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by

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ON n-FOLD HYPERSPACES OF CONTINUA, II

SERGIO MACÍAS

ABSTRACT. We prove that if X is an indecomposable continuum and Y is a hereditarily decomposable continuum, then neither their n-fold hyperspaces nor their n-fold hyperspace suspensions are homeomorphic. We characterize locally connected continua for which their n-fold hyperspaces are dimensionally homogeneous as those such continua X such that X does not contain free arcs or X is either an arc or a simple closed curve. We also prove that if X is a locally connected continuum such that its n-fold hyperspace suspension is dimensionally homogeneous, then X does not contain free arcs or X is either an arc or a simple closed curve. We show that the n-fold hyperspace and the n-fold hyperspace suspension of arc-smooth continua are arc-smooth.

1. Introduction

The notion of n-fold hyperspace suspension was introduced in [10]. This concept is a natural extension of the notion of hyperspace suspension introduced by Sam B. Nadler, Jr. [18].

In [14, Theorem 3.1 and Theorem 4.17], it was proven that indecomposable continua with the property of Kelley share neither n-fold hyperspaces nor n-fold hyperspace suspensions with decomposable continua. Those proofs show more than stated, we

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present those more general statements (Theorem 3.1 and Theorem 4.1, respectively). We prove that indecomposable continua do not share either the n-fold hyperspace or the n-fold hyperspace suspension with the m-fold hyperspace or the m-fold hyperspace suspension hereditarily decomposable continua, respectively (Theorem 3.5 and Theorem 4.5, respectively).

We characterize locally connected continua for which their n-fold hyperspaces are dimensionally homogeneous as those such continua X such that X does not contain free arcs or X is either an arc or a simple closed curve (Theorem 3.6). We also prove that if X is a locally connected continuum such that its n-fold hyperspace suspension is dimensionally homogeneous, then X does not contain free arcs or X is either an arc or a simple closed curve (Theorem 4.8) and prove a partial converse of this theorem (Theorem 4.9).

We show that the n-fold hyperspace and the n-fold hyperspace suspensions of arc-smooth continua are arc-smooth (Theorem 3.7 and Theorem 4.12, respectively).

2. Definitions

If (Z,d) is a metric space, then given $A \subset Z$ and $\varepsilon > 0$, the open ball about A of radius ε is denoted by $\mathcal{V}^d_\varepsilon(A)$, the interior of A is denoted by $Int_Z(A)$. The symbol $\mathbb{R}$ denotes the set of real numbers.

An *arc* is any space homeomorphic to $[0,1]$. The *end points* of an arc are the images of $\{0,1\}$ under a homeomorphism.

Given a metric space Z, the symbol $\dim(Z)$ denotes the topological dimension of Z. Also, if $z \in Z$, then $\dim_z(Z)$ denotes the dimension of the space Z at the point z [6]. A metric space Z is *dimensionally homogeneous* if for any two points $z_1, z_2 \in Z$, $\dim_{z_1}(Z) = \dim_{z_2}(Z)$. An n-dimensional compact connected metric space Z is a *Cantor manifold* provided that for each subset A of Z such that $\dim(A) \leq n-2$, we have that $Z \setminus A$ is connected.

A *continuum* is a nonempty compact, connected metric space. A continuum X is *freely contractible* provided that there exist a point p in X and a homotopy $K\colon X \times [0,1] \to X$ such that for each x in X, (1) $K(x,0) = p$, (2) $K(x,1) = x$, and (3) $K(K(x,s),t) = K(x, \min\{s,t\})$ for all $s,t \in [0,1]$.

Given a continuum X, we consider the following *hyperspaces* of X:

$$2^X = \{A \subset X \mid A \text{ is nonempty and closed}\}$$

and

$$\mathcal{C}_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ components}\},$$

where n is a positive integer. $\mathcal{C}_n(X)$ is called the *n-fold hyperspace* of X. These spaces are topologized with the Hausdorff metric defined as

$$\mathcal{H}(A,B) = \inf\{\varepsilon > 0 \mid A \subset \mathcal{V}^d_\varepsilon(B) \text{ and } B \subset \mathcal{V}^d_\varepsilon(A)\};$$

$\mathcal{H}$ always denotes the Hausdorff metric on 2^X. When $n = 1$, we write $\mathcal{C}(X)$ instead of $\mathcal{C}_1(X)$. An *order arc* in 2^X is a map $\gamma\colon [0,1] \to 2^X$ such that for each $t, s \in [0,1]$ such that $t < s$, we have that $\gamma(t) \subsetneq \gamma(s)$.

