TOPOLOGY
PROCEEDINGS
Volume 38, 2011
Pages 83–98
http://topology.auburn.edu/tp/

Fiberwise Compactness and Quasi-Uniformities

by

Masaki Iwamoto, Yoshifumi Konami, and Takuo Miwa

Electronically published on August 13, 2010

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS
Volume 38 (2011)
Pages 83-98

E-Published on August 13, 2010

FIBERWISE COMPACTNESS AND QUASI-UNIFORMITIES

MASAKI IWAMOTO, YOSHIFUMI KONAMI, AND TAKUO MIWA

ABSTRACT. In this paper, first, we study the relationship between fiberwise compactness and uniformities, and correct the result of I. M. James. Next, we introduce a new notion of fiberwise quasi-uniform spaces over a topological space B, and study the basic properties of fiberwise quasi-uniform spaces and fiberwise quasi-uniformizability of fiberwise spaces. Last, we prove two main theorems of fiberwise quasi-uniform spaces which are extended versions of theorems in both fiberwise compact spaces (as fiberwise uniform spaces) and quasi-uniform spaces.

1. INTRODUCTION

Our motivation of this study is the common extension of fiberwise compact spaces (as fiberwise uniform spaces [3]) and quasi-uniform spaces [2].

Throughout this paper, we use the following notation and terminology. Let B be a fixed topological space (as the base space) with a topology τ. We will use the abbreviation *nbd(s)* for *neighborhood(s)*. For $b \in B$, $N(b)$ is the family of all open nbds of b.

First, in section 2, we consider the relationship between fiberwise compactness and fiberwise uniformities.

2010 *Mathematics Subject Classification.* Primary 55R70, 54E15; 54D30.

Key words and phrases. fiberwise compact space, fiberwise Pervin quasi-uniformity, fiberwise quasi-uniformly continuous, fiberwise quasi-uniform space, fiberwise topology, fiberwise uniform space.

I. M. James obtained the following proposition in [3, Chapter 3].

> **Proposition 17.1**. Let X be a fiberwise compact and fiberwise regular space over B, with B regular. Then there exists a unique fiberwise uniform structure Ω on X, compatible with the fiberwise topology, in which the members of Ω are the nbds of the diagonal.

This proposition is false in a strict sense of the definition of "fiberwise uniform structure." In this section, we remedy this proposition in the form of Theorem 3.

In section 3, we introduce a new notion of fiberwise quasi-uniform spaces which is a common extension of both fiberwise uniform spaces [5] and quasi-uniform spaces [2]. We investigate the basic properties of fiberwise quasi-uniform spaces. In section 4, we investigate the fiberwise quasi-uniformizability of fiberwise spaces.

Last, in section 5, we prove the main theorems. We begin here with a little background of these theorems.

Theorem 1 is a common extension of Theorem 3 and the following theorem.

> **Theorem 1.20** [2]**.** Let (X, τ_X) be a compact Hausdorff space and let G be a closed partial order on X. There exists exactly one quasi-uniformity $\mathcal{U}$ on X such that $\bigcap \mathcal{U} = G$ and $\tau(\mathcal{U}^*) = \tau_X$.

Theorem 1. *Let (X, τ_X) be a fiberwise space X with a topology τ_X, and B a regular space. Let X be a fiberwise compact fiberwise Hausdorff space over B and G be a relation on X such that $G = \bigcup_{b\in B} G_b$, where $G_b = G \cap X_b^2$ for each $b \in B$, and G_b is a closed partial order on X_b. Then there is exactly one fiberwise quasi-uniformity $\mathcal{U}$ on X such that $\tau(\mathcal{U}^*) = \tau_X$ and $(\bigcap \mathcal{U}) \cap X_b^2 = G_b$.*

Theorem 2 is a common extension of Theorem 4 (see page 87) (cf. [3, Corollary 17.2]) and the following theorem.

> **Theorem 1.21** [2]**.** Let $(X,\mathcal{U})$ and $(Y, \mathcal{V})$ be quasi-uniform spaces and suppose that $(X, \tau(\mathcal{U}^*))$ is a compact Hausdorff space. If $f : X \to Y$ is $\tau(\mathcal{U})$-$\tau(\mathcal{V})$ continuous and $\tau(\mathcal{U}^{-1})$-$\tau(\mathcal{V}^{-1})$ continuous, then $f : (X,\mathcal{U}) \to (Y, \mathcal{V})$ is quasi-uniformly continuous.

Theorem 2. *Let* $(X,\mathcal{U})$ *and* $(Y,\mathcal{V})$ *be fiberwise quasi-uniform spaces over* B*, with* B *regular, and* $(X,\tau(\mathcal{U}^*))$ *be the fiberwise compact and fiberwise Hausdorff space over* B*. If for projections* $p : X \to B$ *and* $q : Y \to B$*, a fiberwise function* $f : X \to Y$ *(i.e.,* $p = q \circ f$*) is* $\tau(\mathcal{U})$*-*$\tau(\mathcal{V})$ *continuous and* $\tau(\mathcal{U}^{-1})$*-*$\tau(\mathcal{V}^{-1})$ *continuous, then* $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ *is fiberwise quasi-uniformly continuous.*

For a set X, a function $p : X \to B$, $W \subset B$ and $b \in B$, $p^{-1}(W) = X_W$, $p^{-1}(b) = X_b$, $X_W \times X_W = X_W^2$ and $X \times X = X^2$. For $D, E \subset X^2$, $D \circ E = \{(x,z) | \exists y \in X$ such that $(x,y) \in D, (y,z) \in E\}$, $D^{-1} = \{(y,x) | (x,y) \in D\}$, and $D[x] = \{y | (x,y) \in D\}$. For a quasi-uniformity $\mathcal{U}$ on X, let $\mathcal{U}^{-1} = \{U^{-1} | U \in \mathcal{U}\}$, and $\mathcal{U}^*$ be the fiberwise quasi-uniformity generated by $\{U \cap U^{-1} | U \in \mathcal{U}\}$. For a (fiberwise) quasi-uniform space $(X,\mathcal{U})$, $\tau(\mathcal{U})$, $\tau(\mathcal{U}^{-1})$, and $\tau(\mathcal{U}^*)$ are (fiberwise) topologies induced by $\mathcal{U}$, $\mathcal{U}^{-1}$, and $\mathcal{U}^*$, respectively.

