C(\beta) = 0$, which means that B is an irreducible barrier for β.

(2) Assume that A is an irreducible barrier for some element $\alpha \in H_n(X, X \setminus A; G)$ and let B be a closed subset separating A. Write $A \setminus B = U \cup V$ as the union of two disjoint non-empty open sets $U, V \subset A$. Let $C = V \cup B = A \setminus U$ and $D = U \cup B = A \setminus V$. The irreducibility of A for α yields $i^A_C(\alpha) = i^A_D(\alpha) = 0$. Now, writing a Mayer-Vietoris exact sequence for the pair $(X, X \setminus B) = (X, X \setminus C \cup X \setminus D)$, we get

$$H_{n+1}(X, X \setminus B; G) \xrightarrow{\partial} H_n(X, X \setminus A; G) \xrightarrow{f} H_n(X, X \setminus C; G) \oplus H_n(X, X \setminus D; G).$$

Since $f(\alpha) = (i_C^A(\alpha), -i_D^A(\alpha)) = (0,0)$, $0 \neq \alpha = \partial(\beta)$ for some nontrivial element $\beta \in H_{n+1}(X, X \setminus B; G)$. □

Irreducible barriers help us to prove the following theorem detecting G-homological Z_n-sets.

Theorem 4.2. *A closed subset A of a topological space X is a G-homological Z_n-set in X if each point of A is a G-homological Z_n-point in X and each closed subset B of A with $|B| > 1$ can be separated by a G-homological Z_{n+1}-set.*

Proof: Assume that each point of a closed subset $A \subset X$ is a G-homological Z_n-point and each closed subset $B \subset A$ with $|B| > 1$ can be separated by a G-homological Z_{n+1}-set $B \subset A$ in X. Assuming that A fails to be a G-homological Z_n-set, find an open subset $U \subset X$ and $k < n+1$ with $H_k(U, U \setminus A; G) \neq 0$. By Lemma 4.1(1), the set $A \cap U$ contains an irreducible barrier B for a non-zero element $\alpha \in H_k(U, U \setminus B; G)$. The set B must contain more than one point, since singletons are G-homological Z_n-sets in X. By our assumption the closure $\overline{B}$ in X can be separated by a G-homological Z_{n+1}-set C. Then $C \cap U$ separates $\overline{B} \cap U = B$ and hence $H_{k+1}(U, U \setminus C; G) \neq 0$ by Lemma 4.1(2). But this is not possible because C is a G-homological Z_{n+1}-set in X. □

Theorem 4.2 will be applied to show that a subset $A \subset X$ is a G-homological Z_n-set in X if each point $a \in A$ is a Z_{n+d}-point in X where $d = \mathsf{trt}(A)$ is the separation dimension of A. The separation dimension $\mathsf{t}(\cdot)$ was introduced by Günter Steinke [34] and later was extended to the transfinite separation dimension $\mathsf{trt}(\cdot)$ by F. G. Arenas, V. A. Chatyrko, and M. L. Puertas [2] as follows: given a topological space X, we write

- $\mathsf{trt}(X) = -1$ if and only if $X = \emptyset$;
- $\mathsf{trt}(X) \leq \alpha$ for an ordinal α if any closed subset $B \subset X$ with $|B| \geq 2$ can be separated by a closed subset $P \subset B$ with $\mathsf{trt}(P) < \alpha$.

A space X is defined to be trt-*dimensional* if $\mathsf{trt}(X) \leq \alpha$ for some ordinal α. In this case, the ordinal $\mathsf{trt}(X) = \min\{\alpha : \mathsf{trt}(X) \leq \alpha\}$ is called the (transfinite) separation dimension of X. If X is not trt-dimensional, then we write $\mathsf{trt}(X) = \infty$ and assume that $\alpha < \infty$ for all ordinals α.

By transfinite induction, one can show that $\mathsf{trt}(X) \le \mathrm{trind}(X)$, where $\mathrm{trind}(X)$ is the transfinite extension of the small inductive dimension $\mathrm{ind}(X)$; see [2, Proposition 2.9]. This implies that each countable-dimensional completely-metrizable space is trt-dimensional (because $\mathrm{trind}(X) < \omega_1$ [23, 7.1.9]). On the other hand, each trt-dimensional compact space is a C-space; see [2, Proposition 4.7]. Observe that $\mathsf{trt}(X) \le 0$ if and only if X is hereditarily disconnected space. The strongly infinite-dimensional totally disconnected Polish space X constructed in [23, 6.2.4] has $\mathsf{trt}(X) = 0$ and $\mathrm{trind}\,(X) = \infty$ while a compactification $c(X)$ of X with strongly countable-dimensional remainder (the famous example of Pol) has $\mathsf{trt}(X) = \omega$ and $\mathrm{trind}(X) = \infty$. Thus, (even on the compact level), the gap between $\mathsf{trt}(X)$ and $\mathrm{trind}(X)$ can be huge. However, for finite-dimensional metrizable compacta X, the separation dimension $\mathsf{trt}(X)$ coincides with the usual dimension $\dim(X)$ (and hence with $\mathrm{trind}(X)$); see [34].

Theorem 4.3. *A closed subspace A of a space X with $m = \mathsf{trt}(A) < \omega$ is a G-homological Z_n-set in X provided each point $a \in A$ is a G-homological Z_{n+m}-point in X.*

Proof: The proof is by induction of the number $m = \mathsf{trt}(A)$. The assertion is trivial if $m = -1$ (which means that A is empty). Assume that for some number m the theorem has been proved for all sets A with $\mathsf{trt}(A) < m$. Take a closed subset $A \subset X$ with $\mathsf{trt}(A) = m$ and all points $a \in A$ being G-homological Z_{n+m}-points in X. Assuming that A fails to be a Z_n-set in X, find an open set $U \subset X$ and a number $k < n + 1$ with $H_k(U, U \setminus A; G) \neq 0$. By Lemma 4.1(1), the set $A \cap U$ contains an irreducible barrier $B \subset A \cap U$ for some non-zero element $\beta \in H_k(U, U \setminus B; G)$. Since singletons are G-homological Z_n-sets in X, $|B| > 1$. Since $\mathsf{trt}(A) \le m$, there is a partition $C \subset B$ of B with $d = \mathsf{trt}(C) < m$. Then all points of the set C are G-homological Z_{n+d+1}-points. Now the inductive assumption guarantees that C is a G-homological Z_{n+1}-set in U, which contradicts Lemma 4.1(2) because C separates the irreducible barrier B. □

By the same method one can prove an infinite version of this theorem.

Theorem 4.4. *A closed* trt*-dimensional subspace A of a space X is a G-homological Z_∞-set in X if and only if each point $a \in A$ is a G-homological Z_∞-point in X.*

5. Bockstein Theory for G-homological Z_n-sets

In this section, we study the interplay between G-homological Z_n-sets for various coefficient groups G. Our principal instrument here is the Universal Coefficient Formula (UCF) expressing the homology with respect to an arbitrary coefficient group via homology with respect to the group $\mathbb{Z}$ of integers. The following its form is taken from [25, Ch. 3, Corollary 3A.4].

Lemma 5.1 (Universal Coefficient Formula)**.** *For each pair (X, A) and all $n \geq 1$, there is a natural exact sequence*

$$0 \to H_n(X, A) \otimes G \to H_n(X, A; G) \to H_{n-1}(X, A) * G \to 0$$

and this sequence splits (though non-naturally).

Here $G \otimes H$ and $G * H$ stand for the tensor and torsion products of the groups G and H, respectively. We need some information about those products.

First, some notation. By Π we denote the set of prime numbers. We recall that a group G is *divisible* if it is *divisible by each number* $n \in \mathbb{N}$. The latter means that for any $g \in G$ there is $x \in G$ with $n \cdot x = g$. By $\mathrm{Tor}(G) = \{x \in G : \exists n \in \mathbb{N} \text{ with } nx = 0\}$, we denote the *torsion part* of G. It is well known that $\mathrm{Tor}(G)$ is the direct sum of p-torsion parts $p\text{-}\mathrm{Tor}(G) = \{x \in G : \exists k \in \mathbb{N} \ \ p^k x = 0\}$ where p runs over prime numbers.

The following useful result can be found in [22].

Lemma 5.2. *Let G_0 and G_1 be non-trivial abelian groups and p be a prime number.*

1. *The tensor product $G_0 \otimes G_1$ contains an element of infinite order if and only if both groups G_0 and G_1 contain elements of infinite order.*
2. *The torsion product $G_0 * G_1$ contains an element of order p if and only if G_0 and G_1 contain elements of order p.*
3. *The tensor product $G_0 \otimes G_1$ contains an element of order p if and only if for some $i \in \{0, 1\}$, either G_i is not divisible by*

p and p-$\mathrm{Tor}(G_{1-i})$ is not divisible by p or else G_i/p-$\mathrm{Tor}(G_i)$ is not divisible by p and p-$\mathrm{Tor}(G_{1-i}) \neq 0$ is divisible by p.

The following fact follows immediately from the UCF.

Proposition 5.3. *Each homological Z_n-set in a space X is a G-homological Z_n-set in X for any coefficient group G.*

Next, we show that the study of G-homological Z_n-sets for an arbitrary coefficient group G can be reduced to studying H-homological Z_n-sets for coefficient groups H from the countable family of so-called Bockstein groups.

There are many (more or less standard) notations for Bockstein groups. We follow those of [30] and [22]:

- $\mathbb{Q}$ is the group of rational numbers;
- $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ is the cyclic group of a prime order p;
- R_p is the group of rational numbers whose denominator is not divisible by p.
- $\mathbb{Q}_p = \mathbb{Q}/R_p$ the quasicyclic p-group.

By $\mathfrak{B} = \{\mathbb{Q}, \mathbb{Z}_p, \mathbb{Q}_p, R_p : p \in \Pi\}$, we denote the family of all Bockstein groups.

To each abelian group G, assign the Bockstein family $\sigma(G) \subset \mathfrak{B}$ containing the group

- $\mathbb{Q}$ if and only if the group $G/\mathrm{Tor}(G) \neq 0$ is divisible;
- R_p if and only if G/p-$\mathrm{Tor}(G)$ is not divisible by p;
- $\mathbb{Z}_p$ if and only if p-$\mathrm{Tor}(G)$ is not divisible by p;
- $\mathbb{Q}_p$ if and only if p-$\mathrm{Tor}(G) \neq 0$ is divisible by p.

In particular, $\sigma(G) = \{G\}$ for every Bockstein group $G \in \mathfrak{B}$.

The Bockstein families are helpful because of the following fact which can be easily derived from Lemma 5.2.

Lemma 5.4. *For abelian groups G and H, the tensor product $G \otimes H$ is trivial if and only if $G \otimes B$ is trivial for every group $B \in \sigma(H)$.*

Combining this lemma with the UCF, we will obtain the principal result of this section.

Theorem 5.5. *A closed subset A of a topological space X is a G-homological Z_n-set in X if and only if A is an H-homological Z_n-set in X for every group $H \in \sigma(G)$.*

Proof: Assume that a closed subset $A \subset X$ fails to be a G-homological Z_n-set in X and find $k < n+1$ and an open set $U \subset X$ with $H_k(U, U \setminus A; G) \neq 0$. The UCF implies that either $H_k(U, U \setminus A) \otimes G \neq 0$ or $H_{k-1}(U, U \setminus A) * G \neq 0$.

In the latter case, the group $H_{k-1}(U, U \setminus A) * G$ contains an element of a prime order p and then both the groups $H_{k-1}(U, U \setminus A)$ and G contain elements of order p by Lemma 5.2(2). Consequently, $\sigma(G)$ contains a (quasi)cyclic p-group $H \in \{\mathbb{Z}_p, \mathbb{Q}_p\}$ and then $H_{k-1}(U, U \setminus A) * H \neq 0$. Applying the UCF, we conclude that $H_k(U, U \setminus A; H) \neq 0$.

Next, we assume that $H_k(U, U \setminus A) \otimes G \neq 0$. Then Lemma 5.2 implies that $H_k(U, U \setminus A) \otimes H \neq 0$ for some group $H \in \sigma(G)$. Applying the UCF, we conclude that $H_k(U, U \setminus A; H) \neq 0$, which means that A fails to be an H-homological Z_n-set in X. This proves the "if" part of the theorem.

To prove the "only if" part, assume that A fails to be a H-homological Z_n-set in X for some group $H \in \sigma(G)$. Then for some $k < n+1$ and an open set $U \subset X$, the group $H_k(U, U \setminus A; H)$ is not trivial. The UCF yields that either $H_k(U, U \setminus A) \otimes H \neq 0$ or $H_{k-1}(U, U \setminus A) * H \neq 0$.

In the first case, the tensor product $H_k(U, U \setminus A) \otimes G \neq 0$ by Lemma 5.2. In the second case, $H_{k-1}(U, U \setminus A) * H$ contains an element of a prime order p and so do the groups $H_{k-1}(U, U \setminus A)$ and H. The inclusion $H \in \sigma(G)$ implies then that G contains an element of order p and hence $H_{k-1}(U, U \setminus A) * G \neq 0$. In both cases, the UCF implies that $H_k(U, U \setminus A; G) \neq 0$, which means that A fails to be a G-homological Z_n-set in X. □

Next, we study the interplay between G-homological Z_n-sets for various Bockstein groups G.

Theorem 5.6. *Let A be a closed subset of a space X and p be a prime number.*

1. *If A is an R_p-homological Z_n-set in X, then A is a $\mathbb{Q}$-homological and $\mathbb{Z}_p$-homological Z_n-set in X.*
2. *If A is a $\mathbb{Z}_p$-homological Z_n-set in X, then A is a $\mathbb{Q}_p$-homological Z_n-set in X.*
3. *If A is a $\mathbb{Q}_p$-homological Z_{n+1}-set in X, then A is a $\mathbb{Z}_p$-homological Z_n-set in X.*

(4) *A is an R_p-homological Z_n-set in X, provided that A is a $\mathbb{Q}$-homological Z_n-set in X and a $\mathbb{Q}_p$-homological Z_{n+1}-set in X.*

Proof: (1) Assuming that A is not a $\mathbb{Q}$-homological Z_n-set in X, find a number $k<n+1$ and an open set $U \subset X$ with $H_k(U, U \setminus A; \mathbb{Q}) \neq 0$. Since $\mathbb{Q}$ is torsion free, the UCF implies that $H_k(U, U \setminus A)$ contains an element of infinite order and then $H_k(U, U \setminus A) \otimes R_p \neq 0$. Applying the UCF once more, we obtain that $H_k(U, U \setminus A; R_p) \neq 0$, which means that A is not an R_p-homological Z_n-set in X.