The symbol $\mathcal{F}_n(X)$ denotes the *n-fold symmetric product* of a continuum X; that is,

$$\mathcal{F}_n(X) = \{A \in \mathcal{C}_n(X) \mid A \text{ has at most } n \text{ points}\}.$$

If A is a nonempty subset of X, $\mathcal{C}_n(A)$ denotes the set $\{B \in \mathcal{C}_n(X) \mid B \subset A\}$.

By the *n-fold hyperspace suspension* of a continuum X, which is denoted by $HS_n(X)$, we mean the quotient space

$$HS_n(X) = \mathcal{C}_n(X)/\mathcal{F}_n(X)$$

with the quotient topology. The fact that $HS_n(X)$ is a continuum follows from [19, Theorem 3.10]. Note that $HS_1(X)$ corresponds to the hyperspace suspension $HS(X)$ defined by Nadler in [18].

Notation 2.1. Given a continuum X, $q^n_X\colon \mathcal{C}_n(X) \twoheadrightarrow HS_n(X)$ denotes the quotient map. Also, let F^n_X and T^n_X denote the points $q^n_X(\mathcal{F}_n(X))$ and $q^n_X(X)$, respectively.

Remark 2.2. Note that the sets $HS_n(X) \setminus \{F^n_X\}$ and $HS_n(X) \setminus \{T^n_X, F^n_X\}$ are homeomorphic to $\mathcal{C}_n(X) \setminus \mathcal{F}_n(X)$ and $\mathcal{C}_n(X) \setminus (\{X\} \cup \mathcal{F}_n(X))$, respectively, using the appropriate restriction of q^n_X.

Definitions not included here may be found in [17], [7], or [11].

3. n-fold hyperspaces

We begin by noting that Theorem 3.1 of [14] may be strengthened, with small changes to the proof.

Theorem 3.1. *Let X be an indecomposable continuum with the property of Kelley and let n and m be positive integers. If Y is a continuum such that $\mathcal{C}_m(Y)$ is homeomorphic to $\mathcal{C}_n(X)$, then Y is indecomposable.*

The following lemma is easy to establish.

Lemma 3.2. *Let X be a continuum, let n be a positive integer, and let $A \in \mathcal{C}_n(X)$. Then $\mathcal{C}_n(X) \setminus \{A\}$ is not arcwise connected if and only if $\mathcal{C}_n(X) \setminus (\{A\} \cup \mathcal{F}_n(X))$ is not arcwise connected.*

Theorem 3.3. *Let X be a continuum, let n be a positive integer, and let $A \in \mathcal{C}_n(X)$. Then $\mathcal{C}_n(X) \setminus \{A\}$ is not arcwise connected if and only if $HS_n(X) \setminus \{q_X^n(A), F_X^n\}$ is not arcwise connected.*

Proof: Suppose $\mathcal{C}_n(X) \setminus \{A\}$ is not arcwise connected. Then $A \in \mathcal{C}(X)$ [11, Theorem 6.5.2]. Hence, $\mathcal{C}_n(X) \setminus \mathcal{C}_n(A)$ is arcwise connected [11, Lemma 6.5.1]. Thus, there exists $B \in \mathcal{C}(A) \setminus \mathcal{F}_1(X)$ such that A belongs to each arc in $\mathcal{C}_n(X)$ joining B and X.

Assume $HS_n(X) \setminus \{q_X^n(A), F_X^n\}$ is arcwise connected. Then there exists an arc $\alpha\colon [0,1] \to HS_n(X) \setminus \{q_X^n(A), F_X^n\}$ such that $\alpha(0) = q_X^n(B)$ and $\alpha(1) = T_X^n$. Since q_X^n is a homeomorphism on $\mathcal{C}_n(X) \setminus \mathcal{F}_n(X)$ onto $HS_n(X) \setminus \{F_X^n\}$, $q_X^n \circ \alpha$ is an arc in $\mathcal{C}_n(X) \setminus (\{A\} \cup \mathcal{F}_n(X))$ joining B to X, a contradiction. Therefore, $HS_n(X) \setminus \{q_X^n(A), F_X^n\}$ is not arcwise connected.