For a map $p : X \to B$, X is said to be a *fiberwise* T_0*-space* (*fiberwise Hausdorff space*, respectively) if for any different points $x, y \in X$ with $p(x) = p(y)$, at least one of the points x, y has a nbd in X not containing the other point (the points x and y have disjoint nbds in X, respectively). Further, a fiberwise T_0-space X is said to be *fiberwise regular* if for any point $x \in X$ and a closed subset F of X such that $x \notin F$ there exists a nbd $W \in N(p(x))$ such that x and $F \cap X_W$ have disjoint nbds in X_W.

In this paper, we assume that all maps are continuous. For other terminology and definitions in the topological category TOP and the fiberwise category TOP_B, one can consult [1] and [3], respectively, and for quasi-uniform spaces, see [2].

2. Fiberwise compactness and uniformities

In this section, we discuss the difference of fiberwise uniformities in [3] and [5] and show that the assertion of Proposition 17.1 in [3] is false in the strict sense of its definition and relieve its difficulty by using the notion of "fiberwise entourage uniformity" in [5].

We begin with the definition of fiberwise uniform structure.

Definition 2.1 ([3, Section 12])**.** Let X be a fiberwise set over B. By a *fiberwise uniform structure* on X, we mean a filter Ω on X^2 satisfying three conditions, as follows.

(FU1) Each $D \in \Omega$ contains the diagonal Δ of X.

(FU2) For any $D \in \Omega$ and $b \in B$, there exist $W \in N(b)$ and $E \in \Omega$ such that $X_W^2 \cap E \subset D^{-1}$.

(FU3) For any $D \in \Omega$ and $b \in B$, there exist $W \in N(b)$ and $E \in \Omega$ such that $(X_W^2 \cap E) \circ (X_W^2 \cap E) \subset D$.

Now we can construct the following example.

Example 2.2. Let $X = B$ be the set of all positive real numbers with the usual topology, and let $p : X \to B$ be the identity map. Then X is a fiberwise compact and fiberwise regular space over B. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two families of X^2 constructed as follows:

$\mathcal{B}_1 = \{U_\epsilon | U_\epsilon = \{(x,y) | x - \epsilon < y < x + \epsilon\},\ \epsilon > 0\}$,

$\mathcal{B}_2 = \{U_{\epsilon,a} | U_{\epsilon,a} = \{(x,y) | x - \epsilon < y < \sqrt{x^2 + a}\},\ \epsilon > 0, a > 0\}$.

Let Ω_1 and Ω_2 be the filters on X^2 generated by $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, and let Ω be the filter on X^2 which contains all nbds of the diagonal. Then it is easy to see that Ω_1, Ω_2, and Ω are different from each other.

On the other hand, in [5], we introduced a notion of slightly stronger fiberwise uniformity (called *fiberwise entourage uniformity*) in order to discuss the relationship between the fiberwise uniformities by using entourages and coverings. This notion of fiberwise entourage uniformity seems to relieve the difficulty in the above.

Definition 2.3 ([5])**.** Let X be a fiberwise set over B. By a *fiberwise entourage uniformity* on X, we mean a filter Ω on X^2 satisfying four conditions: (FU1), (FU2), and (FU3), above, and

(FU4) If $D \subset X^2$ satisfies that for each $b \in B$, there exist $W \in N(b)$ and $E \in \Omega$ such that $X_W^2 \cap E \subset D$, then $D \in \Omega$.

We call X with Ω a *fiberwise entourage uniform space*, denoted by (X, Ω).

It is easily verified that, in Example 2.2, Ω_1 and Ω_2 are fiberwise uniform structures but not fiberwise entourage uniformities on X, and Ω is a fiberwise entourage uniformity on X.

To remedy Proposition 17.1 in [3], we shall introduce some notions.

For a fiberwise entourage uniformity Ω on X, a subfamily $\mathcal{B}$ of Ω is said to be a *fiberwise uniform base* (briefly, *fiberwise u-base*) if $\mathcal{B}$ is a filter-base and satisfies the conditions (FU1), (FU2), (FU3), in Definition 2.1, and the following:

For each $D \in \Omega$ and $b \in B$, there exist $W \in N(b)$ and $E \in \mathcal{B}$ such that $X^2_W \cap E \subset D$.

A subfamily $\mathcal{S}$ of Ω is said to be a *fiberwise uniform subbase* (briefly, *fiberwise u-subbase*) if $\mathcal{S}$ is a filter-base and the family of all finite intersections of members of $\mathcal{S}$ is a fiberwise u-base of Ω.

A family $\mathcal{G}$ of subsets of X^2 is said to be a *fiberwise uniform germ* (briefly, *fiberwise u-germ*) if $\mathcal{G}$ is a filter-base and satisfies the conditions (FU1), (FU2), and (FU3). A family $\mathcal{S}$ of subsets of X^2 is said to be a *fiberwise uniform subgerm* (briefly, *fiberwise u-subgerm*) if $\mathcal{S}$ is a filter-base and the family of all finite intersections of members of $\mathcal{S}$ is a fiberwise u-germ.

It is clear that, for a fiberwise u-germ $\mathcal{G}$, the family

$$\Omega = \{D | \forall b \in B, \exists E \in \mathcal{G} \text{ such that } X^2_W \cap E \subset D\}$$

is a fiberwise entourage uniformity on X. Then it is clear that $\mathcal{G}$ is a fiberwise u-base of Ω. (Ω is said to be the fiberwise entourage uniformity *generated* by $\mathcal{G}$.)

In Example 2.2, Ω_1 and Ω_2 are fiberwise u-germs and the fiberwise entourage uniformities generated by Ω_1 and Ω_2 are equal to the fiberwise entourage uniformity Ω.

We can remedy Proposition 17.1 and Corollary 17.2 in [3] in the following theorems. The fiberwise uniform topology is the fiberwise topology induced by the (entourage) uniformity (cf. [3, Section 13] and [5, Section 3]). Proofs of the theorems are omitted because these are almost all the same as those in [3].

Theorem 3. *Let X be a fiberwise compact and fiberwise regular space over B, with B regular. Then there exists a unique fiberwise entourage uniformity Ω on X, compatible with the fiberwise topology, in which the members of Ω are the nbds of the diagonal.*

Theorem 4. *Let $f : X \to Y$ be a fiberwise function, where X and Y are fiberwise entourage uniform spaces over B, with B regular. Suppose that X is fiberwise compact over B in the fiberwise uniform topology. If f is continuous in the fiberwise uniform topology, then f is fiberwise uniformly continuous.*

3. Fiberwise quasi-uniform spaces and basic properties

In this section, we define a new notion of fiberwise quasi-uniform spaces, some related notions, and study some basic properties. We begin with the following definition.