Next, assume that A is not a $\mathbb{Z}_p$-homological Z_n-set in X and find an integer number $k \leq n$ and an open set $U \subset X$ with $H_k(U, U \setminus A; \mathbb{Z}_p) \neq 0$. The UCF implies that either $H_k(U, U \setminus A) \otimes \mathbb{Z}_p \neq 0$ or $H_{k-1}(U, U \setminus A) * \mathbb{Z}_p \neq 0$. In the latter case, the group $H_{k-1}(U, U \setminus A)$ contains an element of order p and by Lemma 5.2, $H_{k-1}(U, U \setminus A) \otimes R_p \neq 0$. The UCF implies that $H_{k-1}(U, U \setminus A; R_p) \neq 0$, which means that A fails to be an R_p-homological Z_{k-1}-set in X.

So assume that $H_k(U, U \setminus A) \otimes \mathbb{Z}_p \neq 0$. If $H_k(U, U \setminus A)$ contains an element of order p, then we can proceed as in the preceding case to prove that A fails to be an R_p-homological Z_k-set in X. So we can assume that p-$\mathrm{Tor}(H_k(U, U \setminus A)) = 0$, which implies that the torsion part $\mathrm{Tor}(H_k(U, U \setminus A))$ is divisible by p. Taking into account that $H_k(U, U \setminus A) \otimes \mathbb{Z}_p \neq 0$, we can apply Lemma 5.2(3) to find an element $a \in H_k(U, U \setminus A)$ not divisible by p. This element cannot belong to the torsion part of $H_k(U, U \setminus A)$. Hence, the group $H_k(U, U \setminus A)$ contains an element of infinite order and so does the tensor product $H_k(U, U \setminus A) \otimes R_p = H_k(U, U \setminus A; R_p)$. This means that A fails to be an R_p-homological Z_k-set in X.

(2) Now assume that A fails to be a $\mathbb{Q}_p$-homological Z_n-set in X and find $k \leq n$ and an open set $U \subset X$ with $H_k(U, U \setminus A; \mathbb{Q}_p) \neq 0$. Applying the UCF, we get $H_k(U, U \setminus A) \otimes \mathbb{Q}_p \neq 0$ or $H_{k-1}(U, U \setminus A) * \mathbb{Q}_p \neq 0$. In the latter case, $H_{k-1}(U, U \setminus A) * \mathbb{Z}_p \neq 0$ and hence $H_k(U, U \setminus A; \mathbb{Z}_p) \neq 0$. If $H_k(U, U \setminus A) \otimes \mathbb{Q}_p \neq 0$, then, by Lemma 5.2, the group $H_k(U, U \setminus A)$ contains an element not divisible by p and then $H_k(U, U \setminus A) \otimes \mathbb{Z}_p \neq 0$ by Lemma 5.2. In both the cases, the UCF implies that $H_k(U, U \setminus A; \mathbb{Z}_p) \neq 0$, which means that A fails to be a $\mathbb{Z}_p$-homological Z_n-set in X.

(3) Assume that A fails to be a $\mathbb{Z}_p$-homological Z_n-set in X and find $k \le n$ and an open set $U \subset X$ with $H_k(U, U \setminus A; \mathbb{Z}_p) \ne 0$. The UCF yields $H_k(U, U \setminus A) \otimes \mathbb{Z}_p \ne 0$ or $H_{k-1}(U, U \setminus A) * \mathbb{Z}_p \ne 0$. In the latter case, $H_{k-1}(U, U \setminus A)$ contains an element of order p and then $H_{k-1}(U, U \setminus A) * \mathbb{Q}_p \ne 0$. Applying the UCF once more, we get $H_k(U, U \setminus A; \mathbb{Q}_p) \ne 0$, which means that A fails to be a $\mathbb{Q}_p$-homological Z_k-set.

Now assume that $H_k(U, U \setminus A) \otimes \mathbb{Z}_p \ne 0$. If $H_k(U, U \setminus A)$ contains an element of order p, then $H_k(U, U \setminus A) * \mathbb{Q}_p \ne 0$ and $H_{k+1}(U, U \setminus A; \mathbb{Q}_p) \ne 0$ by the UCF. This means that A fails to be a $\mathbb{Q}_p$-homological Z_{k+1}-set in X. So it remains to consider the case when the group $H = H_k(U, U \setminus A)$ has trivial p-torsion part p-Tor(H). Since $H \otimes \mathbb{Z}_p \ne 0$, the group $H = H/p\text{-Tor}(H)$ contains an element a not divisible by p; see Lemma 5.2. Then $H_k(U, U \setminus A) \otimes \mathbb{Q}_p = H \otimes \mathbb{Q}_p \ne 0$, according to Lemma 5.2. Now the UCF implies that $H_k(U, U \setminus A; \mathbb{Q}_p) \ne 0$, which means that A fails to be a $\mathbb{Q}_p$-homological Z_k-set in X.

(4) Assume that A fails to be an R_p-homological Z_n-set in X. Then, for some $k \le n$ and an open set $U \subset X$, the homology group $H_k(U, U \setminus A; R_p) = H_k(U, U \setminus A) \otimes R_p$ is not trivial. By Lemma 5.2, the group $H_k(U, U \setminus A)$ contains an element of infinite order or an element of order p. In the first case, the group $H_k(U, U \setminus A; \mathbb{Q}) = H_k(U, U \setminus A) \otimes \mathbb{Q}$ is not trivial, which means that A fails to be a $\mathbb{Q}$-homological Z_k-set. In the second case, the subgroup

$$H_k(U, U \setminus A) * \mathbb{Q}_p \subset H_{k+1}(U, U \setminus A; \mathbb{Q}_p)$$

is not trivial, which means that A fails to be a $\mathbb{Q}_p$-homological Z_{k+1}-set in X. □

Combining Theorem 5.5 and Theorem 5.6, we get the following.

Corollary 5.7. *Let A be a closed subset of a topological space X.*

(1) *If A is a G-homological Z_n-set in X for some coefficient group G, then A is an H-homological Z_n in X for some divisible group $H \in \{\mathbb{Q}, \mathbb{Q}_p : p \in \Pi\}$.*
(2) *A is an F-homological Z_n-set in X for a field F if and only if A is a G-homological Z_n-set in X for the field $G \in \{\mathbb{Q}, \mathbb{Z}_p : p \in \Pi\}$ with $\{G\} = \sigma(F)$.*

6. Multiplication Theorem for homotopical and homological Z_n-sets

We discuss so-called multiplication theorems for homotopical and R-homological Z_n sets with R being a *principal ideal domain*. The latter means that R is a commutative ring with unit and without zero divisors in which each proper ideal is generated by a single element. A typical example of a principal ideal domain is the ring $\mathbb{Z}$ of integers.

According to the Künneth formula [33, Ch. 5, Sec. 3, Theorem 10], for closed subsets $A \subset X$ and $B \subset Y$ in topological spaces X and Y and a principal ideal domain R, the relative homology group

$$H_n(X \times Y, X \times Y \setminus A \times B; R)$$

is isomorphic to the direct sum of the R-modules

$$[H_*(X, X \setminus A; R) \otimes_R H_*(Y, Y \setminus B; R)]_n = \oplus_{i+j=n} H_i(X, X \setminus A; R) \otimes_R H_j(Y, Y \setminus B; R)$$

and

$$[H_*(X, X \setminus A; R) *_R H_*(Y, Y \setminus B; R)]_{n-1} = \oplus_{i+j=n-1} H_i(X, X \setminus A; R) *_R H_j(Y, Y \setminus B; R).$$

Here $G \otimes_R H$ and $G *_R H$ stand for the tensor and torsion products of R-modules G and H over R. If $R = \mathbb{Z}$, then we omit the subscript and write $G \otimes H$ and $G * H$. It is known that the torsion product over a field F always is trivial. In this case,

$$H_n(X \times Y, X \times Y \setminus A \times B; F) = [H_*(X, X \setminus A; F) \otimes_F H_*(Y, Y \setminus B; F)]_n.$$

With help of the Künneth formula and Theorem 5.5, we shall prove multiplication formulas for homological and homotopical Z_n-sets.

Theorem 6.1. *Let $A \subset X$ and $B \subset Y$ be closed subsets in Tychonoff spaces X and Y, and let G be a coefficient group.*

1. *If A is a G-homological Z_n-set in X and B is a G-homological Z_m-set in Y, then $A \times B$ is a G-homological Z_{n+m}-set in $X \times Y$. Moreover, if $\sigma(G) \subset \{\mathbb{Q}, \mathbb{Z}_p, R_p : p \in \Pi\}$, then $A \times B$ is a G-homological Z_{n+m+1}-set in $X \times Y$.*
2. *If A is a homotopical Z_n-set in X and B is a homotopical Z_m-set in Y, then $A \times B$ is a homotopical Z_{n+m+1}-set in $X \times Y$.*

Proof: (1) In light of Theorem 5.5, it suffices to prove the first item only for a Bockstein group $G \in \{\mathbb{Q}, \mathbb{Z}_p, R_p, \mathbb{Q}_p : p \in \Pi\}$. Assume that A is a G-homological Z_n-set in X and B is a G-homological Z_m-set in Y.

First, we prove the second part of the first item, assuming that $G \in \{\mathbb{Q}, \mathbb{Z}_p, R_p : p \in \Pi\}$ is a principal ideal domain. We have to check that $A \times B$ is a G-homological Z_{n+m+1}-set in $X \times Y$, which means that $H_k(W, W \setminus A \times B; G) = 0$ for every $k \leq n+m+1$ and every open set $W \subset X \times Y$. By Lemma 3.6, it suffices to consider the case $W = U \times V$ for some open sets $U \subset X$ and $V \subset Y$.

By the Künneth formula, the homology group $H_k(U \times V, U \times V \setminus A \times B; G)$ is isomorphic to the direct sum of the groups

$\oplus_{i+j=k} H_i(U, U \setminus A; G) \otimes_G H_j(V, V \setminus B; G)$ and

$\oplus_{i+j=k-1} H_i(U, U \setminus A; G) *_G H_j(V, V \setminus B; G)$.

Observe that for every i and j with $i+j \leq n+m+1$, either $i \leq n$ or $j \leq m$. In the first case, the group $H_i(U, U \setminus A; G)$ is trivial (because A is a G-homological Z_n-set in X); in the second case, the group $H_j(V, V \setminus B; G) = 0$. Hence, the above tensor and torsion products are trivial and so is the group $H_k(U \times V, U \times V \setminus A \times B; G)$.

Next, assume that $G = \mathbb{Q}_p$ for a prime p. We need to prove that $A \times B$ is a G-homological Z_{n+m}-set in $X \times Y$. As in the preceding case, this reduces to showing that for every $k \leq n+m$ and open sets $U \subset X$ and $V \subset Y$; the homology group $H_k(U \times V, U \times V \setminus A \times B; \mathbb{Q}_p)$ is trivial. By the Universal Coefficient Formula (UCF), this group is isomorphic to the direct sum of the groups $H_k(U \times V, U \times V \setminus A \times B) \otimes \mathbb{Q}_p$ and $H_{k-1}(U \times V, U \times V \setminus A \times B) * \mathbb{Q}_p$. So it suffices to prove the triviality of these two groups.

The triviality of the group $H_{k-1}(U \times V, U \times V \setminus A \times B) * \mathbb{Q}_p$ will follow as soon as we prove that the group $H_{k-1}(U \times V, U \times V \setminus A \times B)$ has no p-torsion. Assuming the converse and using the Künneth formula, we conclude that either the group

$$\oplus_{i+j=k-1} H_i(U, U \setminus A) \otimes H_j(V, V \setminus B)$$

or the group

$$\oplus_{i+j=k-2} H_i(U, U \setminus A) * H_j(V, V \setminus B)$$

contains an element of order p.

In the latter case, there are i and j with $i+j = k-2$ such that $H_i(U, U \setminus A) * H_j(V, V \setminus B)$ contains an element of order p. By

Lemma 5.2, the groups $H_i(U, U \setminus A)$ and $H_j(V, V \setminus B)$ contain elements of order p and then $H_i(U, U \setminus A) * \mathbb{Q}_p$ and $H_j(U, U \setminus B) * \mathbb{Q}_p$ are not trivial, so neither are the homology groups $H_{i+1}(U, U \setminus A; \mathbb{Q}_p)$ and $H_{j+1}(V, V \setminus B; \mathbb{Q}_p)$ by the UCF. Since A is a $\mathbb{Q}_p$-homological Z_n-set in X and B is a $\mathbb{Q}_p$-homological Z_m-set in Y, the non-triviality of the two latter homology groups implies that $i \geq n$ and $j \geq m$ and hence, $k - 2 = i + j \geq n + m \geq k$, which is a contradiction.

Next, we consider the case when, for some i and j with $i + j = k - 1$, the group $H_i(U, U \setminus A) \otimes H_j(V, V \setminus B)$ contains an element of order p. To simplify notation, let $H_i = H_i(U, U \setminus A)$ and $H_j = H_j(V, V \setminus B)$. Since $H_i \otimes H_j$ contains an element of order p, we can apply Lemma 5.2(3) to conclude that either H_i or H_j contains an element of order p. Without loss of generality, H_i contains an element of order p. It follows from the UCF and the fact that A is a $\mathbb{Q}_p$-homological Z_n-set in X that $i \geq n$. Then $j = k - 1 - i \leq n + m - 1 - i < m$. Since B is a $\mathbb{Q}_p$-homological Z_m-set in Y, the UCF implies that $H_j = H_j(V, V \setminus B)$ has no p-torsion. Taking into account that H_i has p-torsion and $H_i \otimes H_j$ contains an element of order p, we can apply Lemma 5.2(3) to conclude that $H_j = H_j/p\text{-Tor}(H_j)$ is not divisible by p and hence $H_j(V, V \setminus B) \otimes \mathbb{Q}_p = H_j \otimes \mathbb{Q}_p \neq 0$ by Lemma 5.2. The UCF implies now that $H_j(V, V \setminus B; \mathbb{Q}_p) \neq 0$, which contradicts the fact that B is a $\mathbb{Q}_p$-homological Z_m-set (because $j \leq m$). This completes the proof of the triviality of the group $H_{k-1}(U \times V, U \times V \setminus A \times B) * \mathbb{Q}_p$.