Now, suppose $HS_n(X) \setminus \{q_X^n(A), F_X^n\}$ is not arcwise connected. Then $A \in \mathcal{C}(X)$ [12, Theorem 4.4]. Assume $\mathcal{C}_n(X) \setminus \{A\}$ is arcwise connected. Then, by Lemma 3.2, $\mathcal{C}_n(X) \setminus (\{A\} \cup \mathcal{F}_n(X))$ is arcwise connected. Thus, since $HS_n(X) \setminus \{q_X^n(A), F_X^n\} = q_X^n(\mathcal{C}_n(X) \setminus (\{A\} \cup \mathcal{F}_n(X)))$, we have that $HS_n(X) \setminus \{q_X^n(A), F_X^n\}$ is arcwise connected, a contradiction. Therefore, $\mathcal{C}_n(X) \setminus \{A\}$ is not arcwise connected. □

Lemma 3.4. *Let A be a proper decomposable subcontinuum of a continuum X and let n be a positive integer. If $\mathcal{C}_n(X) \setminus \{A\}$ is not arcwise connected, then $\mathcal{C}_n(X) \setminus \{A\}$ has exactly two arc components.*

Proof: Note that $\mathcal{C}_n(X)\setminus\mathcal{C}_n(A)$ is arcwise connected [11, Lemma 6.5.1]. Since A is a decomposable continuum, $\mathcal{C}_n(A)\setminus\{A\}$ is arcwise connected [11, Theorem 6.5.3]. Hence, since $\mathcal{C}_n(X)\setminus\{A\} = (\mathcal{C}_n(X)\setminus \mathcal{C}_n(A)) \cup \mathcal{C}_n(A) \setminus \{A\}$, we have that $\mathcal{C}_n(X) \setminus \{A\}$ has exactly two arc components. □

Theorem 3.5. *Let X be an indecomposable continuum and let n and m be positive integers. If Y is a hereditarily decomposable continuum, then $\mathcal{C}_m(Y)$ is not homeomorphic to $\mathcal{C}_n(X)$.*

Proof: Let us suppose that $\mathcal{C}_m(Y)$ is homeomorphic to $\mathcal{C}_n(X)$. Let $h\colon \mathcal{C}_n(X) \twoheadrightarrow \mathcal{C}_m(Y)$ be a homeomorphism. Since X is indecomposable, $\mathcal{C}_n(X)\setminus\{X\}$ is not arcwise connected. Recall that since X is indecomposable, X has uncountably many composants [5, Theorem 3–46]. Also, for each composant κ of X, $\mathcal{C}_n(\kappa)$ is an arc component of $\mathcal{C}_n(X) \setminus \{X\}$ [11, Theorem 6.5.11]. Hence, we have that $\mathcal{C}_n(X) \setminus \{X\}$ has uncountably many arc components. Then $\mathcal{C}_m(Y) \setminus \{h(X)\}$ has uncountably many arc components. Since $\mathcal{C}_m(Y) \setminus \{h(X)\}$ is not arcwise connected, $h(X) \in \mathcal{C}(Y)$ [11, Theorem 6.5.2]. Since Y is a hereditarily decomposable continuum, $h(X)$ is a decomposable subcontinuum of Y. Hence, by Lemma 3.4, $\mathcal{C}_m(Y) \setminus \{h(X)\}$ has exactly two arc components, a contradiction. Therefore, $\mathcal{C}_m(Y)$ is not homeomorphic to $\mathcal{C}_n(X)$. □

Now, we consider dimensionally homogeneous n-fold hyperspaces. The following result is a generalization to n-fold hyperspaces of [17, Theorem (2.16)].

Theorem 3.6. *Let X be a locally connected continuum, and let n be a positive integer. Then $\mathcal{C}_n(X)$ is dimensionally homogeneous if and only if X does not contain a free arc or X is an arc or a simple closed curve.*

Proof: Suppose $\mathcal{C}_n(X)$ is dimensionally homogeneous. Assume X contains a free arc. Then, by [11, Theorem 6.8.10], $\dim(\mathcal{C}_n(X)) = 2n$. Hence, by [11, Theorem 6.8.3], X is a graph. Thus, X is either an arc of a simple closed curve [14, Theorem 3.5].

Now, suppose X does not contain a free arc. Then $\mathcal{C}_n(X)$ is homeomorphic to the Hilbert cube $\mathcal{Q}$ [9, Theorem 7.1]. Since $\mathcal{Q}$ is homogeneous [16, Theorem 6.1.6], $\mathcal{C}_n(X)$ is dimensionally homogeneous.

If X is an arc or a simple closed curve, then $\mathcal{C}_n(X)$ is a Cantor manifold [15, Theorem 4.6]. By [6, A), pp. 93 and 94], Cantor manifolds are dimensionally homogeneous. Therefore, $\mathcal{C}_n(X)$ is dimensionally homogeneous.