Definition 3.1. Let X be a fiberwise set over B. By a *fiberwise quasi-uniformity* on X, we mean a filter $\mathcal{U}$ on X^2 satisfying conditions (FU1), (FU3) in Definition 2.1 and (FU4) in Definition 2.3.

By a *fiberwise quasi-uniform space* $(X,\mathcal{U})$, we mean a fiberwise set X with a fiberwise quasi-uniformity $\mathcal{U}$.

Fiberwise quasi-uniform spaces over a point can be regarded as quasi-uniform spaces in the ordinary sense. If $\mathcal{U}$ is a fiberwise quasi-uniformity, then $\mathcal{U}^{-1}$ is also a fiberwise quasi-uniformity and is called the *conjugate* of $\mathcal{U}$.

Further, note that our definition of fiberwise quasi-uniformity is an extended version of a fiberwise entourage uniformity (Definition 2.3) but is not an extended one of fiberwise uniform structure (Definition 2.1).

It is easily verified that for a fiberwise quasi-uniformity $\mathcal{U}$ on X the filter $\mathcal{U}^*$ is a fiberwise entourage uniformity on X.

For a fiberwise quasi-uniformity $\mathcal{U}$ on X, a subfamily $\mathcal{B}$ of $\mathcal{U}$ is said to be a *fiberwise quasi-uniform base* (briefly, *fiberwise qu-base*) if $\mathcal{B}$ is a filter-base and satisfies the conditions (FU1), (FU3), and the following:

For each $U \in \mathcal{U}$ and $b \in B$, there exist $W \in N(b)$ and $V \in \mathcal{B}$ such that $X_W^2 \cap V \subset U$.

A subfamily $\mathcal{S}$ of $\mathcal{U}$ is said to be a *fiberwise quasi-uniform subbase* (briefly, *fiberwise qu-subbase*) if $\mathcal{S}$ is a filter-base and the family of all finite intersections of members of $\mathcal{S}$ is a fiberwise qu-base of $\mathcal{U}$.

A family $\mathcal{G}$ of subsets of X^2 is said to be a *fiberwise quasi-uniform germ* (briefly, *fiberwise qu-germ*) if $\mathcal{G}$ is a filter-base and satisfies the conditions (FU1) and (FU3). A family $\mathcal{S}$ of subsets of X^2 is said to be a *fiberwise quasi-uniform subgerm* (briefly, *fiberwise qu-subgerm*) if $\mathcal{S}$ is a filter-base and the family of all finite intersections of members of $\mathcal{S}$ is a fiberwise qu-germ.

It is clear that, for a fiberwise qu-germ $\mathcal{G}$, the family

$$\mathcal{U} = \{U | \forall b \in B, \exists W \in N(b) \exists V \in \mathcal{G} \text{ such that } V \cap X_W^2 \subset U\}$$

is a fiberwise quasi-uniformity on X. Then it is clear that $\mathcal{G}$ is a fiberwise qu-base of $\mathcal{U}$. ($\mathcal{U}$ is said to be the fiberwise quasi-uniformity *generated* by $\mathcal{G}$.)

If $\mathcal{U}_1$ and $\mathcal{U}_2$ are fiberwise quasi-uniformities on a fiberwise set X over B, $\mathcal{U}_1$ is *finer* than $\mathcal{U}_2$ (or $\mathcal{U}_2$ *coarser* than $\mathcal{U}_1$) if each member of $\mathcal{U}_2$ contains a member of $\mathcal{U}_1$.

If $\mathcal{U}$ is a fiberwise quasi-uniformity on X, then the family $\{U \cap U^{-1} \mid U \in \mathcal{U}\}$ is a fiberwise qu-germ and generates the fiberwise entourage uniformity $\mathcal{U}^*$, which is the coarsest fiberwise entourage uniformity containing $\mathcal{U}$.

Let $\{\mathcal{U}_i \mid i \in A\}$ be a family of fiberwise quasi-uniformities on a fiberwise set X over B. The *supremum* of $\{\mathcal{U}_i \mid i \in A\}$ is the coarsest fiberwise quasi-uniformity on X that is finer than every $\mathcal{U}_i$. The *infimum* of $\{\mathcal{U}_i \mid i \in A\}$ is the finest fiberwise quasi-uniformity on X that is coarser than every $\mathcal{U}_i$. We denote the supremum and the infimum of $\{\mathcal{U}_i \mid i \in A\}$ by $\sup\{\mathcal{U}_i\}$ and $\inf\{\mathcal{U}_i\}$, respectively.

The following proposition holds.

Proposition 3.2. *Let $\{\mathcal{U}_i \mid i \in A\}$ be a family of fiberwise quasi-uniformities on a fiberwise set X over B. The supremum and the infimum always exist.*

Proof: Let $\mathcal{B} = \bigcup_{i \in A} \mathcal{U}_i$ and $\mathcal{B}' = \{U_1 \cap \cdots \cap U_n \mid U_j \in \mathcal{B}, j \in \{1, \cdots, n\}, n \in \mathbb{N}\}$. Then it is easy to see that $\mathcal{B}'$ is a fiberwise qu-germ of X, and that the fiberwise quasi-uniformity generated by $\mathcal{B}'$ is $\sup\{\mathcal{U}_i\}$.

For the existence of the infimum of $\{\mathcal{U}_i \mid i \in A\}$, let $\mathcal{U} = \bigcap_{i \in A} \mathcal{U}_i$. Then it is easy to see that $\mathcal{U}$ is the required fiberwise quasi-uniformity. □

Definition 3.3. Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be fiberwise (quasi-, respectively) uniform spaces. A fiberwise function $f : X \to Y$ is *fiberwise* (*quasi-*, respectively) *uniformly continuous* if for each $V \in \mathcal{V}$ and each point $b \in B$, there exist $W \in N(b)$ and $U \in \mathcal{U}$ such that $U \cap X_W^2 \subset (f \times f)^{-1}(V)$.

For fiberwise quasi-uniform spaces $(X, \mathcal{U})$ and $(Y, \mathcal{V})$, let $\mathcal{B}_{\mathcal{U}}$ and $\mathcal{B}_{\mathcal{V}}$ be fiberwise qu-bases for $\mathcal{U}$ and $\mathcal{V}$, respectively. Then it is easy to see that a fiberwise function $f : X \to Y$ is fiberwise quasi-uniformly continuous if and only if, for $V \in \mathcal{B}_{\mathcal{V}}$ and $b \in B$, there exist $U \in \mathcal{U}$ and $W \in N(b)$ such that $U \cap X_W^2 \subset (f \times f)^{-1}(V)$.

