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$C(\tau)$-COSMIC SPACES

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Abstract. In this paper we introduce and study the notion of $C(\tau)$-cosmic space, where τ is an infinite cardinal. Particularly, we prove that in the class of all $C(\tau)$-cosmic spaces of weight $\leq \tau$, there exists a universal element.

1. Preliminaries

In what follows, by ω and $\mathbf{c}$, we denote the first infinite cardinal and the cardinality of the continuum, respectively. Also, by $|X|$, we denote the cardinality of a set X and by $w(X)$, the weight of a space X.

A space T is said to be *universal* (see [3]) in a class $\mathbb{P}$ of spaces if

(1) $T \in \mathbb{P}$ and
(2) for every $X \in \mathbb{P}$ there exists an embedding e of X into T.

A regular space X is called *cosmic* (see [5]) if there exists a collection $\mathcal{P}$ of subsets of X with the properties

(1) for every open subset U of X and every $x \in U$, there exists $P \in \mathcal{P}$ such that $x \in P \subseteq U$,
(2) $|\mathcal{P}| \leq \omega$.

Recall that a family $\mathcal{P}$ of subsets of a space X is called a *network* of X (see [3]) if, for every point $x \in X$ and every open neighborhood

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U_x of x, there exists $P \in \mathcal{P}$ such that $x \in P \subseteq U_x$. The *network weight* of a space X, denoted by $nw(X)$, is defined as the least cardinal number τ such that X has a network of cardinality $\leq \tau$. We note that a space X is cosmic if and only if it is regular and $nw(X) \leq \omega$.

Let τ be an infinite cardinal. A space X is called τ*-monolithic* (see [1] and [2]) if, for every subset A of X with $|A| \leq \tau$, we have $nw(\mathrm{Cl}(A)) \leq \tau$.

In section 2, we give the notion of $C(\tau)$-cosmic space and basic properties for this notion. In section 3, we prove that in the class of all $C(\tau)$-cosmic spaces of weight $\leq \tau$ there exists universal element. Finally, in section 4, we give some open problems.

2. Basic properties

Definition 2.1. Let τ be an infinite cardinal. A space X is called *Closed(τ)-cosmic* ($C(\tau)$*-cosmic*) if there exists a collection $\mathcal{K}$ of closed subsets of X with the properties

(1) for every open subset U of X and every $x \in U$, there exists $K \in \mathcal{K}$ such that $x \in K \subseteq U$, and
(2) $|\mathcal{K}| \leq \tau$.

Remark 2.2. (1) It is clear that for every $C(\tau)$-cosmic space X we have $nw(X) \leq \tau$. Particularly, if X is Hausdorff and compact, then we have $w(X) = nw(X) \leq \tau$ (see [3, Theorem 3.1.19]).
(2) Every $C(\tau)$-cosmic space is $C(\tau_1)$-cosmic space for every $\tau_1 \geq \tau$.
(3) Every regular space of weight less than or equal to τ is $C(\tau)$-cosmic.
(4) Every regular $C(\omega)$-cosmic space is cosmic.
(5) Every $C(\tau)$-cosmic is τ-monolithic.
(6) Every T_1-space X with $|X| \leq \tau$ is $C(\tau)$-cosmic.
(7) Every T_1-space X with $w(X) \leq \tau$ is $C(2^\tau)$-cosmic. Indeed, for every T_1-space X with $w(X) \leq \tau$, we have that $|X| \leq 2^\tau$. Thus, by (6) the space X is $C(2^\tau)$-cosmic.
(8) Let X be the set of real numbers with the topology

$$t = \{(\alpha, +\infty) : \alpha \in X\} \cup \{\emptyset\}.$$

For the space (X, t), we have that $nw(X) \leq \omega$, X is not regular, and X is not $C(\tau)$-cosmic for every infinite cardinal τ.
(9) Let $X = [-1, 1]$ and $t = \{U \subseteq X : 0 \notin U \text{ or } (-1, 1) \subseteq U\}$. Obviously, t is a topology on X. The space X is T_0 and T_4, but

neither T_1 nor T_2. The space (X, t) is not $C(\tau)$-cosmic for every infinite cardinal τ.

(10) Let X be a discrete space with $|X| = \tau$, where τ is an infinite cardinal. The space X is $C(\tau)$-cosmic. However, X is not $C(\nu)$-cosmic for every infinite cardinal $\nu < \tau$.