Next, we consider arc-smoothness of n-fold hyperspaces. Recall that a continuum X is *arc-smooth* provided that there exist a point p and a map $\alpha\colon X \to \mathcal{C}(X)$ such that (i) $\alpha(p) = \{p\}$; (ii) for each $x \in X \setminus \{p\}$, $\alpha(x)$ is an arc joining p and x; and (iii) if $z \in \alpha(x)$, then $\alpha(z) \subset \alpha(x)$. The map α is called an *arc map* for X.

Theorem 3.7. *If X is an arc-smooth continuum and n is a positive integer, then $\mathcal{C}_n(X)$ is arc-smooth.*

Proof: By [4, Theorem II-3-B], X is freely contractible. Hence, there exist a point p and a homotopy $R\colon X \times [0,1] \to X$ such that for each x in X, (1) $R(x,0) = p$, (2) $R(x,1) = x$, and (3) $R(R(x,s),t) = R(x, \min\{s,t\})$ for all $s,t \in [0,1]$.

Define $G\colon \mathcal{C}_n(X) \times [0,1] \to \mathcal{C}_n(X)$ by

$$G(A,t) = \{R(a,t) \mid a \in A\}.$$

Note that G is continuous. Also observe that for each $A \in \mathcal{C}_n(X)$, $G(A,0) = \{p\}$ and $G(A,1) = A$. It is easy to verify that $G(G(A,s),t) = G(A, \min\{s,t\})$ for all $s,t \in [0,1]$ and each $A \in \mathcal{C}_n(X)$. Hence, $\mathcal{C}_n(X)$ is freely contractible. Therefore, $\mathcal{C}_n(X)$ is arc-smooth [4, Theorem II-3-B]. □

4. n-FOLD HYPERSPACE SUSPENSIONS

We start by noting that Theorem 4.17 of [14] may be strengthened, with few changes to the proof.

Theorem 4.1. *Let X be an indecomposable continuum with the property of Kelley and let n and m be positive integers. If Y is a continuum such that $HS_m(Y)$ is homeomorphic to $HS_n(X)$, then Y is indecomposable.*

Lemma 4.2. *Let A be a proper decomposable subcontinuum of a continuum X and let n be a positive integer. If $HS_n(X) \setminus \{q^n_X(A), F^n_X\}$ is not arcwise connected, then $HS_n(X) \setminus \{q^n_X(A), F^n_X\}$ has exactly two arc components.*

Proof: The lemma follows from Lemma 3.2 and the following facts:

$$\mathcal{C}_n(A) \setminus (\{A\} \cup \mathcal{F}_n(A)) = \mathcal{C}_n(A) \setminus (\{A\} \cup \mathcal{F}_n(X)),$$
$$q_X^n(\mathcal{C}_n(A) \setminus (\{A\} \cup \mathcal{F}_n(X))) = q_X^n(\mathcal{C}_n(A)) \setminus \{q_X^n(A), F_X^n\}, \text{ and}$$
$$HS_n(X) \setminus \{q_X^n(A), F_X^n\} = (q_X^n(\mathcal{C}_n(X)) \setminus q_X^n(\mathcal{C}_n(A))) \cup (q_X^n(\mathcal{C}_n(A)) \setminus \{q_X^n(A), F_X^n\}).$$ □

Lemma 4.3. *If X is an indecomposable continuum, then $HS_n(X) \setminus \{T_X^n, F_X^n\}$ has uncountably many arc components.*

Proof: Since X is an indecomposable continuum, by [10, Theorem 6.2], $HS_n(X) \setminus \{T_X^n, F_X^n\}$ is not arcwise connected. Since indecomposable continua have uncountably many composants [5, Theorem 3–46], and for each composant κ of X, $q_X^n(\mathcal{C}_n(\kappa) \setminus \mathcal{F}_n(\kappa))$ is an arc component of $HS_n(X) \setminus \{T_X^n, F_X^n\}$ [10, Theorem 6.4] (compare with [13, Theorem 2]), we have that $HS_n(X) \setminus \{T_X^n, F_X^n\}$ has uncountably many arc components. □

Lemma 4.4. *Let X be a continuum and let n be a positive integer. If χ_1 and χ_2 are two points of $HS_n(X)$ such that $HS_n(X) \setminus \{\chi_1, \chi_2\}$ is not arcwise connected, then $F_X^n \in \{\chi_1, \chi_2\}$.*

Proof: Suppose $F_X^n \notin \{\chi_1, \chi_2\}$. By [12, Theorem 4.3], we may assume that $T_X^n \notin \{\chi_1, \chi_2\}$.