Let X, Y, and Z be fiberwise quasi-uniform spaces over B and let $f : X \to Y$ and $g : Y \to Z$ be fiberwise functions. Since $gf \times gf = (g \times g) \circ (f \times f)$, fiberwise quasi-uniformly continuities of f and g imply that gf is fiberwise quasi-uniformly continuous.

Let X and Y be fiberwise quasi-uniform spaces over a space B and let $f : X \to Y$ be a fiberwise bijection. Then f is a *fiberwise quasi-unimorphism* if f and f^{-1} are fiberwise quasi-uniformly continuous.

The following propositions can be easily proved, so we omit the proofs.

Proposition 3.4. *Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be fiberwise quasi-uniform spaces over a space B. If $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is fiberwise quasi-uniformly continuous, then $f : (X,\mathcal{U}^{-1}) \to (Y,\mathcal{V}^{-1})$ is fiberwise quasi-uniformly continuous and $f : (X,\mathcal{U}^*) \to (Y,\mathcal{V}^*)$ is fiberwise uniformly continuous.*

Proposition 3.5. *Let X be a fiberwise set over B. For each $i \in A$, let $(Y_i,\mathcal{V}_i)$ be a fiberwise quasi-uniform space over B and let $f_i : X \to Y_i$ be a fiberwise function. Then the family $\{(f_i \times f_i)^{-1}(V) \mid V \in \mathcal{V}_i, i \in A\}$ forms a fiberwise qu-subgerm, which generates the coarsest fiberwise quasi-uniformity $\mathcal{U}$ on X such that $f_i : (X,\mathcal{U}) \to (Y_i,\mathcal{V}_i)$ is fiberwise quasi-uniformly continuous for each $i \in A$.*

Proposition 3.6. *Let X and Y be fiberwise sets over B and let $\{\mathcal{U}_i \mid i \in A\}$ and $\{\mathcal{V}_i \mid i \in A\}$ be families of fiberwise quasi-uniformities on X and Y, respectively. Let $\mathcal{U} = \sup\{\mathcal{U}_i\}$ and $\mathcal{V} = \sup\{\mathcal{V}_i\}$. If for each $i \in A$, $f : (X,\mathcal{U}_i) \to (Y,\mathcal{V}_i)$ is fiberwise quasi-uniformly continuous, then $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is fiberwise quasi-uniformly continuous.*

Let $(X,\mathcal{U})$ be a fiberwise quasi-uniform space over B and let $E \subset X$. The fiberwise quasi-uniformity $\{U \cap E^2 \mid U \in \mathcal{U}\}$ on E is called the fiberwise quasi-uniformity *induced* by $\mathcal{U}$ and denoted by $\mathcal{U}|_{E\times E}$.

Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be fiberwise quasi-uniform spaces over B, let $f : X \to Y$ be a fiberwise function, and let $E \subset X$. If $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is fiberwise quasi-uniformly continuous, then $f|_E : (E,\mathcal{U}|_{E\times E}) \to (Y,\mathcal{V})$ is fiberwise quasi-uniformly continuous. Let $(X,\mathcal{U})$ be a fiberwise quasi-uniform space over B. If $F \subset E \subset X$, then $\mathcal{U}|_{F\times F} = (\mathcal{U}|_{E\times E})|_{F\times F}$.

Let $\{(X_i,\mathcal{U}_i) \mid i \in A\}$ be a family of fiberwise quasi-uniform spaces over B and let $X = \prod_B X_i$. The *product fiberwise quasi-uniformity* is the coarsest fiberwise quasi-uniformity on X for which all the projections $\pi_i : X \to X_i$ are fiberwise quasi-uniformly continuous. The family of all sets of the form $(\pi_i \times \pi_i)^{-1}(U_i)$, for each $U_i \in \mathcal{U}_i$, $i \in A$, is a fiberwise qu-subgerm for the product fiberwise quasi-uniformity of $\{\mathcal{U}_i \mid i \in A\}$.

The following is obvious.

Proposition 3.7. *Let $(X,\mathcal{U})$ and $(Y_i, \mathcal{V}_i)$ be fiberwise quasi-uniform spaces over B for each $i \in A$ and $\mathcal{V}$ the product fiberwise quasi-uniformity of $\{(Y_i, \mathcal{V}_i) \mid i \in A\}$. Then a fiberwise function $f : (X,\mathcal{U}) \to (\prod_B Y_i, \mathcal{V})$ is fiberwise quasi-uniformly continuous if and only if $\pi_i f : (X,\mathcal{U}) \to (Y_i, \mathcal{V}_i)$ is fiberwise quasi-uniformly continuous for each $i \in A$.*

For a fiberwise uniform space (fiberwise entourage uniform space, respectively) $(X,\mathcal{U})$ over B, the fiberwise uniform topology (fiberwise topology, respectively) induced by $\mathcal{U}$ was discussed in [3, Section 13] ([5, Section 3], respectively). For a fiberwise quasi-uniform space $(X,\mathcal{U})$, the fiberwise quasi-uniform topology $\tau(\mathcal{U})$ is defined below.

Definition 3.8. Let $(X,\mathcal{U})$ be a fiberwise quasi-uniform space over B. We denote the topology generated by the nbd system $\{\mathcal{N}(x)|x \in X\}$ where $\mathcal{N}(x) = \{U[x] \cap X_W | U \in \mathcal{U}, W \in N(p(x))\}$ as $\tau(\mathcal{U})$ and we call it the *fiberwise quasi-uniform topology.*

In fact, we can prove that $\{\mathcal{N}(x)|x \in X\}$ satisfies the axiom of nbd system. The only condition which may not be entirely obvious is the coherence condition. To verify this, for each $U[x] \cap X_W \in \mathcal{N}(x)$, where $x \in X$, $W \in N(p(x))$, and $U \in \mathcal{U}$, there exist $V \in \mathcal{U}$ and $W' \in N(p(x))$ such that $(X^2_{W'} \cap V) \circ (X^2_{W'} \cap V) \subset U$. Let $O = W \cap W'$ and $V[x] \cap X_O \in \mathcal{N}(x)$. For each $y \in V[x] \cap X_O$, it is easy to see that $V[y] \cap X_O \subset U[x]$. Therefore, $U[x] \cap X_W \in \mathcal{N}(y)$, which completes the proof.

We shall show some propositions which are used in section 5.