Next, we check the triviality of the group

$$H_k(U \times V, U \times V \setminus A \times B) \otimes \mathbb{Q}_p.$$

By the Künneth formula, the group $H_k(U \times V, U \times V \setminus A \times B)$ is isomorphic to the direct sum of the groups

$$\oplus_{i+j=k} H_i(U, U \setminus A) \otimes H_j(V, V \setminus B)$$

and

$$\oplus_{i+j=k-1} H_i(U, U \setminus A) * H_j(V, V \setminus B).$$

The second sum is a torsion group and hence, its tensor product with $\mathbb{Q}_p$ is trivial. So, it suffices to prove that

$$(H_i(U, U \setminus A) \otimes H_j(V, V \setminus B)) \otimes \mathbb{Q}_p = 0$$

for any i and j with $i + j = k$. Since $k \leq n + m$, either $i \leq n$ or $j \leq m$. If $i \leq n$, we can use the fact that A is a $\mathbb{Q}_p$-homological Z_n-set in X to conclude that the homology group $H_i(U, U \setminus A; \mathbb{Q}_p)$ is trivial and so is the tensor product $H_i(U, U \setminus A) \otimes \mathbb{Q}_p$ according to the UCF. The associativity of the tensor product implies that $\big(H_i(U, U\setminus A)\otimes H_j(V, V\setminus B)\big)\otimes \mathbb{Q}_p$ is trivial as well. If $j\le m$, then the triviality of the above tensor product follows from the $\mathbb{Q}_p$-homological Z_m-set property of B.

(2) Let A be a homotopical Z_n-set in X and let B be a homotopical Z_m-set in Y. It will be convenient to uniformize the notations and put $X_0=X$, $X_1=Y$, $A_0=A$, $A_1=B$, $k_0=n$, $k_1=m$, and $k=k_0+k_1+1$. So we need to prove that the product $A_0\times A_1$ is a homotopical Z_k-set in $X_0\times X_1$ provided A_i is a homotopical Z_{k_i}-set in X_i for $i\in\{0,1\}$.

Let $\mathcal{U}$ be an open cover of $X_0\times X_1$ and $f=(f_0,f_1):I^k\to X_0\times X_1$ be a map of the k-dimensional cube. Then the sets $f_i(I^k)\subset X_i$, $i\in\{0,1\}$, are compact and so is their product. A standard compactness argument yields finite covers $\mathcal{U}_i$ of $f_i(I^k)$ by open subsets of X_i for $i\in\{0,1\}$ such that for any sets $U_0\in\mathcal{U}_0$ and $U_1\in\mathcal{U}_1$, the product $U_0\times U_1$ lies in some set $U\in\mathcal{U}$.

By Lemma 1.1, each space X_i has a continuous pseudometric ρ_i such that any 1-ball $B(x,1)=\{x'\in X_i:\rho_i(x,x')<1\}$ centered at a point $x\in f_i(I^k)$ lies in some set $U\in\mathcal{U}_i$. Then the pseudometric $\rho=\max\{\rho_0,\rho_1\}$, i.e.,

$$\rho((x_0,x_1),(x_0',y_1'))=\max\{\rho_0(x_0,x_0'),\rho_1(x_1,x_1')\}$$

on $X_0\times X_1$ has a similar feature: any 1-ball $B(a,1)$ centered at a point $a\in f_0(I^k)\times f_1(I^k)$ lies in some set $U\in\mathcal{U}$.

Let T be a triangulation of I^k so fine that the image $f(\sigma)$ of any simplex $\sigma\in T$ has ρ-diameter $<1/3$. Let K_0 be the k_0-dimensional skeleton of the triangulation T and K_1 be the dual skeleton consisting of all simplexes of the barycentric subdivision of T that do not meet the skeleton K_0. It is well known (and easy to see) that K_1 has dimension $k_1=k-k_0-1$ and each point $z\in I^k$ lying in a simplex $\sigma\in T$ can be written as $z=(1-\lambda(z))z_0+\lambda(z)z_1$ for some points $z_i\in K_i$, $i\in\{0,1\}$, and some real number $\lambda(z)\in[0,1]$. This number is uniquely determined by the point z and is equal to zero if and only if $z\in K_0$ and equal to 1 if and only if $z\in K_1$. Moreover, the point z_i is uniquely determined by z if and only if $z\notin K_{1-i}$ for $i\in\{0,1\}$. This means that the cube I^k has the structure of a subset of the join K_0*K_1. This structure allows us to write the cube I^k as $I^k=K_{\le 1/2}\cup K_{\ge 1/2}$, where $K_{\le 1/2}=\{z\in I^k:\lambda(z)\le 1/2\}$ and $K_{\ge 1/2}=\{z\in I^k:\lambda(z)\ge 1/2\}$.

Let $\ell : [0,1] \to [0,1]$ be the piecewise linear map determined by the conditions $\ell(0) = 0 = \ell(1/2)$ and $\ell(1) = 1$. Combined with the joint structure $K_0 * K_1$, the map ℓ induces two piecewise-linear maps $h_i : I^k \to I^k$, $i \in \{0,1\}$, assigning to each point $z = \lambda_0 z_0 + \lambda_1 z_1$ with $z_i \in K_i$, $\lambda_0 + \lambda_1 = 1$ the point $h_i(z) = \lambda'_0 z_0 + \lambda'_1 z_1$ where $\lambda'_i = \ell(\lambda_i)$ and $\lambda'_{1-i} = 1 - \lambda'_i$. The crucial property of the maps h_i is that $h_0(K_{\le 1/2}) \subset K_0$, $h_1(K_{\ge 1/2}) \subset K_1$ and both h_0 and h_1 are $\mathcal{S}$-homotopic to the identity map of I^k with respect to the cover $\mathcal{S}$ of I^k by maximal simplexes of the triangulation T.

Applying Theorem 2.1 to the homotopical Z_{k_i}-set $A_i \subset X_i$, find a map $g_i : K_i \to X_i \setminus A_i$, 1/6-homotopic to the map $f_i|K_i : K_i \to X_i$ with respect to the pseudometric ρ_i. Since K_i is a subcomplex of I^k, we may apply Borsuk's homotopy extension theorem (see [33, Ch. 1, Exercise D.2]) and extend the map $g_i : K_i \to X_i$ to a map $\bar{g}_i : I^k \to X_i$, 1/6-homotopic to f_i. Finally, consider the map $\tilde{f} = (\tilde{f}_0, \tilde{f}_1) : I^k \to X_0 \times X_1$, where $\tilde{f}_i = \bar{g}_i \circ h_i$. We claim that $\tilde{f}$ is $\mathcal{U}$-homotopic to f and $\tilde{f}(I^k) \cap (A_0 \times A_1) = \emptyset$.

The $\mathcal{S}$-homotopy of the maps h_i to the identity implies the $\bar{g}_i(\mathcal{S})$-homotopy of $\bar{g}_i \circ h_i$ to $\bar{g}_i$. Now observe that for each simplex σ of the triangulation T, we get

$$\operatorname{diam}(\bar{g}_i(\sigma)) \le \operatorname{diam}(f_i(\sigma)) + 2\operatorname{dist}(f_i, \bar{g}_i) < \tfrac{1}{3} + 2\tfrac{1}{6} = \tfrac{2}{3}.$$

Consequently, $\tilde{f}_i$ is 2/3-homotopic to $\bar{g}_i$. Since $\bar{g}_i$ is 1/6-homotopic to f_i, we get that $\tilde{f}_i$ is 1-homotopic to f_i and consequently, $\tilde{f}$ is 1-homotopic to f. Now the choice of the pseudometric ρ implies that $\tilde{f}$ is $\mathcal{U}$-homotopic to f.

So it remains to prove that $\tilde{f}(z) \notin A_1 \times A_2$ for every point $z \in I^k$. Indeed, if $z \in K_{\le 1/2}$, then $h_0(z) \in K_0$ and $\tilde{f}_0(z) = \bar{g}_0 \circ h_0(z) \in \bar{g}_0(K_0) \subset X_0 \setminus A_0$. A similar argument yields $\tilde{f}_1(z) \notin A_1$, provided $z \in K_{\ge 1/2}$. □

7. Functions $z(A,X)$ and $z^G(A,X)$

It will be convenient to write Theorem 6.1 in terms of the functions

$$z(A,X) = \sup\{n \in \omega : A \text{ is a homotopical } Z_n\text{-set in } X\} \text{ and}$$
$$z^G(A,X) = \sup\{n \in \omega : A \text{ is a } G\text{-homological } Z_n\text{-set in } X\}$$

defined for a closed subset A of a space X and a coefficient group G. In this definition, we put $\sup\emptyset = -1$. So $z^G(A,X) = -1$ if and only if A is not a G-homotopical Z_0-set in X. For a point $x \in X$ of a topological space X, we write $z^G(x, X)$ instead of $z^G(\{x\}, X)$.

Rewriting theorems 3.2, 3.3, 4.3, 5.5, and 5.6 in the terms of the functions $z(A, X)$ and $z^G(A, X)$, we obtain the following.

Theorem 7.1. *Let A be a closed subset of a topological space X, and G be a coefficient group. Then*

(1) $z(A, X) \le z^{\mathbb{Z}}(A, X) \le z^G(A, X) = \min_{H \in \sigma(G)} z^H(A, X)$;
(2) $z(A, X) = z^{\mathbb{Z}}(A, X)$ *if X is an* LC^1*-space and* $z(A, X) \ge 2$;
(3) $z^G(A, X) + \mathsf{trt}(A) \ge \min_{a \in A} z^G(a, X)$;
(4) $\min\{z^{\mathbb{Q}}(A, X), z^{\mathbb{Q}_p}(A, X) - 1\} \le z^{R_p}(A, X)$ *and* $z^{R_p}(A, X) \le \min\{z^{\mathbb{Q}}(A, X), z^{\mathbb{Z}_p}(A, X)\}$;
(5) $z^{\mathbb{Z}_p}(A, X) \le z^{\mathbb{Q}_p}(A, X) \le z^{\mathbb{Z}_p}(A, X) + 1$.

Also, Theorem 6.1 can be rewritten in the form of multiplication formulas.

Theorem 7.2 (Multiplication Formulas)**.** *Let A and B be closed subsets in Tychonoff spaces X and Y, and let G be a coefficient group. Then*

(1) $z^G(A \times B, X \times Y) \ge z^G(A, X) + z^G(B, Y)$;
(2) $z^G(A \times B, X \times Y) \ge z^G(A, X) + z^G(B, Y) + 1$ *if* $\sigma(G) \subset \{\mathbb{Q}, \mathbb{Z}_p, R_p : p \in \Pi\}$;
(3) $z(A \times B, X \times Y) \ge z(A, X) + z(B, Y) + 1$.

8. Division and k-root formulas for homological Z_n-sets

It turns out that inequalities in Theorem 7.2 can be partly reversed, which leads to so-called division and k-root formulas. We start with division formulas for Bockstein coefficient groups.

Lemma 8.1. *Let $A \subset X$, $B \subset Y$, and $C \subset Z$ be closed subsets in topological spaces X, Y, and Z, and let p be a prime number. Then*

(1) $z^F(A \times B, X \times Y) = z^F(A, X) + z^F(B, X) + 1$ *for a field F*;
(2) $z^{\mathbb{Q}_p}(A \times B, X \times Y) \le z^{\mathbb{Q}_p}(A, X) + z^{\mathbb{Z}_p}(B, Y) + 2$;
(3) $z^{R_p}(A, X) + 1 \ge \min\{z^{R_p}(A \times B, X \times Y) - z^{\mathbb{Q}}(B, Y), z^{R_p}(A \times C, X \times Z) - z^{\mathbb{Q}_p}(C, Z)\}$ *if* $\max\{z^{\mathbb{Q}}(B, Y), z^{\mathbb{Q}_p}(C, Z)\} < \infty$.

Proof: (1) Let F be a field. By Theorem 6.1(1), $z^F(A\times B, X\times Y)\geq z^F(A,X)+z^F(B,Y)+1$. So it remains to prove the reverse inequality, which is trivial if one of the numbers $n = z^F(A,X)$ or $m = z^F(B,Y)$ is infinite. So assume that $n, m < \infty$ and find open sets $U \subset X$ and $V \subset Y$ such that the homology groups $H_{n+1}(U, U\setminus A; F)$ and $H_{m+1}(V, V\setminus B; F)$ are not trivial. Their tensor product $H_{n+1}(U, U\setminus A; F)\otimes_F H_{m+1}(V, V\setminus B; F)$ over the field F is not trivial as well. Now the Künneth formula implies that the group $H_{n+m+2}(U\times V, U\times V\setminus A\times B; F)$ is not trivial, which means that $A\times B$ is not an F-homological Z_{n+m+2}-set in $X\times Y$ and hence, $z^F(A\times B, X\times Y)\leq n+m+1 = z^F(A,X)+z^F(B,Y)+1$.

(2) follows from (1) and Theorem 7.1(5):

$$z^{\mathbb{Q}_p}(A\times B, X\times Y)\leq z^{\mathbb{Z}_p}(A\times B, X\times Y)+1 = \\ z^{\mathbb{Z}_p}(A,X)+z^{\mathbb{Z}_p}(B,Y)+2\leq z^{\mathbb{Q}_p}(A,X)+z^{\mathbb{Z}_p}(B,Y)+2.$$

(3) Let $n = z^{R_p}(A,X)$ and find an open set $U\subset X$ and $i\leq n+1$ with $H_i(U, U\setminus A; R_p)\neq 0$. The Universal Coefficient Formula (UCF) yields $H_i(U, U\setminus A)\otimes R_p\neq 0$. By Lemma 5.2, the group $H_i(U, U\setminus A)$ contains an element of infinite order or of order p.

In the first case, use the fact that B is not a $\mathbb{Q}$-homological Z_{m+1}-set in Y for $m = z^{\mathbb{Q}}(B,Y) < \infty$ to find an open set $V\subset Y$ with $H_{m+1}(V, V\setminus B;\mathbb{Q})\neq 0$. By the UCF, $H_{m+1}(V, V\setminus B)\otimes\mathbb{Q}\neq 0$ and hence, $H_{m+1}(V, V\setminus B)$ contains an element of infinite order and so does the tensor product

$$H_i(U, U\setminus A)\otimes H_{m+1}(V, V\setminus B),$$

which lies in the homology group $H_{i+m+1}(U\times V, U\times V\setminus A\times B)$ according to the Künneth formula. Then the tensor product

$$H_{i+m+1}(U\times V, U\times V\setminus A\times B)\otimes R_p = H_{i+m+1}(U\times V, U\times V\setminus A\times B; R_p)$$

is not trivial. This means that $A\times B$ fails to be an R_p-homological Z_{i+m+1}-set in $X\times Y$ and thus $z^{R_p}(A\times B, X\times Y)\leq i+m\leq 1+n+m$. Consequently, $z^{R_p}(A,X)+1 = n+1\geq z^{R_p}(A\times B, X\times Y) - z^{\mathbb{Q}}(B,Y)$.