First, we show that there exists an arc in $HS_n(X) \setminus \{\chi_1, \chi_2\}$ joining F_X^n and T_X^n.

If $(q_X^n)^{-1}(\chi_1) \cup (q_X^n)^{-1}(\chi_2) \neq X$, then there exists a point $x \in X \setminus ((q_X^n)^{-1}(\chi_1) \cup (q_X^n)^{-1}(\chi_2))$. Let α be an order arc in $\mathcal{C}(X)$ from $\{x\}$ to X [17, Theorem (1.8)]. Note that

$$\alpha \cap \{(q_X^n)^{-1}(\chi_1), (q_X^n)^{-1}(\chi_2)\} = \emptyset.$$

Hence, $q_X^n(\alpha)$ is an arc in $HS_n(X) \setminus \{\chi_1, \chi_2\}$ from F_X^n to T_X^n.

Now, assume that $(q_X^n)^{-1}(\chi_1) \cup (q_X^n)^{-1}(\chi_2) = X$. Since $X \notin \{(q_X^n)^{-1}(\chi_1), (q_X^n)^{-1}(\chi_2)\}$, there exist nondegenerate components A_1 and B_1 of $(q_X^n)^{-1}(\chi_1)$ and $(q_X^n)^{-1}(\chi_2)$, respectively, such that $A_1 \cap B_1 \neq \emptyset$. Let $x \in A_1 \cap B_1$ and let $\varepsilon = \frac{1}{2}\min\{\mathrm{diam}(A_1), \mathrm{diam}(B_1)\}$. By [19, Corollary 5.5], there exist two nondegenerate continua A and B such that $x \in A \subset A_1 \cap \mathcal{V}_\varepsilon(x)$ and $x \in B \subset B_1 \cap \mathcal{V}_\varepsilon(x)$. Let α_1 be an order arc in $\mathcal{C}(X)$ from $\{x\}$ to A [17, Theorem (1.8)]. Let α_2 be an order arc in $\mathcal{C}(X)$ from A to $A \cup B$. Let α_3 be an order arc in $\mathcal{C}(X)$ from $A \cup B$ to $= X$. Then

$\alpha_1 \cup \alpha_2 \cup \alpha_3$ is an arc in $\mathcal{C}_n(X) \setminus \{(q_X^n)^{-1}(\chi_1), (q_X^n)^{-1}(\chi_2)\}$ joining $\{x\}$ and X. Hence, $q_X^n(\alpha_1 \cup \alpha_2 \cup \alpha_3)$ is an arc in $HS_n(X) \setminus \{\chi_1, \chi_2\}$ joining F_X^n and T_X^n.

Let $\chi \in HS_n(X) \setminus \{\chi_1, \chi_2\}$. Now it is easy to construct an arc in $HS_n(X) \setminus \{\chi_1, \chi_2\}$ joining χ with either F_X^n or T_X^n. Therefore, $HS_n(X) \setminus \{\chi_1, \chi_2\}$ is arcwise connected. □

The following theorem shows that indecomposable continua and hereditarily decomposable continua do not share n-fold hyperspace suspensions.

Theorem 4.5. *Let X be an indecomposable continuum and let n and m be positive integers. If Y is a hereditarily decomposable continuum, then $HS_m(Y)$ is not homeomorphic to $HS_n(X)$.*

Proof: Suppose that $HS_m(Y)$ is homeomorphic to $HS_n(X)$ and let $h\colon HS_n(X) \twoheadrightarrow HS_m(Y)$ be a homeomorphism.

Since X is indecomposable, $HS_n(X) \setminus \{T_X^n, F_X^n\}$ is not arcwise connected [10, Theorem 6.2]. In fact, by Lemma 4.3, $HS_n(X) \setminus \{T_X^n, F_X^n\}$ has uncountably many arc components. Then $HS_m(Y) \setminus \{h(T_X^n), h(F_X^n)\}$ has uncountably many arc components. Since $HS_m(Y) \setminus \{h(T_X^n), h(F_X^n)\}$ is not arcwise connected, by Lemma 4.4, $F_Y^m \in \{h(T_X^n), h(F_X^n)\}$. Let $\chi \in HS_m(Y)$ be such that $\{\chi, F_Y^m\} = \{h(T_X^n), h(F_X^n)\}$. Since $HS_m(Y) \setminus \{\chi, F_Y^m\}$ is not arcwise connected, $(q_Y^m)^{-1}(\chi) \in \mathcal{C}(Y)$. Since Y is hereditarily decomposable, $(q_Y^m)^{-1}(\chi)$ is a decomposable subcontinuum of Y. Hence, by Lemma 4.2, $HS_m(Y) \setminus \{\chi, F_Y^m\}$ has exactly two arc components, a contradiction. Therefore, $HS_m(Y)$ is not homeomorphic to $HS_n(X)$. □

Next, we note that Theorem 7.1 of [12] may be strengthened, with small changes to the proof.