Proposition 3.9. *Let $(X,\mathcal{U})$ be a fiberwise quasi-uniform space over B.*

(1) *$(X, \tau(\mathcal{U}))$ is a fiberwise T_0-space if and only if $(\bigcap \mathcal{U}) \cap X^2_b$ is a partial order on X_b for each $b \in B$.*
(2) *$(X, \tau(\mathcal{U}))$ is a fiberwise T_0-space if and only if $(X, \tau(\mathcal{U}^*))$ is a fiberwise Hausdorff space.*

Thus, $(\bigcap \mathcal{U}) \cap X_b^2$ is a partial order on X_b for each $b \in B$ if and only if $(X, \tau(\mathcal{U}^*))$ is a fiberwise Hausdorff space.

Proof: (1) ($\Rightarrow$): For each $b \in B$, we show that $(\bigcap \mathcal{U}) \cap X_b^2$ is a partial order on X_b. First, it is clear that $(x, x) \in (\bigcap \mathcal{U}) \cap X_b^2$ for every $x \in X_b$. Next, let $(x, y), (y, z) \in (\bigcap \mathcal{U}) \cap X_b^2$. Then for any $U \in \mathcal{U}$, there exist $W \in N(b)$ and $V \in \mathcal{U}$ such that $(X_W^2 \cap V) \circ (X_W^2 \cap V) \subset U$; it is easy to show $(x, z) \in U$, which shows $(x, z) \in (\bigcap \mathcal{U}) \cap X_b^2$. Finally, for each x,$y \in X_b (x \neq y)$, since $(X, \tau(\mathcal{U}))$ is a fiberwise T_0-space, there exists $U \in \mathcal{U}$ such that $x \notin U[y]$ or $y \notin U[x]$. Therefore, $x \notin U[y] \cap X_b$ or $y \notin U[x] \cap X_b$, and $(x, y) \notin (\bigcap \mathcal{U}) \cap X_b^2$ or $(y, x) \notin (\bigcap \mathcal{U}) \cap X_b^2$. Thus, $(\bigcap \mathcal{U}) \cap X_b^2$ is a partial order on X_b.

($\Leftarrow$): For each x,$y \in X_b$ $(x \neq y)$ where $b \in B$, since $(\bigcap \mathcal{U}) \cap X_b^2$ is a partial order on X_b, $(x, y) \notin (\bigcap \mathcal{U}) \cap X_b^2$ or $(y, x) \notin (\bigcap \mathcal{U}) \cap X_b^2$. There exists $U \in \mathcal{U}$ such that $x \notin U[y]$ or $y \notin U[x]$. Therefore, $(X, \tau(\mathcal{U}))$ is a fiberwise T_0-space.

(2) ($\Rightarrow$): For each $b \in B$ and $x, x' \in X_b$ $(x \neq x')$, there exists a $\tau(\mathcal{U})$-nbd O of x such that $x' \notin O$. So, there exists $U \in \mathcal{U}$ such that $U[x] \subset O$. There exist $V \in \mathcal{U}$ and $W \in N(b)$ such that $(V \cap X_W^2) \circ (V \cap X_W^2) \subset U$. Then $(V \cap V^{-1} \cap X_W^2)[x']$ and $(V \cap V^{-1} \cap X_W^2)[x] \in \tau(\mathcal{U}^*)$, and it is easy to see $(V \cap V^{-1} \cap X_W^2)[x'] \cap (V \cap V^{-1} \cap X_W^2)[x] = \emptyset$. Thus, $(X, \tau(\mathcal{U}^*))$ is a fiberwise Hausdorff space.

($\Leftarrow$): For each $b \in B$ and $x, x' \in X_b$ $(x \neq x')$, there exist a $\tau(\mathcal{U}^*)$-nbd O of x and a $\tau(\mathcal{U}^*)$-nbd O' of x' such that $O \cap O' = \emptyset$. So, there exist $U \in \mathcal{U}^*$ and $W \in N(b)$ such that $U[x] \cap X_W \subset O$. There exists $V \in \mathcal{U}$ such that $V \cap V^{-1} \subset U$. Since $x' \notin U[x] \cap X_W$, $x' \notin (V \cap V^{-1})[x] \cap X_W$. Therefore, $(x, x') \notin V \cap X_W^2$ or $(x, x') \notin V^{-1} \cap X_W^2$. Thus, $x' \notin V[x] \cap X_W$ or $x \notin V[x'] \cap X_W$. □

Proposition 3.10. *Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be fiberwise quasi-uniform spaces over B. If a fiberwise function $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is fiberwise quasi-uniformly continuous, then f is $\tau(\mathcal{U})$-$\tau(\mathcal{V})$ continuous and $\tau(\mathcal{U}^{-1})$-$\tau(\mathcal{V}^{-1})$ continuous and $\tau(\mathcal{U}^*)$-$\tau(\mathcal{V}^*)$ continuous.*

Proof: To prove $\tau(\mathcal{U})$-$\tau(\mathcal{V})$ continuity, let $q : Y \to B$ be the projection. For each $O \in \tau(\mathcal{V})$ and $x \in f^{-1}(O)$, there exists $V \in \mathcal{V}$ and a nbd W of $q(f(x))$ such that $V[f(x)] \cap Y_W \subset O$. Since $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is fiberwise quasi-uniformly continuous, there exists $U \in \mathcal{U}$ and a nbd W' of $q(f(x))$ such that $X^2_{W'} \cap U \subset (f \times f)^{-1}(V)$. Therefore,

$$\begin{array}{rcl} f^{-1}(V[f(x)] \cap Y_W) & \subset & f^{-1}(O) \\ ((f \times f)^{-1}(V))[x] \cap X_W & \subset & f^{-1}(V[f(x)] \cap Y_W) \\ (X^2_{W'} \cap U)[x] \cap X_W & \subset & ((f \times f)^{-1}(V))[x] \cap X_W. \end{array}$$

Therefore, $U[x] \cap X_{W \cap W'} \subset f^{-1}(O)$, $f^{-1}(O) \in \tau(\mathcal{U})$. It follows that f is $\tau(\mathcal{U})$-$\tau(\mathcal{V})$ continuous. The other continuities follow from Proposition 3.4. □

Proposition 3.11. *Let τ_1 and τ_2 be fiberwise topologies on X, and let $(Y, \mathcal{V})$ be a fiberwise quasi-uniform space. Let $f : X \to Y$ be a fiberwise function such that f is τ_1-$\tau(\mathcal{V})$ continuous and τ_2-$\tau(\mathcal{V}^{-1})$ continuous. Then for each $V \in \mathcal{V}$, $(f \times f)^{-1}(V)$ is a $\tau_2 \times \tau_1$-nbd of the diagonal Δ_X.*