Next, assume that $H_i(U, U\setminus A)$ contains an element of order p and use the fact that the set C fails to be a $\mathbb{Q}_p$-homological Z_{m+1}-set in Z for the number $m = z^{\mathbb{Q}_p}(C,Z) < \infty$ to find an open set $V\subset Z$ with $H_{m+1}(V, V\setminus C;\mathbb{Q}_p)\neq 0$. Then either $H_{m+1}(V, V\setminus C)\otimes\mathbb{Q}_p\neq 0$ or $H_m(V, V\setminus C)*\mathbb{Q}_p\neq 0$. In the latter case, $H_m(V, V\setminus C)$

contains an element of order p and so does the torsion product $H_i(U,U\setminus A)*H_m(V,V\setminus C)$ which lies in $H_{i+m+1}(U\times V,U\times V\setminus A\times C)$ by the Künneth formula. Applying Lemma 5.2, we see that

$$H_{i+m+1}(U\times V,U\times V\setminus A\times C)\otimes R_p = H_{i+m+1}(U\times V,U\times V\setminus A\times C;R_p)$$

is not trivial. This means that $A\times C$ is not an R_p-homological Z_{i+m+1}-set in $X\times Z$.

Finally, assume that $H_{m+1}(V,V\setminus C)\otimes\mathbb{Q}_p\neq 0$. By Lemma 5.2, the group $H_{m+1}(V,V\setminus C)/p\text{-Tor}(H_{m+1}(V,V\setminus C))$ is not divisible by p and hence, the tensor product $H_i(U,U\setminus A)\otimes H_{m+1}(V,V\setminus C)$ contains an element of order p. By the Künneth formula, the latter tensor product lies in $H_{i+m+1}(U\times V,U\times V\setminus A\times C)$. Applying Lemma 5.2 again, we obtain that the tensor product

$$H_{i+m+1}(U\times V,U\times V\setminus A\times C)\otimes R_p = H_{i+m+1}(U\times V,U\times V\setminus A\otimes C;R_p)$$

is not trivial. In both cases, $A\times C$ is not an R_p-homological Z_{i+m+1}-set in $X\times Z$ and hence, $z^{R_p}(A\times C,X\times Z)\le i+m\le n+1+z^{\mathbb{Q}_p}(C,Z)$. Then $z^{R_p}(A,X)+1=n+1\ge z^{R_p}(A\times C,X\times Z)-z^{\mathbb{Q}_p}(C,Z)$. □

Now we can establish division formulas in the general case. First, to each coefficient group G, assign two families $d(G),\varphi(G)\subset\{\mathbb{Q},\mathbb{Z}_p,\mathbb{Q}_p : p\in\Pi\}$ as follows. Put

- $d(\mathbb{Q})=\varphi(\mathbb{Q})=\{\mathbb{Q}\}$,
- $d(\mathbb{Z}_p)=\varphi(\mathbb{Z}_p)=\{\mathbb{Z}_p\}$,
- $d(\mathbb{Q}_p)=\varphi(\mathbb{Q}_p)=\{\mathbb{Z}_p\}$,
- $d(R_p)=\{\mathbb{Q},\mathbb{Q}_p\}$, $\varphi(R_p)=\{\mathbb{Q},\mathbb{Z}_p\}$

and let

$$d(G)=\bigcup_{H\in\sigma(G)}d(H)\quad\text{and}\quad\varphi(G)=\bigcup_{H\in\sigma(G)}\varphi(H).$$

In particular, $d(\mathbb{Z})=\{\mathbb{Q},\mathbb{Q}_p : p\in\Pi\}$ is the family of divisible Bockstein groups and $\varphi(\mathbb{Z})=\{\mathbb{Q},\mathbb{Z}_p : p\in\Pi\}$ is the family of Bockstein fields.

Theorem 8.2 (Division Formulas). *Let $A\subset X$ and $B\subset Y$ be closed subsets in topological spaces and let G be a coefficient group. Then*

(1) $z^G(A\times B,X\times Y)\le 2+z^G(A,X)+\sup_{F\in\varphi(G)}z^F(B,Y)$;
(2) $z^G(A\times B,X\times Y)\le 1+z^G(A,X)+\sup_{H\in d(G)}z^H(B,Y)$ *provided* $\sigma(G)\subset\{\mathbb{Q},\mathbb{Z}_p,R_p : p\in\Pi\}$.

Proof: (1) The inequality $z^G(A\times B, X\times Y)\le 2+z^G(A,X)+\sup_{F\in\varphi(G)} z^F(B,Y)$ is trivial if $n=z^G(A,X)$ is infinite. So assume that n is finite and using Theorem 7.1(1), find a Bockstein group $D\in\sigma(G)$ with $z^D(A,X)=n$. Theorem 5.5 implies that $z^G(A\times B, X\times Y)\le z^D(A\times B, X\times Y)$. If D is a field, then $\{D\}=\varphi(D)\subset\varphi(G)$ and, by Lemma 8.1,

$$z^G(A\times B, X\times Y)\le z^D(A\times B, X\times Y)= 1+z^D(A,X)+z^D(B,Y)\le 1+z^G(A,X)+\sup_{H\in\varphi(G)} z^H(B,Y).$$

If $D=R_p$ for some p, then $\varphi(R_p)=\{\mathbb{Q},\mathbb{Z}_p\}\subset\varphi(G)$ and by Lemma 8.1, we get

$$\begin{aligned} z^G(A\times B, X\times Y) &\le z^{R_p}(A\times B, X\times Y)\\ &\le 1+z^{R_p}(A,X)+\max\{z^{\mathbb{Q}}(B,Y), z^{\mathbb{Q}_p}(B,Y)\}\\ &\le 1+z^{R_p}(A,X)+\max\{z^{\mathbb{Q}}(B,Y), z^{\mathbb{Z}_p}(B,Y)+1\}\\ &\le 2+z^{D}(A,X)+\max\{z^{\mathbb{Q}}(B,Y), z^{\mathbb{Z}_p}(B,Y)\}\\ &\le 2+z^{G}(A,X)+\sup_{F\in\varphi(G)} z^F(B,Y). \end{aligned}$$

If $D=\mathbb{Q}_p$ for some p, then, according to Lemma 8.1(2),

$$z^G(A\times B, X\times Y)\le z^{\mathbb{Q}_p}(A\times B, X\times Y)\le 2+z^{\mathbb{Q}_p}(A,X)+z^{\mathbb{Z}_p}(B,Y)\le 2+z^G(A,X)+\sup_{F\in\varphi(G)} z^F(B,Y).$$

(2) If $\sigma(G)\subset\{\mathbb{Q},\mathbb{Z}_p,R_p : p\in\Pi\}$, then the last case in the preceding item is excluded and we can repeat the preceding argument to prove the inequality

$$z^G(A\times B, X\times Y)\le 1+z^G(A,X)+\sup_{H\in d(G)} z^H(B,Y). \qquad \square$$

Theorem 8.3. *Let $A\subset X$ and $B\subset Y$ be closed nowhere dense subsets in Tychonoff spaces X and Y. Then*

(1) $1+z(A,X)+z(B,Y)\le z(A\times B, X\times Y)\le z^{\mathbb{Z}}(A\times B, X\times Y)$;
(2) $z(A\times B, X\times Y)=z^{\mathbb{Z}}(A\times B, X\times Y)$ *provided X and Y are* LC^1*-spaces.*

Proof: (1) follows from Theorem 6.1(2) and Theorem 7.1(1).

(2) Assume that X and Y are LC^1-spaces and consider three cases.

Case (i): $z^{\mathbb{Z}}(A,X) = z^{\mathbb{Z}}(B,X) = 0$. In this case, $z(A,X) = z^{\mathbb{Z}}(A,X) = z^{\mathbb{Q}}(A,X)$ and $z(B,Y) = z^{\mathbb{Z}}(B,Y) = z^{\mathbb{Q}}(B,Y)$. So by Lemma 8.1(1) and Theorem 6.1(2),

$$z(A,X)+z(B,X)+1 = z^{\mathbb{Q}}(A,X)+z^{\mathbb{Q}}(B,Y)+1 = z^{\mathbb{Q}}(A\times B, X\times Y) \ge z(A\times B, X\times Y) \ge z(A,X)+z(B,Y)+1.$$

Case (ii): $z^{\mathbb{Z}}(A,X) > 0$. In this case, A is a homological Z_1-set in X and a homotopical Z_1-set in the LC^1-space X, by Theorem 3.2(6) and (3). Set B, being nowhere dense in the LC^1-space Y, is a homotopical Z_0-set in Y. Then $A\times B$, being the product of a homotopical Z_1-set and a homotopical Z_0-set, is a homotopical Z_2-set in $X\times Y$ by Theorem 6.1(2). By Theorem 3.3, $A\times B$ is a homotopical Z_n-set in $X\times Y$ if and only if it is a homological Z_n-set in $X\times Y$, which implies the desired equality $z(A\times B, X\times Y) = z^{\mathbb{Z}}(A\times B, X\times Y)$.

Case (iii): $z^{\mathbb{Z}}(B,Y) > 0$ can be considered by analogy. □

Theorem 8.3 implies the following.

Corollary 8.4. *A subset A of a Tychonoff LC^1-space X is a homological Z_n-set in X if and only if $A\times\{0\}$ is a homotopical Z_{n+1}-set in $X\times[-1,1]$.*

Next we turn to the k-root theorem.

Theorem 8.5 (k-Root Theorem)**.** *Let A be a closed subset in a topological space X, $k\in\mathbb{N}$, and G be a coefficient group. Then*

(1) $k\cdot z^G(A,X) \le z^G(A^k,X^k) \le k\cdot z^G(A,X)+2k-2$;
(2) $k\cdot z^G(A,X)+k-1 \le z^G(A^k,X^k)$ *provided* $\sigma(G)\subset\{\mathbb{Q},\mathbb{Z}_p,R_p : p\in\Pi\}$;
(3) $z^G(A^k,X^k) \le k\cdot z^G(A,X)+k$, *provided* $\sigma(G)\subset\{\mathbb{Q},\mathbb{Z}_p,\mathbb{Q}_p : p\in\Pi\}$;
(4) $z^G(A^k,X^k) = k\cdot z^G(A,X)+k-1$, *provided* $\sigma(G)\subset\{\mathbb{Q},\mathbb{Z}_p : p\in\Pi\}$.

Proof: The items of this theorem will be proved in the following order: (2), (4), (3), (1). There is nothing to prove if $k=1$. So we assume that $k\ge 2$.

(2) follows by induction from Theorem 7.2(2).

(4) Assume that $\sigma(G) \subset \{\mathbb{Q}, \mathbb{Z}_p : p \in \Pi\}$. By Theorem 7.1(1), there is a Bockstein group $F \in \sigma(G)$ with $z^F(A,X) = z^G(A,X)$. Since F is a field, we may apply Lemma 8.1(1) $k-1$ times and get the equality $z^F(A^k, X^k) = k \cdot z^F(A,X) + k - 1$. Then

$$k \cdot z^G(A,X) + k - 1 = k \cdot z^F(A,X) + k - 1 = \\ z^F(A^k, X^k) \geq z^G(A^k, X^k) \geq k \cdot z^G(A,X) + k - 1,$$

(the last inequality follows from the preceding item) which yields the equality from the fourth item of the theorem.

(3) Assume that $\sigma(G) \subset \{\mathbb{Q}, \mathbb{Z}_p, \mathbb{Q}_p : p \in \Pi\}$ and, using Theorem 7.1(1), find a Bockstein group $H \in \sigma(G)$ with $z^G(A,X) = z^H(A,X)$. For this group, we also get $z^G(A^k, X^k) \leq z^H(A^k, X^k)$ according to Theorem 7.1(1). If $H \in \{\mathbb{Q}, \mathbb{Z}_p : p \in \Pi\}$, then

$$z^G(A^k, X^k) \leq z^H(A^k, X^k) = k \cdot z^H(A,X) + k - 1 = k \cdot z^G(A,X) + k - 1$$

by the preceding case.

So it remains to consider the case $H = \mathbb{Q}_p$ for a prime p. Applying Theorem 7.1(5) and the second item of this theorem, we get

$$z^G(A^k, X^k) \leq z^{\mathbb{Q}_p}(A^k, X^k) \leq z^{\mathbb{Z}_p}(A^k, X^k) + 1 = \\ k \cdot z^{\mathbb{Z}_p}(A,X) + k \leq k \cdot z^{\mathbb{Q}_p}(A,X) + k = k \cdot z^G(A,X) + k.$$

(1) The inequality $k \cdot z^G(A,X) \leq z^G(A^k, X^k)$ follows by induction from Theorem 7.2(1). To prove the inequality $z^G(A^k, X^k) \leq k \cdot z^G(A,X) + 2k - 2$, apply Theorem 7.1(1) to find a group $H \in \sigma(G)$ such that $z^H(A,X) = z^G(A,X)$. If $H \subset \{\mathbb{Q}, \mathbb{Z}_p, \mathbb{Q}_p : p \in \Pi\}$, then

$$z^G(A^k, X^k) \leq z^H(A^k, X^k) \leq k \cdot z^H(A,X) + k = \\ k \cdot z^G(A,X) + k \leq z^G(A,X) + 2k - 2.$$

So it remains to consider the case of $H = R_p$ for a prime p and prove that $z^G(A^k, X^k) \leq k \cdot z^G(A,X) + 2k - 2$. Assuming the converse, we conclude that A^k is a G-homological Z_{kn+k-1}-set in X^k for $n = z^G(A,X) + 1 = z^{R_p}(A,X) + 1$. It follows from the definition of $z^{R_p}(A,X) = n - 1$ that the homology group $H_n(U, U \setminus A; R_p) = H_n(U, U \setminus A) \otimes R_p$ is not trivial for some open set $U \subset X$. Applying

Lemma 5.2, we get that $H_n(U, U \setminus A)$ contains an element a that has either infinite order or order p.