Theorem 4.6. *Let X be a hereditarily indecomposable continuum, and let n and m be integers greater than or equal to two. If Y is a continuum such that $HS_m(Y)$ is homeomorphic to $HS_n(X)$, then Y is homeomorphic to X.*

Next, we consider dimensionally homogeneous n-fold hyperspace suspensions. First, we note that as a consequence of [14, Lemma 3.5] and [10, Theorem 3.6], we have the following.

Lemma 4.7. *Let X be a graph topologically different from an arc and a simple closed curve, and let n be a positive integer. Then* $\dim(HS_n(X)) \geq 2n+1$.

Theorem 4.8. *Let X be a locally connected continuum and let n be a positive integer. If $HS_n(X)$ is dimensionally homogeneous, then X does not contain a free arc or X is either an arc or a simple closed curve.*

Proof: Suppose X contains a free arc. Then $\dim(HS_n(X)) = 2n$ [10, Corollary 3.10]. Hence, by [10, Theorem 3.6], $\dim(HS_n(X)) = \dim(\mathcal{C}_n(X))$. Thus, X is a graph [11, Theorem 6.8.3]. This implies, by Lemma 4.7, that X is either an arc or a simple closed curve. □

The following theorem is a partial converse of Theorem 4.8.

Theorem 4.9. *If X is an arc or a simple closed curve, then $HS_n(X)$ is dimensionally homogeneous.*

Proof: If X is an arc or a simple closed curve, by [10, Corollary 3.10], $HS_n(X)$ is a $2n$-dimensionally Cantor manifold. The theorem now follows from [6, A), pp. 93 and 94]. □

Let us recall that there exists a locally connected continuum X without free arcs such that $HS(X)$ is not homeomorphic to the Hilbert cube [3, Example 5.3]. Hence, we have the following.

Question 4.10. If X is a locally connected continuum without free arcs and n is a positive integer, then is $HS_n(X)$ dimensionally homogeneous?

Note that for locally connected and contractible continua without free arcs, it is known that their n-fold hyperspace suspension is homeomorphic to the Hilbert cube [10, Theorem 5.3]. Hence, in this case, we have a positive answer to Question 4.10.

Now, we consider arc-smoothness on n-fold hyperspace suspensions.

Remark 4.11. Note that if X is the unit circle, then, since $\mathcal{C}(X)$ is a 2-cell and $\mathcal{F}_1(X)$ is the manifold boundary of the cell [17, Example (0.55)], $\mathcal{C}(X)$ is arc-smooth [4, Introduction, p. 545], but $HS(X)$ is not arc-smooth, because $HS(X)$ is a 2-sphere, which is not contractible.

Theorem 4.12. *If X is an arc-smooth continuum and n is a positive integer, then $HS_n(X)$ is arc-smooth.*

Proof: Let $G\colon \mathcal{C}_n(X)\times[0,1]\to\mathcal{C}_n(X)$ be the map defined in the proof of Theorem 3.7. Let $K\colon HS_n(X)\times[0,1]\to HS_n(X)$ be given by

$$K(\chi,t)=\begin{cases}F_X^n, & \text{if } \chi=F_X^n;\\ q_X^n\left(G\left(\left(q_X^n\right)^{-1}(\chi),t\right)\right), & \text{if } \chi\neq F_X^n.\end{cases}$$

Then K is continuous by [2, Theorem 4.3, p. 126]. Observe that for each $\chi\in HS_n(X)$, $K(\chi,0)=F_X^n$ and $K(\chi,1)=\chi$. It is also easy to see that $K(K(\chi,s),t)=K(\chi,\min\{s,t\})$ for all $s,t\in[0,1]$ and each $\chi\in HS_n(X)$. Hence, $HS_n(X)$ is freely contractible. Therefore, by [4, Theorem II-3-B], $HS_n(X)$ is arc-smooth. □

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