Proof: Let $V \in \mathcal{V}$, $x \in X$, and $b = p(x)$. Then there exist $W \in N(b)$ and $V_1 \in \mathcal{V}$ such that $(Y^2_W \cap V_1) \circ (Y^2_W \cap V_1) \subset V$. Since f is τ_1-$\tau(\mathcal{V})$ continuous and τ_2-$\tau(\mathcal{V}^{-1})$ continuous, there exist $G_1 \in \tau_1$ and $G_2 \in \tau_2$ such that $x \in G_1 \cap G_2$, $f(G_1) \subset V_1[f(x)] \cap Y_W$, and $f(G_2) \subset V_1^{-1}[f(x)] \cap Y_W$. Then for every $(y, z) \in G_2 \times G_1$, $(f(x), f(y)) \in V_1^{-1}$, $f(y) \in Y_W$, and $(f(x), f(z)) \in V_1$, $f(z) \in Y_W$. Thus, $(f(y), f(z)) = (f(y), f(x)) \circ (f(x), f(z)) \in (Y^2_W \cap V_1) \circ (Y^2_W \cap V_1) \subset V$, which shows $(f \times f)(G_2 \times G_1) \subset V$. □

4. Fiberwise quasi-uniformizability of fiberwise spaces

In this section, we prove that every fiberwise space is fiberwise quasi-uniformizable; that is, there exists a fiberwise quasi-uniformity $\mathcal{U}$ on X such that $\tau(\mathcal{U}) = \tau_X$. This idea is analogous to Pervin quasi-uniformity [2]. Further, we refer to the definition of "quasi-uniform space over B" in [6].

Let X be a set. For every subset A of X, let

$$S(A) := A \times A \cup (X - A) \times X.$$

Theorem 5. *Let (X, τ_X) be a fiberwise space over B. Then $\mathcal{S} = \{S(A) | A \in \tau_X\}$ is a fiberwise qu-subgerm for a fiberwise quasi-uniformity on X compatible with τ_X.*

Proof: For each $A \in \tau_X$, it is clear that $\Delta \subset S(A)$, and we can easily show that $S(A) \circ S(A) = S(A)$. Thus, $\mathcal{S}$ is a fiberwise qu-subgerm for a fiberwise quasi-uniformity on X.

Let $\tau(\mathcal{U})$ be the topology defined by the fiberwise quasi-uniformity $\mathcal{U}$ which is generated by the qu-subgerm $\mathcal{S}$.

Now we shall show that $\tau(\mathcal{U}) = \tau_X$. Let $O \in \tau_X$ and $x \in O$. Then $x \in S(O)[x] = O$. Thus, $O \in \tau(\mathcal{U})$.

Conversely, let $O \in \tau(\mathcal{U})$ and $x \in O$. Then there exist $W \in N(p(x))$ and $O_1, \cdots, O_n \in \tau_X$ such that $x \in \bigcap_{i=1}^{n} S(O_i)[x] \cap X_W \subset O$. In fact, if $x \notin \bigcup_{i=1}^{n} O_i$, then $X = \bigcap_{i=1}^{n} S(O_i)[x] \subset U[x]$. Therefore, $U[x] = X \in \tau_X$. If $x \in \bigcup_{i=1}^{n} O_i$, then $\bigcap_{i=1}^{n} S(O_i)[x] = \bigcap_{i=1}^{n} \{O_i | x \in O_i\}$ is a τ_X-open set and X_W is also τ_X-open. Thus, $\bigcap_{i=1}^{n} S(O_i)[x] \cap X_W$ is a τ-open set. Hence, $O \in \tau_X$. □

We call the fiberwise quasi-uniformity constructed in this theorem *fiberwise Pervin quasi-uniformity.*

Last, we shall note the definition of "quasi-uniform space over B" as presented by Jin Won Park and Byung Sik Lee [6]: A *quasi-uniform space X over B* is a function $p : X \to B$ in which both X and B are quasi-uniform spaces and p is a quasi-uniformly continuous map. This definition is a generalization of James in [4], where he studied $p : X \to B$ in the situation that both X and B are uniform spaces and p is a uniformly continuous map. On the other hand, our definition of fiberwise quasi-uniformity in section 3 is a generalization along the lines of Y. Konami and T. Miwa in [5], as well as James in [3].

In connection with the Pervin quasi-uniformity [2], the following proposition was obtained.

Proposition 2.17 [2]**.** For every continuous map $f : (X, \tau_X) \to (B, \tau_B)$, let $\mathcal{U}$ and $\mathcal{V}$ be the Pervin quasi-uniformities on X and B, respectively, then $f : (X,\mathcal{U}) \to (B, \mathcal{V})$ is quasi-uniformly continuous.

If we consider this proposition, we can say that every fiberwise space X over B can be considered as "quasi-uniform space X over B" (as in [6]), if we introduce the Pervin quasi-uniformities to X and B.

5. Proofs of theorems 1 and 2

In this section, we shall prove the main theorems.

Proof of Theorem 1: Let $\mathcal{U} = \{U \subset X^2 \mid U$ is a $(\tau_X \times \tau_X)$-nbd of G_b for any $b \in B\}$. First, we shall show that $\mathcal{U}$ is a fiberwise quasi-uniformity on X. Since $\mathcal{U}$ is a nbd filter of $G, \mathcal{U}$ is a filter on X^2. It is easy to see that Definition 3.1(FU1) (see Definition 2.1) is satisfied. To show Definition 3.1(FU3) (see Definition 2.1), we assume that there exist $b \in B$ and an open entourage $U \in \mathcal{U}$ such that $(X^2_W \cap V) \circ (X^2_W \cap V) \not\subset U$ for each $W \in N(b)$ and each $V \in \mathcal{U}$.