If a has infinite order, then the tensor product $H_n(U, U \setminus A) \otimes H_n(U, U \setminus A)$ contains an element of infinite order and so does the group $H_{2n}(U^2, U^2 \setminus A^2)$ according to the Künneth formula. Now, we can show by induction that the group $H_{kn}(U^k, U^k \setminus A^k)$ contains an element of infinite order, which implies that $H_{kn}(U^k, U^k \setminus A^k) \otimes R_p = H_{kn}(U^k, U^k \setminus A^k; R_p) \neq 0$ and hence, A^k fails to be an R_p-homological Z_{kn}-set in X^k. Then we get a contradiction:

$$z^G(A^k, X^k) \leq z^{R_p}(A^k, X^k) < kn \leq kn + k - 1 \leq z^G(A^k, X^k)\,.$$

If a is of order p, then the torsion product $H_n(U, U \setminus A) * H_n(U, U \setminus A)$ contains an element of order p and, according to the Künneth formula, so does the group $H_{2n+1}(U^2, U^2 \setminus A^2)$. By induction, we can show that the homology group $H_{i(n+1)-1}(U^i, U^i \setminus A^i)$ contains an element of order p. For $i = k$, the group $H_{kn+k-1}(U^k, U^k \setminus A^k)$ contains an element of order p and hence, $H_{kn+k-1}(U^k, U^k \setminus A^k) \otimes R_p = H_{kn+k-1}(U^k, U^k \setminus A^k; R_p)$ is not trivial by Lemma 5.2. This means that A^k is not an R_p-homological Z_{kn+k-1}-set in X and thus,

$$z^G(A^k, X^k) \leq z^{R_p}(A^k, X^k) \leq kn + k - 2 < z^G(A^k, X^k),$$

which is a contradiction. □

9. Z_n-POINTS

A point x of a space X is defined to be a *Z_n-point* if its singleton $\{x\}$ is a Z_n-set in X. By analogy we define homotopical and G-homological Z_n-points. It is clear that all results proved in the preceding sections for Z_n-sets concern also Z_n-points. For Z_n-points there are, however, some simplifications. In particular, the excision property for singular homology theory (see in [33, Ch. 4, Sec. 6, Theorem 4]) allows us to characterize homological Z_n-points as follows.

Proposition 9.1. *For a point x of a regular topological space X and a coefficient group G, the following conditions are equivalent.*

(1) *x is a G-homological Z_n-point in X;*
(2) *$H_k(X, X \setminus \{x\}; G) = 0$ for all $k < n + 1$;*
(3) *there is an open neighborhood $U \subset X$ of x such that $H_k(U, U \setminus \{x\}; G) = 0$ for all $k < n + 1$.*

In this section, given a topological space X and a coefficient group G we shall study the Borel complexity of the sets

- $\mathcal{Z}_n(X)$ of all homotopical Z_n-points in X, and
- $\mathcal{Z}_n^G(X)$ of all G-homological Z_n-points in X.

Theorem 9.2. *Let X be a metrizable separable space and G be a coefficient group. Then*

(1) *the set of Z_n-points in X is a G_δ-set in X;*
(2) *the set $\mathcal{Z}_n(X)$ of homotopical Z_n-points is a G_δ-set in X provided X is an* LC^n*-space;*
(3) *the set of G-homological Z_n-points is a G_δ-set in X if $|H_k(U;G)| \le \aleph_0$ for all open sets $U \subset X$ and all $k < n+1$;*
(4) *the set $\mathcal{Z}_n^G(X)$ of G-homological Z_n-points is a G_δ-set in X provided X is an lc^n-space;*
(5) *the set of homotopical Z_n-points is a G_δ-set in X if X is an* LC^2*-space and $|H_k(U)| \le \aleph_0$ for all open subsets $U \subset X$ and all $k < n+1$.*

Proof: (1) Let ρ be a metric on X generating the topology of X. Since the space X is metrizable and separable, so is the function space $C(I^n, X)$ endowed with the sup-metric

$$\tilde{\rho}(f,g) = \sup_{z \in I^n} \rho(f(z), g(z)).$$

So, we may fix a countable dense set $\{f_i\}_{i=1}^\infty$ in $C(I^n, X)$. Observe that a closed subset $A \subset X$ fails to be a Z_n-set if and only if there are a function $f \in C(I^k, X)$ and $\varepsilon > 0$ such that for each function $g \in C(I^n, X)$ with $\tilde{\rho}(f,g) < \varepsilon$ the image $g(I^n)$ meets the set A. The density of $\{f_i\}$ implies that $\tilde{\rho}(f, f_i) < \varepsilon/2$ for some $i \in \omega$. Consequently, $g(I^k) \cap A \neq \emptyset$ for all $g \in C(I^n, X)$ with $\tilde{\rho}(g, f_i) < \varepsilon/2$.

Now for each $i, k \in \mathbb{N}$, consider the set

$$F_{i,k} = \{x \in X : \forall g \in C(I^n, X) \quad \tilde{\rho}(g, f_i) < 1/k \Rightarrow x \in g(I^n)\}.$$

It is easy to see that the set $F_{i,k}$ is closed in X. Moreover, the preceding discussion implies that $F = \bigcup_{i,k=1}^\infty F_{i,k}$ is the set of all points of X that fail to be Z_n-points in X. Then its complement is a G_δ-set coinciding with the set of all Z_n-points in X.

(2) If X is an LC^n-space, then each Z_n-point is a homotopical Z_n-point in X and consequently, the set $\mathcal{Z}_n(X)$ of homotopical Z_n-points coincides with the G_δ-set $X \setminus F$ of all Z_n-points.

(3) Assume that the homology groups $H_k(U;G)$ are countable for all $k < n+1$ and all open sets $U \subset X$. Let $\mathcal{B} = \{U_i : i \in \omega\}$ be a countable base of the topology for X with $U_0 = \emptyset$. For every $i \in \omega$ and $k < n+1$, use the countability of the groups $H_k(X \setminus \overline{U}_i; G)$ to find a countable sequence $(\alpha_{i,j})_{j\in\omega}$ of cycles in $X \setminus \overline{U}_i$ whose representatives $[\alpha_{i,j}]$ exhaust all non-zero elements of the groups $H_k(X \setminus \bar{U}_i; G)$, $k < n+1$.

For every $j \in \omega$, consider the open set $W_{0,j} = \{x \in X : \alpha_{0,j}$ is homologous to some cycle in $X \setminus \{x\}\}$. Also for every $i \in \mathbb{N}$ and $j \in \omega$, consider the open set $W_{i,j} = \{x \in U_i :$ if $\alpha_{i,j}$ is null-homological in X, then it is null-homological in $X \setminus \{x\}\}$. It remains to prove that the G_δ-set $Z = \bigcap_{i,j\in\omega} W_{i,j}$ coincides with the set $\mathcal{Z}_n^G(X)$ of all G-homological Z_n-points in X. It is clear that Z contains all G-homological Z_n-points of X. Now take any point $x \in Z$. Assuming that x is not a G-homological Z_n-point, find $k < n+1$ such that $H_k(X, X \setminus \{x\}; G) \neq 0$. The exact sequence

$$H_k(X \setminus \{x\}; G) \to H_k(X; G) \to H_k(X, X \setminus \{x\}; G) \to \\ H_{k-1}(X \setminus \{x\}; G) \to H_{k-1}(X; G)$$

of the pair $(X, X \setminus \{x\})$ now implies that for $m = k$ or $m = k-1$ the inclusion homomorphism $i : H_m(X \setminus \{x\}; G) \to H_m(X; G)$ fails to be an isomorphism.

If i is not onto, then, for some $j \in \omega$, the element $\alpha_{0,j}$ is homologous to no cycle in $X \setminus \{x\}$. This means that $x \notin W_{0,j}$ and thus, $x \notin Z$, which is a contradiction.

If i is not injective, then there is a k-cycle α in $X \setminus \{x\}$ which is homologous to zero in X but not in $X \setminus \{x\}$. Since α has compact support, there is a basic neighborhood U_i of x such that α is supported by the set $X \setminus \overline{U}_i$. Find $j \in \omega$ such that the cycle $\alpha_{i,j}$ is homologous to α in $X \setminus \overline{U}_i$. It follows that $\alpha_{i,j}$ is homologous to zero in X but not in $X \setminus \{x\}$. Then $x \notin W_{i,j}$ and hence $x \notin Z$. This contradiction completes the proof of the third item.

(4) Assuming that X is an lc^n-space, we shall show that the set $\mathcal{Z}_n^G(X)$ is of type G_δ in X. Since $\mathcal{Z}_n^G(X) = \bigcap_{H\in\sigma(G)} \mathcal{Z}_n^H(X)$, it suffices to check that $\mathcal{Z}_n^H(X)$ is a G_δ-set in X for every countable group H. This will follow from the preceding item as soon as we show that for every open set $U \subset X$, the homology groups $H_i(U; H)$, $i \leq n$, are at most countable. In its turn, this will follow

from the Universal Coefficient Formula as soon as we check that the homology groups $H_i(U)$, $i \le n$, are at most countable.

Fix a countable family $\mathcal{K}$ of compact polyhedra containing a topological copy of each compact polyhedron. For each polyhedron $K \in \mathcal{K}$, fix a countable dense set $\mathcal{F}_K$ in the function space $C(K, U)$. Note that for each polyhedron $K \in \mathcal{K}$, the homology group $H_*(K)$ is finitely generated and hence at most countable.

For every homology element $\alpha \in H_i(U)$ with $i \le n$, there is a continuous map $f : K \to U$ of a compact polyhedron $K \in \mathcal{K}$ such that $\alpha \in f_*(H_i(K))$. Moreover, according to Lemma 1.3, we can assume that $f \in \mathcal{F}_K$. Then the homology group

$$H_i(U) = \bigcup_{K \in \mathcal{K}} \bigcup_{f \in \mathcal{F}_K} f_*(H_i(K))$$

is countable, being the countable union of finitely generated groups.

(5) Follows from items (2) and (4) and the characterization of homotopical Z_n-sets given by Theorem 3.3. □

10. On spaces all of whose points are Z_n-points

In this section, we introduce three classes of Tychonoff spaces related to Z_n-points:

- $\mathcal{Z}_n$ the class of spaces X with $X = \mathcal{Z}_n(X)$,
- $\mathcal{Z}_n^G$ the class of spaces X with $X = \mathcal{Z}_n^G(X)$,
- $\cup_G \mathcal{Z}_n^G$ the union of classes $\mathcal{Z}_n^G$ over all coefficient groups.

For example, $\mathbb{R}^{n+1}$ belongs to all of these classes, while $\mathbb{R}^n$ belongs to none of them. The classes $\mathcal{Z}_n$ play an important role in studying the general position properties from [9].

By LC^n (lc^n, respectively), we shall denote the class of metrizable LC^n-spaces (lc^n-spaces, respectively).

The following corollary describes the relation between the introduced classes and can be easily derived from theorems 3.2, 3.3, 5.5, and 5.6.

Corollary 10.1. *Let $n \in \omega \cup \{\infty\}$ and G be a coefficient group. Then*

(1) $\mathcal{Z}_n \subset \mathcal{Z}_n^{\mathbb{Z}} \subset \mathcal{Z}_n^G = \bigcap_{H \in \sigma(G)} \mathcal{Z}_n^H$;
(2) $\mathcal{Z}_0 = \mathcal{Z}_0^{\mathbb{Z}} = \mathcal{Z}_0^G$;
(3) $\mathrm{LC}^1 \cap \mathcal{Z}_1^G \subset \mathcal{Z}_1$;
(4) $\mathrm{LC}^1 \cap \mathcal{Z}_2 \cap \mathcal{Z}_n^{\mathbb{Z}} \subset \mathcal{Z}_n$;

(5) $\cup_G \mathcal{Z}_n^G = \mathcal{Z}_n^{\mathbb{Q}} \cup \bigcup_{p\in\Pi} \mathcal{Z}_n^{\mathbb{Q}_p}$;

(6) $\mathcal{Z}_n^{\mathbb{Z}} = \bigcap_{p\in\Pi} \mathcal{Z}_n^{R_p}$;

(7) $\mathcal{Z}_n^{R_p} \subset \mathcal{Z}_n^{\mathbb{Q}} \cap \mathcal{Z}_n^{\mathbb{Z}_p}$, $\mathcal{Z}_n^{\mathbb{Z}_p} \subset \mathcal{Z}_n^{\mathbb{Q}_p} \subset \mathbb{Z}_{n-1}^{\mathbb{Z}_p}$, *and* $\mathcal{Z}_n^{\mathbb{Q}} \cap \mathcal{Z}_{n+1}^{\mathbb{Q}_p} \subset \mathcal{Z}_n^{R_p}$ *for every prime number* p.

For a better visual presentation of our subsequent results, let us introduce the following operations on subclasses $\mathcal{A}, \mathcal{B} \subset \mathsf{Top}$ of the class Top of topological spaces:

$$\mathcal{A} \times \mathcal{B} = \{A \times B : A \in \mathcal{A},\ B \in \mathcal{B}\},$$

$$\frac{\mathcal{A}}{\mathcal{B}} = \{X \in \mathsf{Top} : \exists B \in \mathcal{B} \text{ with } X \times B \in \mathcal{A}\},$$

$$\mathcal{A}^k = \{A^k : A \in \mathcal{A}\} \text{ and } \sqrt[k]{\mathcal{A}} = \{A \in \mathsf{Top} : A^k \in \mathcal{A}\}.$$

Theorem 7.2 implies the following three multiplication formulas for the classes $\mathcal{Z}_n$ and $\mathcal{Z}_n^G$.

Theorem 10.2 (Multiplication Formulas)**.** *Let* $n, m \in \omega \cup \infty$ *and* X *and* Y *be Tychonoff spaces.*

(1) *If* $X \in \mathcal{Z}_n$ *and* $Y \in \mathcal{Z}_m$, *then* $X \times Y \in \mathcal{Z}_{n+m+1}$:

$$\boxed{\mathcal{Z}_n \times \mathcal{Z}_m \subset \mathcal{Z}_{n+m+1}.}$$

(2) *If* $X \in \mathcal{Z}_n^G$ *and* $Y \in \mathcal{Z}_m^G$ *for a coefficient group* G, *then* $X \times Y \in \mathcal{Z}_{n+m}^G$:

$$\boxed{\mathcal{Z}_n^G \times \mathcal{Z}_m^G \subset \mathcal{Z}_{n+m}^G.}$$

(3) *If* $X \in \mathcal{Z}_n^R$ *and* $Y \in \mathcal{Z}_m^R$ *for a coefficient group* R *with* $\sigma(R) \subset \{\mathbb{Q}, \mathbb{Z}_p, R_p : p \in \Pi\}$, *then* $X \times Y \in \mathcal{Z}_{n+m+1}^R$:

$$\boxed{\mathcal{Z}_n^R \times \mathcal{Z}_m^R \subset \mathcal{Z}_{n+m+1}^R.}$$

The multiplication formulas for the classes $\mathcal{Z}_n^G$ can be reversed.