For each $V \in \mathcal{U}$ and $W \in N(b)$ let

$$V(W) = \{((x, y), z) \mid (x, z), (z, y) \in X^2_W \cap V,\ (x, y) \in U^c\}.$$

It is easy to see that $\mathcal{B} = \{V(W) \mid V \in \mathcal{U},\ W \in N(b)\}$ is a filter base on $U^c \times X$. Let $\mathcal{F}$ be the filter on $U^c \times X$ generated by $\mathcal{B}$. Since X is fiberwise compact and U^c is closed in $X \times X$, $(U^c \times X)$ is fiberwise compact over $(B \times B) \times B$. Further, since we can prove easily that $\mathcal{F}$ is a $((b, b), b)$-filter on $U^c \times X$, from [3, Proposition 4.3], there exists an adherence point $((r, s), t)$ of $\mathcal{F}$ such that $((r, s), t) \in (U^c \times X)_{((b,b),b)}$. We assert that $(r, t) \in G_b$. Suppose that $(r, t) \notin G_b$. Since X is fiberwise regular over B, $X \times X$ is fiberwise regular over $B \times B$. Further, since G_b is closed in X^2_b (hence in X^2), there exists an open nbd W' of (b, b), a nbd A of (r, t), and a nbd A' of G_b such that $A \cap A' = \emptyset$. From regularity of B, there exists $W \in N(b)$ such that $\overline{W} \times \overline{W} \subset W'$. Let $D = \{((x, y), z) \in U^c \times X \mid (x, z) \in A\}$. It is easy to see that D is a nbd of $((r, s), t)$. Let $V = A' \cup (X_{B-\overline{W}})^2$. Then it is easily verified that V is a nbd of G_b, $V \in \mathcal{U}$, and $V(W) \in \mathcal{B}$. Since D is a nbd of $((r, s), t)$ and $((r, s), t) \in \overline{V(W)}$, we have $D \cap V(W) \neq \emptyset$, which contradicts the constructions of D and $V(W)$. Thus, $(r, t) \in G_b$. By this same argument, we have $(t, s) \in G_b$. Since G_b is transitive, $(r, s) \in G_b \subset U$. This contradicts to $(r, s) \in U^c$. Thus, $\mathcal{U}$ satisfies Definition 2.1(FU3) and $\mathcal{U}$ is a fiberwise quasi-uniformity on X.

Now we shall show that

(i) $(\cap\mathcal{U}) \cap X^2_b = G_b$,
(ii) $\tau(\mathcal{U}^*) = \tau_X$, and
(iii) the uniqueness of $\mathcal{U}$ satisfying these conditions.

Proof of (i): (i) is trivial.

Proof of (ii): It is clear that $\tau(\mathcal{U}^*) \subset \tau_X$.

By Proposition 3.9, we have that $\tau(\mathcal{U}^*)$ is fiberwise Hausdorff. Now let $i : (X, \tau_X) \to (X, \tau(\mathcal{U}^*))$ be the identity map, then Corollary 3.20 and the comment after that in [3] show i is a fiberwise topological equivalence. That is $\tau(\mathcal{U}^*) = \tau_X$.

Proof of (iii): Let $\mathcal{V}$ be another fiberwise quasi-uniformity on X such that $(\cap\mathcal{V}) \cap X_b^2 = G_b$ and $\tau(\mathcal{V}^*) = \tau_X$.

Firstly, we show that $\mathcal{V}$ consists of all $\tau_X \times \tau_X$-nbds of G_b for all $b \in B$. Since $(\cap\mathcal{V}) \cap X_b^2 = G_b$, it is clear that $G_b \subset V$ for every $V \in \mathcal{V}$ and for every $b \in B$. Let $(x, y) \in G_b$. For every $V \in \mathcal{V}$, there exist $V' \in \mathcal{V}$ and $W \in N(b)$ such that $(V' \cap X_W^2) \circ (V' \cap X_W^2) \circ (V' \cap X_W^2) \subset V$. Then a $\tau_X \times \tau_X$-nbd $((V' \cap V'^{-1})[x] \cap X_W) \times ((V' \cap V'^{-1})[y] \cap X_W)$ of (x, y) is contained in V. Since for $(p, q) \in ((V' \cap V'^{-1})[x] \cap X_W) \times ((V' \cap V'^{-1})[y] \cap X_W)$, noting $p, q, x, y \in X_W$, we have $(p, x), (x, y), (y, q) \in V' \cap X_W^2$. Therefore, $(p, q) \in (V' \cap X_W^2)^3 \subset V$. This shows that V is a $\tau_X \times \tau_X$-nbd of G_b for every $b \in B$, i.e., $\mathcal{V} \subset \mathcal{U}$.

Next, suppose that $\mathcal{V} \neq \mathcal{U}$. This means there exists $U \in \mathcal{U}$ such that $U \notin \mathcal{V}$. Note that $V_\alpha - U \neq \emptyset$ for all $V_\alpha \in \mathcal{V}$.

For every $b \in B$, let

$$\mathcal{F}_b := \{(V_\alpha - U) \cap X_W^2 | V_\alpha \in \mathcal{V}, W \in N(b)\}.$$

Since $[(V_\alpha - U) \cap X_{W_1}^2] \cap [(V_\beta - U) \cap X_{W_2}^2] \neq \emptyset$ for every $V_\alpha, V_\beta \in \mathcal{V}$ and $W_1, W_2 \in N(b)$, if $\mathcal{F}_b$ is not a filter, then $\emptyset \in \mathcal{F}_b$. Then we have that $V_{\alpha_b} \cap X_{W_b}^2 \subset U$ for some $V_{\alpha_b} \in \mathcal{V}$ and $W_b \in N(b)$. By Definition 2.3(FU4), we have $U \in \mathcal{V}$, which is a contradiction.

Therefore, $\mathcal{F}_b$ is a filter for some $b \in B$. It is clear that $\mathcal{F}_b$ is a (b, b)-filter on $X \times X$. Since $X \times X$ is fiberwise compact, $\mathcal{F}_b$ has a $\tau_X \times \tau_X$-cluster point (x, y) that does not belong to G_b.

On the other hand, with a method similar to the proof of Proposition 13.5 in [3], we have that for each $V \in \mathcal{V}$, there exist $V' \in \mathcal{V}$ and $W \in N(b)$ such that $\mathrm{Cl}\ V' \cap X_W^2 \subset V$, where Cl is the closure operator of the topology $\tau_X \times \tau_X$. Then we have

$G_b = (\bigcap \mathcal{V}) \cap X_b^2 = (\bigcap\{\mathrm{Cl}\ V \cap X_W^2 | W \in N(b), V \in \mathcal{V}\}) \cap X_b^2$.

This contradicts the fact that (x, y) does not belong to G_b. Thus, $\mathcal{V} = \mathcal{U}$.

The proof of Theorem 1 is complete. □

Next, we shall prove the second main theorem.

Proof of Theorem 2: The proof consists of the following four steps. Let $G_b = (\bigcap \mathcal{U}) \cap X_b^2$ for each $b \in B$.

(1) For each $b \in B$, G_b is a partial order on X_b, and for each $U \in \mathcal{U}$, U is a $\tau(\mathcal{U}^*)^2$-nbd of G_b for each $b \in B$.

Proof of (1): The first part follows from Proposition 3.9. The second part follows from the definition of the fiberwise quasi-uniform topology (Definition 3.8).

(2) For each $b \in B$, G_b is closed in $(X^2, \tau(\mathcal{U}^*)^2)$.

Note that by the facts (1) and (2) and the construction of $\mathcal{U}$ in the proof of Theorem 1, $\mathcal{U}$ in this theorem satisfies the conditions in Theorem 1.

Proof of (2): To show that $(\bigcap \mathcal{U}) \cap X_b^2$ is closed in $(X^2, \tau(\mathcal{U}^*)^2)$, for every $(x, y) \notin (\bigcap \mathcal{U}) \cap X_b^2$ (so $x \neq y$), we shall show that there exist $W \in N(b)$ and $D \in \mathcal{U}$ such that

(*) $((D \cap D^{-1})[x] \times (D \cap D^{-1})[y] \cap X_W^2) \bigcap ((\bigcap \mathcal{U}) \cap X_b^2) = \emptyset.$

Assume that (*) does not hold. Since for every $D \in \mathcal{U}$ and $W \in N(b)$, (*) does not hold, for $D \in \mathcal{U}$ and $W \in N(b)$, there exist $E \in \mathcal{U}$ and $W_1 \in N(b)$ such that $W_1 \subset W$, $(X_{W_1}^2 \cap E) \circ (X_{W_1}^2 \cap E) \circ (X_{W_1}^2 \cap E) \subset D$. Therefore, there exists $(s, t) \in ((E \cap E^{-1})[x] \times (E \cap E^{-1})[y] \cap X_{W_1}^2) \cap ((\bigcap \mathcal{U}) \cap X_b^2)$. This shows $(x, y) = (x, s) \circ (s, t) \circ (t, y) \in (X_{W_1}^2 \cap E) \circ (X_{W_1}^2 \cap E) \circ (X_{W_1}^2 \cap E) \subset D$. Therefore, for any $D \in \mathcal{U}$, $(x, y) \in D$. Thus, $(x, y) \in (\bigcap \mathcal{U}) \cap X_b^2$, which is a contradiction.

(3) For each $V \in \mathcal{V}$, $(f \times f)^{-1}(V)$ is a $\tau(\mathcal{U}^{-1}) \times \tau(\mathcal{U})$-nbd of Δ_X in X^2.

Proof of (3): This follows from Proposition 3.11.

(4) For each $V \in \mathcal{V}$ and each $b \in B$, $(f \times f)^{-1}(V)$ is a $\tau(\mathcal{U}^{-1}) \times \tau(\mathcal{U})$-nbd of G_b in X^2.

Proof of (4): For this, we will prove the next two facts:

(i) $(\bigcap \mathcal{U}) \cap X_b^2 \subset (f \times f)^{-1}(V)$;

(ii) $(f \times f)^{-1}(V)$ is a $\tau(\mathcal{U}^{-1}) \times \tau(\mathcal{U})$-nbd of $(\bigcap \mathcal{U}) \cap X_b^2$.

Proof of (i): Assume that (i) does not hold. Then there exists $(x, y) \in (\bigcap \mathcal{U}) \cap X_b^2 - (f \times f)^{-1}(V)$. Since $V[f(x)]$ is a $\tau(\mathcal{V})$-nbd of $f(x)$, from the $\tau(\mathcal{U})$-$\tau(\mathcal{V})$-continuity, we have that there exist $U \in \mathcal{U}$ and $W \in N(b)$ such that $f(U[x] \cap X_W) \subset V[f(x)]$. This means that $U[x] \cap X_W \subset ((f \times f)^{-1}(V))[x]$. But by $(x, y) \in \bigcap \mathcal{U}$, we have $(x, y) \in U$. Therefore, $(x, y) \in (f \times f)^{-1}(V)$, which is a contradiction.

Proof of (ii): Let $(x, y) \in (\bigcap \mathcal{U}) \cap X_b^2$. By Proposition 3.11, $(f \times f)^{-1}(V)$ is a nbd of (x, x) and (y, y). Then there exist $U \in \mathcal{U}$ and $W \in N(b)$ such that $(U^{-1}[x] \cap X_W) \times (U[x] \cap X_W) \subset (f \times f)^{-1}(V)$, $(U^{-1}[y] \cap X_W) \times (U[y] \cap X_W) \subset (f \times f)^{-1}(V)$. For this U, there exist $U_1 \in \mathcal{U}$ and $W_1 \in N(b)$ such that $W_1 \subset W$, $(X_{W_1}^2 \cap U_1) \circ (X_{W_1}^2 \cap U_1) \circ (X_{W_1}^2 \cap U_1) \subset U$. Then $(U_1^{-1}[x] \cap X_{W_1}) \times (U_1[y] \cap X_{W_1}) \subset (f \times f)^{-1}(V)$.

Thus, from the fact $\tau(\mathcal{U}) \cup \tau(\mathcal{U}^{-1}) \subset \tau(\mathcal{U}^*)$, we have $(f \times f)^{-1}(V) \in \mathcal{U}$, so f is fiberwise quasi-uniformly continuous.

The proof of Theorem 2 is complete. □

References

[1] Ryszard Engelking, *General Topology*. Translated from the Polish by the author. 2nd ed. Sigma Series in Pure Mathematics, 6. Berlin: Heldermann Verlag, 1989.

[2] Peter Fletcher and William F. Lindgren, *Quasi-Uniform Spaces*. Lecture Notes in Pure and Applied Mathematics, 77. New York: Marcel Dekker, Inc., 1982.

[3] I. M. James, *Fibrewise Topology*. Cambridge Tracts in Mathematics, 91. Cambridge: Cambridge University Press, 1989.

[4] ———, *Introduction to Uniform Spaces*. London Mathematical Society Lecture Note Series, 144. Cambridge: Cambridge University Press,1990.

[5] Y. Konami and T. Miwa, *Fibrewise extensions, Shanin compactification and extensions of fibrewise maps*, Acta Math. Hungar. **122** (2009), no. 1-2, 1–28.

[6] Jin Won Park and Byung Sik Lee, *Fibrewise complete quasi-uniform spaces*, JP J. Geom. Topol. **4** (2004), no. 2, 157–166.

(Iwamoto) JFE Systems Inc.; Chiba, Japan

(Konami) Department of Mathematics; Shimane University; Matsue 690-8504, Japan

E-mail address: yoshi@math-konami.com

(Miwa) Department of Mathematics; Shimane University; Matsue 690-8504, Japan

Current address: Hamanogi 7-14-25; Matsue 690-0044, Japan

E-mail address: takuo-miwa@mable.ne.jp