Theorem 10.3 (Division Formulas)**.** *Let* $n, m \in \omega \cup \infty$ *and* X *and* Y *be Tychonoff spaces.*

(1) *If* $X \times Y \in \mathcal{Z}_{n+m+1}^G$ *for a coefficient group* G, *then either* $X \in \mathcal{Z}_n^G$ *or* $Y \in \bigcup_{F\in\varphi(G)} \mathcal{Z}_m^F$. *This can be written as*

$$\boxed{\frac{\mathcal{Z}_{n+m+1}^G}{\mathsf{Top} \setminus \mathcal{Z}_n^G} \subset \bigcup_{F\in\varphi(G)} \mathcal{Z}_m^F.}$$

(2) *If* $X \times Y \in \mathcal{Z}^R_{n+m}$ *for a coefficient group* R *with* $\sigma(R) \subset \{\mathbb{Q}, \mathbb{Z}_p, R_p : p \in \Pi\}$*, then either* $X \in \mathcal{Z}^R_n$ *or* $Y \in \bigcup_{H \in d(R)} \mathcal{Z}^H_m$*. This can be written as*

$$\boxed{\frac{\mathcal{Z}^R_{n+m}}{\mathsf{Top} \setminus \mathcal{Z}^R_n} \subset \bigcup_{H \in d(R)} \mathcal{Z}^H_m.}$$

Proof: (1) Assume that $X \notin \mathcal{Z}^G_n$ and $Y \notin \bigcup_{F \in \varphi(G)} \mathcal{Z}^F_m$. Then there is a point $x \in X$ with $z^G(x, X) < n$, and for every field $F \in \varphi(G)$, there is a point $y_F \in Y$ with $z^F(y_F, Y) < m$. By Theorem 5.5, $z^G(x, X) = z^H(x, X)$ for some group $H \in \sigma(G)$. If H is a field, then $H \in \varphi(H) \subset \varphi(G)$, and by Lemma 8.1,

$$\begin{aligned} z^H((x, y_H), X \times Y) &= z^H(x, X) + z^H(y_H, Y) + 1 \\ &\le (n-1) + (m-1) + 1 = n + m - 1, \end{aligned}$$

which means that (x, y_H) is not an H-homological Z_{n+m}-set in $X \times Y$ and thus, $X \times Y \notin \mathcal{Z}^H_{n+m} \supset \mathcal{Z}^G_{n+m}$.

If $H = R_p$ for some p, then $\{\mathbb{Q}, \mathbb{Q}_p\} = d(R_p)$ and $\mathbb{Z}_p \in \varphi(G)$. By Lemma 8.1(3),

$$\begin{aligned} z^H(x, X) + 1 \ge \\ \min\{& z^H((x, y_{\mathbb{Q}}), X \times Y) - z^{\mathbb{Q}}(y_{\mathbb{Q}}, Y), \\ & z^H((x, y_{\mathbb{Z}_p}), X \times Y) - z^{\mathbb{Q}_p}(y_{\mathbb{Z}_p}, Y)\}. \end{aligned}$$

Therefore, either

$$\begin{aligned} z^H((x, y_{\mathbb{Q}}), X \times Y) &\le 1 + z^H(x, X) + z^{\mathbb{Q}}(y_{\mathbb{Q}}, Y) \\ &\le (n-1) + (m-1) + 1 \text{ or} \\ z^H((x, y_{\mathbb{Z}_p}), X \times Y) &\le 1 + z^H(x, X) + z^{\mathbb{Q}_p}(y_{\mathbb{Z}_p}, Y) \\ &\le 1 + (n-1) + z^{\mathbb{Z}_p}(y_{\mathbb{Z}_p}, Y) + 1 = n + m. \end{aligned}$$

In both cases, $X \times Y \notin \mathcal{Z}^H_{n+m+1}$. If $H = \mathbb{Q}_p$, then $\mathbb{Z}_p \in \varphi(G)$. By Lemma 8.1,

$$\begin{aligned} z^H((x, y_{\mathbb{Z}_p}), X \times Y) &\le z^H(x, X) + z^{\mathbb{Z}_p}(y_{\mathbb{Z}_p}, Y) + 2 \\ &\le (n-1) + (m-1) + 2 = n + m, \end{aligned}$$

which implies that $X \times Y \notin \mathcal{Z}^H_{n+m+1}$.

(2) Assume that $X \notin \mathcal{Z}^R_n$ and $Y \notin \bigcup_{F \in d(R)} \mathcal{Z}^F_m$ for a group R with $\sigma(R) \subset \{\mathbb{Q}, \mathbb{Z}_p, R_p : p \in \Pi\}$. Then there is a point $x \in X$ with $z^R(x, X) < n$ and for every group $H \in d(R)$ there is a point $y_H \in Y$ with $z^H(y_H, Y) < m$. By Theorem 5.5, $z^R(x, X) = z^H(x, X)$ for some group $H \in \sigma(R) \subset \{\mathbb{Q}, \mathbb{Z}_p, R_p : p \in \Pi\}$. Theorem 5.5 implies that $\mathcal{Z}^R_{n+m} \subset \mathcal{Z}^H_{n+m}$.

If H is a field, then repeating the reasoning from the preceding item, we can prove that

$$z^H((x, y_H), X \times Y) = z^H(x, X) + z^H(y_H, Y) + 1 \\ \leq (n-1) + (m-1) + 1 = n + m - 1,$$

which means that $X \times Y \notin \mathcal{Z}^H_{n+m}$.

If $H = R_p$ for some p, then $\{\mathbb{Q}, \mathbb{Q}_p\} = d(R_p) \subset d(G)$. By Lemma 8.1(3),

$$z^H(x, X) + 1 \geq \\ \min\{z^H((x, y_{\mathbb{Q}}), X \times Y) - z^{\mathbb{Q}}(y_{\mathbb{Q}}, Y), \\ z^H((x, y_{\mathbb{Z}_p}), X \times Y) - z^{\mathbb{Q}_p}(y_{\mathbb{Q}_p}, Y)\}.$$

Therefore, either

$$z^H((x, y_{\mathbb{Q}}), X \times Y) \leq 1 + z^H(x, X) + z^{\mathbb{Q}}(y_{\mathbb{Q}}, Y) \\ \leq (n-1) + (m-1) + 1 = n + m - 1 \text{ or} \\ z^H((x, y_{\mathbb{Z}_p}), X \times Y) \leq 1 + z^H(x, X) + z^{\mathbb{Q}_p}(y_{\mathbb{Q}_p}, Y) \\ \leq 1 + (n-1) + (m-1) = n + m - 1.$$

In both cases, $X \times Y \notin \mathcal{Z}^H_{n+m} \supset \mathcal{Z}^R_{n+m}$. □

Finally, we prove k-root formulas for the classes $\mathcal{Z}^G_n$.

Theorem 10.4 (k-Root Formulas)**.** *Let $n \in \omega \cup \infty$, $k \in \mathbb{N}$, X be a topological space, and G be a coefficient group.*

(1) *If $X^k \in \mathcal{Z}^G_{kn+k-1}$, then $X \in \mathcal{Z}^G_n$:*

$$\boxed{\sqrt[k]{\mathcal{Z}^G_{nk+k-1}} \subset \mathcal{Z}^G_n.}$$

(2) *If $\sigma(G) \subset \{\mathbb{Q}, \mathbb{Z}_p, \mathbb{Q}_p : p \in \Pi\}$ and $X^k \in \mathcal{Z}^G_{kn+1}$, then $X \in \mathcal{Z}^G_n$:*

$$\boxed{\sigma(G) \subset \{\mathbb{Q}, \mathbb{Z}_p, \mathbb{Q}_p : p \in \Pi\} \Rightarrow \sqrt[k]{\mathcal{Z}^G_{nk+1}} \subset \mathcal{Z}^G_n.}$$

(3) *If $\sigma(G) \subset \{\mathbb{Q}, \mathbb{Z}_p : p \in \Pi\}$ and $X^k \in \mathcal{Z}^G_{kn}$, then $X \in \mathcal{Z}^G_n$:*

$$\boxed{\sigma(G) \subset \{\mathbb{Q}, \mathbb{Z}_p : p \in \Pi\} \Rightarrow \sqrt[k]{\mathcal{Z}^G_{nk}} = \mathcal{Z}^G_n.}$$

Proof: Assume that $X \notin \mathcal{Z}^G_n$ and find a point $x \in X$ with $z^G(\{x\}, X) < n$.

(1) Applying Theorem 8.5(1), we get $z^G(\{x\}^k, X^k) < k \cdot z^G(x, X) + 2k - 1 \leq k(n-1) + 2k - 1 = kn + k - 1$ and hence $X^k \notin \mathcal{Z}^G_{kn+k-1}$.

(2) If $\sigma(G) \subset \{\mathbb{Q}, \mathbb{Z}_p, \mathbb{Q}_p : p \in \Pi\}$, then we can apply Theorem 8.5(3) to conclude that $z^G(\{x\}^k, X^k) \le k \cdot z^G(x, X) + k \le k(n-1) + k = kn$ and $X^k \notin \mathcal{Z}^G_{kn+1}$.

(3) If $\sigma(G) \subset \{\mathbb{Q}, \mathbb{Z}_p : p \in \Pi\}$, then we can apply Theorem 8.5(4) to conclude that $z^G(\{x\}^k, X^k) = k \cdot z^G(x, X) + k - 1 \le k(n-1) + k - 1 = kn - 1$ and hence $X^k \notin \mathcal{Z}^G_{kn}$. □

11. On spaces containing a dense set of (homological) Z_n-points

In this section we consider classes of Tychonoff spaces containing dense sets of (homological) Z_n-points. More precisely, let

- $\overline{\mathcal{Z}}_n$ be the class of spaces X with dense set $\mathcal{Z}_n(X)$ of homotopical Z_n-points in X;
- $\overline{\mathcal{Z}}^G_n$ be the class of spaces X with dense set $\mathcal{Z}^G_n(X)$ of G-homological Z_n-points;
- $\cup_G \overline{\mathcal{Z}}^G_n$ be the union of classes $\overline{\mathcal{Z}}^G_n$ over all coefficient groups.

The classes $\overline{\mathcal{Z}}_n$ play an important role in [9], a paper devoted to general position properties.

It is clear that $\mathcal{Z}_n \subset \overline{\mathcal{Z}}_n$. On the other hand, a dendrite D with dense set of end-points belongs to $\overline{\mathcal{Z}}_\infty$ but not to $\mathcal{Z}_1$. By Br we shall denote the class of metrizable separable Baire spaces.

The following proposition describes the relation between the introduced classes.

Proposition 11.1. *Let $n \in \omega \cup \{\infty\}$ and G be a coefficient group. Then*

(1) $\overline{\mathcal{Z}}_n \subset \overline{\mathcal{Z}}^{\mathbb{Z}}_n \subset \overline{\mathcal{Z}}^G_n \subset \bigcap_{H \in \sigma(G)} \overline{\mathcal{Z}}^H_n$;
(2) $\mathsf{Br} \cap lc^n \cap \bigcap_{H \in \sigma(G)} \overline{\mathcal{Z}}^H_n \subset \overline{\mathcal{Z}}^G_n$;
(3) $\overline{\mathcal{Z}}_0 = \overline{\mathcal{Z}}^G_0$;
(4) $\mathrm{LC}^1 \cap \overline{\mathcal{Z}}^G_1 \subset \overline{\mathcal{Z}}_1$;
(5) $\mathrm{LC}^1 \cap \mathcal{Z}_2 \cap \overline{\mathcal{Z}}^{\mathbb{Z}}_n \subset \overline{\mathcal{Z}}_n$;
(6) $\mathrm{LC}^n \cap \mathsf{Br} \cap \overline{\mathcal{Z}}_2 \cap \overline{\mathcal{Z}}^{\mathbb{Z}}_n \subset \overline{\mathcal{Z}}_n$.

Proof: (1) Follows from Theorem 3.2(4), Proposition 5.3, and Theorem 5.5.

(2) If $X \in \bigcap_{H \in \sigma(G)} \overline{\mathcal{Z}}^H_n$ is a metrizable separable Baire lc^n-space, then for every group $H \in \sigma(G)$, the set $\mathcal{Z}^H_n(X)$ of H-homological

Z_n-points is a dense G_δ-set in X according to Theorem 9.2(4). By Theorem 5.5, the intersection $\bigcap_{H\in\sigma(G)} \mathcal{Z}_n^H(X)$ coincides with $\mathcal{Z}_n^G(X)$ and is dense in X, being the countable intersection of dense G_δ-sets in the Baire space X. Hence, $X \in \overline{\mathcal{Z}}_n^G$.

(3)–(5) Follow immediately from Theorem 3.2 and Theorem 3.3.

(6) Assume that $X \in \overline{\mathcal{Z}}_2 \cap \overline{\mathcal{Z}}^{\mathbb{Z}}$ is a metrizable separable Baire LC^n-space. By Theorem 9.2, the sets $\mathcal{Z}_2(X)$ and $\mathcal{Z}_n^{\mathbb{Z}}(X)$ are dense G_δ in X. Since X is Baire, the intersection $\mathcal{Z}_2(X) \cap \mathcal{Z}_n^{\mathbb{Z}}(X)$ is dense G_δ in X. By Theorem 3.3, the latter intersection consists of homotopical Z_n-points. So $X \in \overline{\mathcal{Z}}_n$. □

Multiplication formulas for the classes $\overline{\mathcal{Z}}_n$ and $\overline{\mathcal{Z}}_n^G$ follow immediately from Theorem 6.1 for homotopical and homological Z_n-sets.

Theorem 11.2 (Multiplication Formulas)**.** *Let $n, m \in \omega \cup \infty$ and X and Y be Tychonoff spaces.*

(1) *If $X \in \overline{\mathcal{Z}}_n$ and $Y \in \overline{\mathcal{Z}}_m$, then $X \times Y \in \overline{\mathcal{Z}}_{n+m+1}$:*

$$\boxed{\overline{\mathcal{Z}}_n \times \overline{\mathcal{Z}}_m \subset \overline{\mathcal{Z}}_{n+m+1}.}$$

(2) *If $X \in \overline{\mathcal{Z}}_n^G$ and $Y \in \overline{\mathcal{Z}}_m^G$ for a coefficient group G, then $X \times Y \in \overline{\mathcal{Z}}_{n+m}^G$:*

$$\boxed{\overline{\mathcal{Z}}_n^G \times \overline{\mathcal{Z}}_m^G \subset \overline{\mathcal{Z}}_{n+m}^G.}$$

(3) *If $X \in \overline{\mathcal{Z}}_n^R$ and $Y \in \overline{\mathcal{Z}}_m^R$ for a coefficient group R with $\sigma(R) \subset \{\mathbb{Q}, \mathbb{Z}_p, R_p : p \in \Pi\}$, then $X \times Y \in \overline{\mathcal{Z}}_{n+m+1}^R$:*

$$\boxed{\overline{\mathcal{Z}}_n^R \times \overline{\mathcal{Z}}_m^R \subset \overline{\mathcal{Z}}_{n+m+1}^R.}$$

Division and k-root formulas for the classes $\overline{\mathcal{Z}}_n^G$ look as follows. For a class C of topological spaces, we denote by $\exists_\circ \mathsf{C}$ the class of topological spaces X containing a non-empty open subspace $U \in \mathsf{C}$.

Theorem 11.3 (Division Formulas)**.** *Let X and Y be topological spaces.*

(1) *If $X \times Y \in \overline{\mathcal{Z}}_{n+m}^F$ for some field F, then either $X \in \overline{\mathcal{Z}}_n^F$ or $Y \in \overline{\mathcal{Z}}_m^F$:*

$$\boxed{\frac{\overline{\mathcal{Z}}^F_{n+m}}{\mathsf{Top}\setminus\overline{\mathcal{Z}}^F_m}=\overline{\mathcal{Z}}^F_n.}$$

(2) *Assume that* $X\times Y\in\overline{\mathcal{Z}}^R_{n+m}$ *for a coefficient group* R *with* $\sigma(R)\subset\{\mathbb{Q},\mathbb{Z}_p,R_p:p\in\Pi\}$ *and either* $\sigma(R)$ *is finite or* X *is a metrizable separable Baire* lc^n*-space. Then either* $X\in\overline{\mathcal{Z}}^R_n$ *or else some non-empty open set* $U\subset Y$ *belongs to the class* $\bigcup_{H\in d(R)}\overline{\mathcal{Z}}^H_m$*:*

$$\boxed{\frac{\overline{\mathcal{Z}}^R_{n+m}}{\mathsf{Br}\cap lc^n\setminus\overline{\mathcal{Z}}^R_n}\subset\exists_\circ\bigcup_{H\in d(R)}\overline{\mathcal{Z}}^H_m.}$$

(3) *Assume that* $X\times Y\in\overline{\mathcal{Z}}^G_{n+m+1}$ *for a coefficient group* G *and either* $\sigma(G)$ *is finite or* X *is a metrizable separable Baire* lc^n*-space. Then either* $X\in\overline{\mathcal{Z}}^G_n$ *or else some non-empty open set* $U\subset Y$ *belongs to the class* $\bigcup_{F\in\varphi(G)}\overline{\mathcal{Z}}^F_m$*:*

$$\boxed{\frac{\overline{\mathcal{Z}}^G_{n+m+1}}{\mathsf{Br}\cap lc^n\setminus\overline{\mathcal{Z}}^G_n}\subset\exists_\circ\bigcup_{F\in\varphi(G)}\overline{\mathcal{Z}}^F_m.}$$

Proof: (1) Assume that $X\times Y\in\overline{\mathcal{Z}}^F_{n+m}$ for some field F, but $X\notin\mathcal{Z}^F_n$ and $Y\notin\overline{\mathcal{Z}}^F_m$. Let U be the interior of the set $X\setminus\mathcal{Z}^F_n(X)$ and V be the interior of the set $Y\setminus\mathcal{Z}^F_m(Y)$. The product $U\times V$ is a non-empty open subset of $X\times Y$ and thus contains some F-homological Z_{n+m}-point (x,y) in $X\times Y$. By Lemma 8.1(1), either $x\in\mathcal{Z}^F_n(X)$ or $y\in\mathcal{Z}^F_m(Y)$. Both cases are not possible by the choice of U and V. This contradiction shows that $X\in\overline{\mathcal{Z}}^F_n$ or $Y\in\overline{\mathcal{Z}}^F_m$.

(2) Assume that $X\times Y\in\overline{\mathcal{Z}}^R_{n+m}$ for a group R with $\sigma(R)\subset\{\mathbb{Q},\mathbb{Z}_p,R_p:p\in\Pi\}$ but $X\notin\overline{\mathcal{Z}}^R_n$. The latter means that X contains a non-empty open set $W\subset X$ disjoint with the set $\mathcal{Z}^R_n(X)$ of R-homological Z_n-points of X. It is necessary to find an open set $U\in\bigcup_{H\in d(R)}\overline{\mathcal{Z}}^H_m$. Assume conversely that no such set U exists. This means that for every $H\in d(G)$, the set $\mathcal{Z}^H_m(Y)$ is nowhere dense in Y.

If $\sigma(R)$ is finite, then so is the set $d(R)$ and we can find a non-empty open set $U\subset X$ disjoint with $\bigcup_{H\in d(R)}\mathcal{Z}^H_m(Y)$.

By Theorem 8.2(2), no point $(x,y)\in W\times U$ is an R-homological Z_{n+m}-point in $X\times Y$. But this contradicts the density of the set $\mathcal{Z}^R_{n+m}(X\times Y)$ in $X\times Y$.

Next, we consider the case when $\sigma(R)$ is infinite and X is a metrizable separable Baire lc^n-space. By Theorem 9.2, for every group $H\in\sigma(R)$, the set $\mathcal{Z}^H_n(X)$ is of type G_δ in X and by Theorem 5.5, $\mathcal{Z}^G_n(X)=\bigcap_{H\in\sigma(R)}\mathcal{Z}^H_n(X)$. Assuming that $\mathcal{Z}^R_n(X)$ is not dense in the Baire space X, we conclude that for some group $H\in\sigma(R)$, the set $\mathcal{Z}^H_n(X)$ is not dense in X. Since $\sigma(H)=\{H\}$ is finite, we may apply the preceding case to conclude that some nonempty open set $U\subset Y$ belongs to the class

$$\bigcup_{D\in d(H)}\overline{\mathcal{Z}}^D_m\subset\bigcup_{D\in d(G)}\overline{\mathcal{Z}}^D_m.$$

(3) Assume that $X\times Y\in\overline{\mathcal{Z}}^G_{n+m+1}$ for a coefficient group G, but $X\notin\overline{\mathcal{Z}}^G_n$. The latter means that X contains a non-empty open set $W\subset X$ disjoint with the set $\mathcal{Z}^G_n(X)$ of G-homological Z_n-points of X. It is necessary to find an open set $U\in\bigcup_{F\in\varphi(G)}\overline{\mathcal{Z}}^F_m$. Assume conversely that no such set U exists. This means that for every $F\in\varphi(G)$, the set $\mathcal{Z}^F_m(Y)$ is nowhere dense in Y.

If $\sigma(G)$ is finite, then the set $\varphi(G)$ is also infinite, and we can find a non-empty open set $U\subset X$ disjoint with $\bigcup_{F\in\varphi(G)}\mathcal{Z}^F_m(Y)$. By Theorem 8.2(1), no point $(x,y)\in W\times U$ is a G-homological Z_{n+m+1}-point in $X\times Y$. But this contradicts the density of the set $\mathcal{Z}^G_{n+m+1}(X\times Y)$ in $X\times Y$.

The case of infinite $\sigma(G)$ can be reduced to the previous case by the same argument as in (2). □

Theorem 11.4 (k-Root Formulas)**.** *Let $n\in\omega\cup\{\infty\}$, $k\in\mathbb{N}$, X be a topological space, and G be a coefficient group.*

(1) *$X\in\overline{\mathcal{Z}}^F_n$ for some field F if and only if $X^k\in\overline{\mathcal{Z}}^F_{nk}$:*

$$\boxed{\sqrt[k]{\overline{\mathcal{Z}}^F_{nk}}=\overline{\mathcal{Z}}^F_n.}$$

(2) *If X is a metrizable separable Baire lc^n-space and $X^k\in\overline{\mathcal{Z}}^G_{nk+k}$, then $X\in\overline{\mathcal{Z}}^G_n$:*

$$\boxed{\mathsf{Br} \cap lc^n \cap \sqrt[k]{\overline{\mathcal{Z}}^G_{nk+k}} \subset \overline{\mathcal{Z}}^G_n.}$$

(3) *If X is a metrizable separable Baire lc^{nk+k-1}-space and $X^k \in \overline{\mathcal{Z}}^G_{nk+k-1}$, then $X \in \overline{\mathcal{Z}}^G_n$:*

$$\boxed{\mathsf{Br} \cap lc^{kn+k-1} \cap \sqrt[k]{\overline{\mathcal{Z}}^G_{nk+k-1}} = \overline{\mathcal{Z}}^G_n.}$$

Proof: (1) Can be derived by induction from Theorem 11.2(2) and Theorem 11.3(1).

(2) Assume that X is a metrizable separable Baire lc^n-space with $X^k \in \overline{\mathcal{Z}}^G_{kn+k}$. It follows from Theorem 9.2 that for every group $H \in \sigma(G)$, the set $\mathcal{Z}^H_n(X)$ is of type G_δ in X. Since $\mathcal{Z}^G_n(X) = \bigcap_{H\in\sigma(G)} \mathcal{Z}^H_n(X)$, the density of $\mathcal{Z}^G_n(X)$ in the Baire space X will follow as soon as we prove the density of $\mathcal{Z}^H_n(X)$ in X for each group $H \in \sigma(G)$.

If H is a field, then the inclusion $X^k \in \overline{\mathcal{Z}}^G_{kn+k} \subset \overline{\mathcal{Z}}^H_{kn+k}$ combined with the first item implies that $X \in \overline{\mathcal{Z}}^H_{n+1} \subset \overline{\mathcal{Z}}^H_n$, which means that the set $\mathcal{Z}^H_n(X)$ is dense in X.

If $H = R_p$ for some p, then Theorem 5.6(1) implies that $X^k \in \overline{\mathcal{Z}}^{R_p}_{nk+k} \subset \overline{\mathcal{Z}}^{\mathbb{Q}}_{nk+k} \cap \overline{\mathcal{Z}}^{\mathbb{Z}_p}_{nk+k}$ and the first item yields $X \in \overline{\mathcal{Z}}^{\mathbb{Q}}_{n+1} \cap \overline{\mathcal{Z}}^{\mathbb{Z}_p}_{n+1}$. Applying Theorem 5.6(2) and (4), we get

$$X \in \overline{\mathcal{Z}}^{\mathbb{Q}}_{n+1} \cap \overline{\mathcal{Z}}^{\mathbb{Z}_p}_{n+1} \subset \overline{\mathcal{Z}}^{\mathbb{Q}}_n \cap \overline{\mathcal{Z}}^{\mathbb{Q}_p}_{n+1} \subset \overline{\mathcal{Z}}^{R_p}_n = \overline{\mathcal{Z}}^H_n.$$

If $H = \mathbb{Q}_p$ for some p, then Theorem 5.6(3) implies that $X^k \in \overline{\mathcal{Z}}^{\mathbb{Q}_p}_{nk+k} \subset \overline{\mathcal{Z}}^{\mathbb{Z}_p}_{nk+k-1} \subset \overline{\mathcal{Z}}^{\mathbb{Z}_p}_{nk}$ and the first item combined with Theorem 5.6(2) implies $X \in \overline{\mathcal{Z}}^{\mathbb{Z}_p}_n \subset \overline{\mathcal{Z}}^{\mathbb{Q}_p}_n = \overline{\mathcal{Z}}^H_n$.

(3) Assume that X is a metrizable separable Baire lc^{nk+k-1}-space and $X^k \in \overline{\mathcal{Z}}^G_{nk+k-1}$. Arguing as in the preceding case, we can reduce the problem to the case $G = R_p$ for some prime number p. By Theorem 9.2, $\mathcal{Z}^{R_p}_{kn+k-1}(X^k)$ is a dense G_δ-set in X^k. Assuming that $\mathcal{Z}^{R_p}_n(X)$ is not dense in X, find a non-empty open set $U \subset X$ disjoint with $\mathcal{Z}^{R_p}_n(X)$. Since $\overline{\mathcal{Z}}^{R_p}_{nk+k-1} \subset \overline{\mathcal{Z}}^{\mathbb{Q}}_{nk+k-1} \subset \overline{\mathcal{Z}}^{\mathbb{Q}}_{nk}$, the first item implies that $X \in \overline{\mathcal{Z}}^{\mathbb{Q}}_n$. Then $D = U \cap \overline{\mathcal{Z}}^{\mathbb{Q}}_n(X)$ is a dense G_δ-set in U by Theorem 9.2 and hence, D^k is a dense G_δ set in U^k. Since

X^k is a Baire space, there is a point $\vec{x} \in D^k \cap \mathcal{Z}^{R_p}_{nk+k-1}(X^k)$ which can be written as $\vec{x} = (x_1, \dots, x_k)$.

Each point x_i is a $\mathbb{Q}$-homological but not an R_p-homological Z_n-point in X. This means that for some $n_i \le n$ the group $H_{n_i}(X, X\setminus\{x_i\}; R_p) = H_{n_i}(X, X\setminus\{x_i\})\otimes R_p$ is not trivial. Since x_i is a $\mathbb{Q}$-homological Z_n-set in X, the group $H_{n_i}(X, X \setminus \{x_i\})$ cannot contain an element of infinite order. Since $H_{n_i}(X, X\setminus\{x_i\})\otimes R_p \neq 0$, the group $H_{n_i}(X, X \setminus \{x_i\})$ contains an element of order p.

Let $m_i = i - 1 + \sum_{j=1}^{i} n_j$ for $i \le k$. The torsion product $H_{n_1}(X, X \setminus \{x_1\}) * H_{n_2}(X, X \setminus \{x_2\})$ contains an element of order p and so does the group $H_{m_2}(X^2, X^2 \setminus \{(x_1, x_2)\})$ by the Künneth formula. Now by induction, we can show that for every $i \le k$, the homology group $H_{m_i}(X^i, X^i \setminus \{(x_1, \dots, x_i)\})$ contains an element of order p. For $i = k$, we get that the group $H_{m_k}(X^k, X^k \setminus \{\vec{x}\})$ contains an element of order p and hence, the tensor product

$$H_{m_k}(X^k, X^k \setminus \{\vec{x}\}) \otimes R_p = H_{m_k}(X^k, X^k \setminus \{\vec{x}\}; R_p)$$

is not trivial, which is not possible because $m_k \le nk + k - 1$ and $\vec{x}$ is an R_p-homological Z_{nk+k-1}-point in X^k. □

Remark 11.5. A dendrite D with a dense set of end-points is an example of a space in which Z_∞-points form a dense G_δ-set. Being large in the sense of a Baire category, the set of Z_∞-points of D is small in a geometric sense: It is locally ∞-negligible in D. On the other hand, Włodzimierz Kuperberg [29] has constructed a finite polyhedron P in which the set of all Z_∞-points fails to be locally ∞-negligible. Yet, according to I. Namioka [32], the set Z of Z_∞-points in a finite-dimensional LC^∞-space X is small in a homological sense: $H_k(U, U\setminus Z) = 0$ for any open set $U \subset X$ and any $k \in \omega$. This result of Namioka is similar in spirit to our Theorem 4.4 asserting that a closed trt-dimensional subspace $A \subset X$ consisting of G-homological Z_∞-points of X is a G-homological Z_∞-set in X.

12. Z_n-points and dimension

In this section we study the dimension properties of spaces all of whose points are homological Z_n-points. First, we note a simple corollary of Theorem 4.3.

Theorem 12.1. *If a (separable metrizable) topological space* $X \in \cup_G \mathcal{Z}^G_n$*, then* $\mathrm{trind}(X) \ge \mathsf{trt}(X) \ge 1+n$ *(and hence* $\dim(X) \ge 1+n$*).*

A similar lower bound holds also for the cohomological and extension dimensions. Given a space X and a CW-complex L, we write e-dim$X \le L$ if any map $f : A \to L$ defined on a closed subset $A \subset X$ extends to a map $\bar{f} : X \to L$. The extension dimension (e-dim) generalizes both the usual covering dimension dim and the cohomological dimension $\dim_G$ because

- $\dim X \le n$ if and only if e-dim$X \le S^n$ and
- $\dim_G X \le n$ if and only if e-dim$X \le K(G, n)$,

where $K(G, n)$ is an Eilenberg-MacLane complex, i.e., a CW-complex K with a unique non-trivial homotopy group $\pi_n(K) = G$.

The main (and technically most difficult) result of this section follows.

Theorem 12.2. *If $X \in \mathcal{Z}_n^{\mathbb{Z}}$ is a locally compact lc^n-space, then $\dim_G X \ge n + 1$ for any non-trivial abelian group G.*

Proof: Assume that $\dim_G X = n < \infty$ for some abelian group G. By [30, Theorem 2] (or [20, Theorem 1.8]), the space X contains a point x having an open neighborhood $U \subset X$ with compact closure such that for any smaller neighborhood $V \subset U$ of x, the homomorphism in the relative Čech cohomology groups

$$i_{V,U} : \check{H}^n(X, X \setminus \bar{V}; G) \to \check{H}^n(X, X \setminus \bar{U}; G),$$

induced by the inclusion $(X, X \setminus \bar{U}) \subset (X, X \setminus \bar{V})$, is non-trivial. The complete regularity of the locally compact space X allows us to find a compact G_δ-set $K_1 \subset U$ containing x in its interior.

It is well known (see [33, Ch. 6, Sec. 9]) that in paracompact lc^n-spaces, Čech cohomology coincides with singular cohomology. Singular cohomology relates to singular homology via the following exact sequence, see [25, Sec. 3.1]:

$$0 \to \mathrm{Ext}(H_{n-1}(X, A); G) \to H^n(X, A; G) \to \mathrm{Hom}(H_n(X, A); G) \to 0.$$

This sequence will be applied to the pairs

$$(X, X \setminus K_1) \subset (X, X \setminus K_2) \subset (X, X \setminus K_3),$$

where $K_3 \subset K_2 \subset K_1$ are compact G_δ-neighborhoods of x so small that the inclusion homomorphisms

$$H_k(X, X \setminus K_1) \to H_k(X, X \setminus K_2) \to H_k(X, X \setminus K_3)$$

are trivial for all $k \le n$ (the existence of such neighborhoods K_2 and K_3 follows from Lemma 1.6).

These trivial homomorphisms induce trivial homomorphisms

$$e_{2,1} : \mathrm{Ext}(H_{n-1}(X, X \setminus K_2); G) \to \mathrm{Ext}(H_{n-1}(X, X \setminus K_1); G)$$

and

$$h_{3,2} : \mathrm{Hom}(H_n(X, X \setminus K_3); G) \to \mathrm{Hom}(H_n(X, X \setminus K_2); G).$$

Now consider the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Ext}(H_{n-1}(X, X\setminus K_3);G) & \longrightarrow & H^n(X, X\setminus K_3;G) & \longrightarrow & \mathrm{Hom}(H_n(X, X\setminus K_3);G) \\
\downarrow & & \downarrow {\scriptstyle i_{3,2}} & & \downarrow {\scriptstyle h_{3,2}} \\
\mathrm{Ext}(H_{n-1}(X, X\setminus K_2);G) & \longrightarrow & H^n(X, X\setminus K_2;G) & \longrightarrow & \mathrm{Hom}(H_n(X, X\setminus K_2);G) \\
{\scriptstyle e_{2,1}} \downarrow & & \downarrow {\scriptstyle i_{2,1}} & & \downarrow \\
\mathrm{Ext}(H_{n-1}(X, X\setminus K_1);G) & \longrightarrow & H^n(X, X\setminus K_1;G) & \longrightarrow & \mathrm{Hom}(H_n(X, X\setminus K_1);G).
\end{array}$$

The exactness of rows of the diagram and the triviality of the homomorphisms $e_{2,1}$ and $h_{3,2}$ imply the triviality of the homomorphism $i_{3,1} = i_{2,1} \circ i_{3,2} : H^n(X, X \setminus K_3; G) \to H^n(X, X \setminus K_1; G)$. The local compactness of X allows us to find an open σ-compact subset $W \subset X$ containing the compact set K_1. Since the sets K_i, $i \in \{1, 3\}$, are of type G_δ in X, the spaces $W \setminus K_i$ are σ-compact and thus paracompact.

The excision axiom for singular cohomology (see [33, Ch. 5, Sec. 4, 14]) implies that $H^n(X, X \setminus K_i; G) = H^n(W, W \setminus K_i; G)$ for $i \in \{1, 3\}$. This observation implies that the inclusion homomorphism $i_{3,1} : H^n(W, W \setminus K_3; G) \to H^n(W, W \setminus K_1; G)$ is trivial. Since W and $W \setminus K_i$, $i \in \{1, 3\}$, are paracompact lc^n-spaces, the singular cohomology group $H^n(W, W \setminus K_i; G)$ coincides with the Čech cohomology group $\check{H}^n(W, W \setminus K_i; G)$. Consequently, the inclusion homomorphism $\check{H}^n(W, W \setminus K_3; G) \to \check{H}^n(W, W \setminus K_1; G)$ is trivial. By the excision axiom for Čech cohomology, $\check{H}^n(W, W \setminus K_i; G) = \check{H}^n(X, X \setminus K_i; G)$ for $i \in \{1, 3\}$. Consequently, the inclusion homomorphism $\check{H}^n(X, X \setminus K_3; G) \to \check{H}^n(X, X \setminus K_1; G)$ is trivial and so is the inclusion homomorphism $\check{H}^n(X, X \setminus \bar{V}; G) \to \check{H}^n(X, X \setminus \bar{U}; G)$, where V is the interior of K_3. But this contradicts the choice of the neighborhood U. □

Theorem 12.2 will help us to evaluate the extension dimension of a locally compact LC^n-space all of whose points are homological Z_n-points.

Theorem 12.3. *If a metrizable locally compact* LC^n*-space* $X \in \mathcal{Z}_n^{\mathbb{Z}}$ *has e-dim*$(X) \le L$ *for some CW-complex* L*, then* $\pi_k(L) = 0$ *for all* $k \le n$*.*

Proof: Separately we shall consider the cases of $n = 0$, $n = 1$, and $n \ge 2$.

$n = 0$: It suffices to check that the CW-complex L is connected. Since each point of the LC^0-space X is a homological Z_0-point, X contains no isolated point and thus, X contains an arc J connecting two distinct points $a, b \in X$. Assuming that the complex L is disconnected, consider any map $f : \{a, b\} \to L$ sending the points a and b to different components of L. Because of the connectedness of $J \subset X$, the map f does not extend to X, which contradicts e-dim$X \le L$.

$n = 1$: We should prove the simple-connectedness of L. Theorem 12.1 implies that $\dim(X) > 1$. Then there is a point $x \in X$ whose any neighborhood $U \subset X$ has dimension $\dim U > 1$. Since X is an LC^1-space, the point x has a closed neighborhood N such that any map $f : \partial I^2 \to N$ is null-homotopic in X. Moreover, we can assume that N is a Peano continuum. Since $\dim N > 1$, the continuum N is not a dendrite and, consequently, contains a simple closed curve $S \subset N$. Assuming that the CW-complex L is not simply-connected, we can find a map $f : S \to L$ that is not homotopic to a constant map. Then the map f cannot be extended over X since the identity map of S is null-homotopic in X. But this contradicts e-dim$X \le L$.

$n \ge 2$: Suppose that $\pi_k(L) \ne 0$ for some $k \le n$. We can assume that k is the smallest number with $\pi_k(L) \ne 0$. The simple connectedness of L implies that $k > 1$. Applying the Hurewicz isomorphism theorem, we conclude that $H_k(L) = \pi_k(L) \ne 0$. Since e-dim$X \le L$, we may apply [21, Theorem 7.14] to conclude that $\dim_{H_n(L)} X \le n$. But this contradicts Theorem 12.2. □

13. Dimension of spaces all of whose points are Z_∞-points

In this section we study the dimension properties of spaces all of whose points are homological Z_∞. We shall show that locally compact ANR's with this property are infinite-dimensional in a rather strong sense: They cannot be C-spaces and have infinite cohomological dimension.

We recall that a topological space X is defined to be a *C-space* if for any sequence $\{\mathcal{V}_n\colon n\in\omega\}$ of open covers of X there exists a sequence $\{\mathcal{U}_n\colon n\in\omega\}$ of disjoint families of open sets in X such that each $\mathcal{U}_n$ refines $\mathcal{V}_n$ and $\bigcup\{\mathcal{U}_n\colon n\in\omega\}$ is a cover of X. By [23, 6.3.8], each metrizable countable-dimensional space is a C-space.

Theorem 13.1. *If $X\in\mathcal{Z}^{\mathbb{Z}}_\infty$ is a locally compact metrizable* LC^0-*space, then*

1. *X is neither* trt-*dimensional nor countable-dimensional;*
2. *if X is an lc^∞-space, then $\dim_G X=\infty$ for any abelian group G;*
3. *if X is an* LC^∞-*space, then e-dim$X\not\le L$ for any non-contractible CW-complex L;*
4. *if X is locally contractible, then X fails to be a C-space.*

Proof: (1)–(3) Follow from Theorem 4.4, Theorem 12.2, and Theorem 12.3, respectively.

(4) First, we prove the fourth item under a stronger assumption that all points of X are homotopical Z_∞-points. Assume that X is a C-space. By a theorem of John H. Gresham [24], the C-space X, being locally contractible, is an ANR. Then the product $X\times Q$ is a Hilbert cube manifold according to the ANR theorem of Edwards, see [15, Theorem 44.1]. The product $X\times Q$, being a Q-manifold, contains an open subset $U\subset X\times Q$ whose closure $\overline{U}$ is homeomorphic to the Hilbert cube Q and whose boundary ∂U is a Z_∞-set in $\overline{U}$. Now consider the multivalued map $\Phi: X\to\bar{U}$, assigning to each point $x\in X$ the set $\Phi(x)=\overline{U}\setminus(\{x\}\times Q)$. Our goal is to show that the map Φ has a continuous selection $s: X\to\overline{U}$. Assuming that this is done, consider the map $s\circ\mathrm{pr}:\overline{U}\to\overline{U}$, where $\mathrm{pr}:\overline{U}\to X$ stands for the natural projection of $\overline{U}\subset X\times Q$ onto the first factor. This map is continuous and has no fixed point, which is a contradiction.

So, it remains to construct the continuous selection s of the multivalued map Φ. The existence of such a selection will follow from selection theorem of V. V. Uspenskij [39] as soon as we verify that

- for each $x\in X$, the complement $\overline{U}\setminus\Phi(x)=\overline{U}\cap(\{x\}\times Q)$ is a Z_∞-set in $\overline{U}$;

- for any compact set $K \subset \bar{U}$, the set $\{x \in X : \Phi(x) \supset K\}$ is open in K.

The first condition holds since each point of X is a Z_∞-point in X and the boundary ∂U is a Z_∞-set in $\overline{U}$. The second condition holds because $\{x \in X : F(x) \supset K\} = X \setminus \mathrm{pr}(K)$. This completes the proof of the special case when all points of X are Z_∞-points.

Now assume that all points of X are merely homological Z_∞-points in X. Then the points of the product $X \times [0,1]$ are (homotopical) Z_∞-points by Corollary 8.4. Now the preceding discussion implies that $X \times [0,1]$ fails to be a C-space. Taking into account that the product of a metrizable C-space with the interval is a C-space [1, Theorem 2.2.3], we conclude that X is not a C-space. □

Remark 13.2. The last item of Theorem 13.1 is true in a bit stronger form: Each locally compact locally contractible space $X \in \cup_G \mathcal{Z}_\infty^G$ fails to be a C-space; see [5]. However, the proof of this stronger result requires non-elementary tools like a homological version of Uspenskij's selection theorem combined with a homological version of the Brouwer fixed point theorem. On the other hand, this stronger result would follow from Theorem 4.4 if each compact C-space were trt-dimensional. However, we are not sure that this is true.

Remark 13.3. In light of Theorem 13.1, it is interesting to mention that a compact AR all of whose points are Z_∞-points need not be homeomorphic to the Hilbert cube. A suitable counterexample was constructed in [17, Corollary 9.3].

Problem 13.4. *Let X be a compact absolute retract all of whose points are Z_∞-points. Is X strongly infinite-dimensional? Is $X \times I$ homeomorphic to the Hilbert cube?*

In this respect let us mention the following characterization of Hilbert cube manifolds [7] which can be deduced from Theorem 4.3 and the homological characterization of Q-manifolds due to Robert J. Daverman and John J. Walsh [17]; see also [9].

Theorem 13.5. *A locally compact ANR-space X is a Hilbert cube manifold if and only if*

(1) *X has the disjoint disks property;*
(2) *each point of X is a homological Z_∞-point;*

(3) *each map $f : K \to X$ of a compact polyhedron can be approximated by a map with* trt*-dimensional image.*

We recall that a space X has the *disjoint disks property* if any two maps $f, g : I^2 \to X$ can be approximated by maps with disjoint images.

Theorem 4.4 implies that each closed trt-dimensional subspace of the Hilbert cube Q is a homological Z_∞-set in Q. By [2, Proposition 4.7], each compact trt-dimensional space is a C-space.

Question 13.6. *Is a closed subset $A \subset Q$ a homological Z_∞-set in Q if A is weakly infinite-dimensional or a C-space?*

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