(11) Consider a dense subspace D of cardinality τ of the Cantor cube 2^τ. Then D is a $C(\tau)$-cosmic regular space. Also, it has character 2^τ, and hence $w(D) > \tau$.

(12) Consider the space $X = (\omega \times \omega) \cup \{\infty\}$ in which a subset U is open if and only if U is empty or $\infty \in U$ and $U \setminus (\{n\} \times \omega)$ is finite for all $n \in \omega$. Then X is T_0, cosmic (being countable), has uncountable weight (the neighborhoods of ∞ are the same as in the countable Fréchet fan), and has no closed network (because the singleton $\{\infty\}$ is dense in X).

(13) If, in Definition 2.1, the elements K of the family $\mathcal{K}$ are regular closed sets (that is, Cl(Int(K)) = K), then the class of all $C(\tau)$-cosmic spaces is exactly the class of all regular spaces of weight less than or equal to τ.

(14) Let X be a non-compact locally compact space of weight less than or equal to τ. Then the Alexandroff compactification ωX of X is a $C(\tau)$-cosmic space.

In what follows we give special examples of $C(\tau)$-cosmic spaces. In particular, we give Hausdorff $C(\tau)$-cosmic spaces which are not regular.

Example 2.3. (1) Let X be the set of real numbers with the topology

$$t = \{U \subseteq X : |X \setminus U| \leq \omega\} \cup \{\emptyset\}.$$

The space (X, t) is T_1-space of weight $\mathbf{c}$ and it is not regular. Also, setting $\mathcal{K} = \{\{x\} : x \in X\}$, we take a family of closed subsets of X such that $|\mathcal{K}| = \mathbf{c}$ and for every open subset U of X and every $x \in U$, we have $x \in \{x\} \subseteq U$. This means that the space (X, t) is $C(\mathbf{c})$-cosmic.

(2) Let $P = \{(\alpha, \beta) \in \mathbb{R}^2 : \beta > 0\}$ be the open upper half-plane with the Euclidean topology T and $L = \{(\alpha, \beta) \in \mathbb{R}^2 : \beta = 0\}$. We consider the set $X = P \cup L$ with the topology

$$t = T \cup \{\{x\} \cup (P \cap U_x) : x \in L,\ U_x \in T,\ \text{and } x \in U_x\}.$$

The space (X, t) is Hausdorff and it is not regular. Moreover, since the subspace L of X is discrete, the weight of X is $\mathbf{c}$. Setting $\mathcal{K} = \{\{x\} : x \in X\}$, we take a family of closed subsets of X such that $|\mathcal{K}| = \mathbf{c}$ and for every open subset U of X and every $x \in U$, we have $x \in \{x\} \subseteq U$. Therefore, the space (X, t) is $C(\mathbf{c})$-cosmic.

(3) Let X be the set of real numbers with open neighborhoods of every nonzero point being as in the usual topology, while open neighborhoods of 0 will have the form $U \setminus \{\frac{1}{n} : n = 1, 2, \ldots\}$, where U is an open neighborhood of 0 in the usual topology. The space X is Hausdorff of weight ω, is $C(\omega)$-cosmic, and is not regular.

Proposition 2.4. *Every subspace of a $C(\tau)$-cosmic space is a $C(\tau)$-cosmic space.*

Proof: Let X be a $C(\tau)$-cosmic space and $A \subseteq X$. Then there exists a collection $\mathcal{K}$ of closed subsets of X with the properties

(1) for every open subset U of X and every $x \in U$, there exists $K \in \mathcal{K}$ such that $x \in K \subseteq U$, and
(2) $|\mathcal{K}| \leq \tau$.

We consider the collection $\mathcal{K}_A \equiv \{A \cap K : K \in \mathcal{K}\}$ of closed subsets of A. The family $\mathcal{K}_A$ satisfies the properties

(3) for every open subset U^A of A and every $x \in U^A$, there exists $K^A \in \mathcal{K}_A$ such that $x \in K^A \subseteq U^A$, and
(4) $|\mathcal{K}_A| \leq \tau$.

The fourth property is clear. For the third, we consider an open subset U^A of A and an element x of U^A. Then there exists an open neighborhood U of x in X such that $U^A = A \cap U$. By the first property, there exists an element K of $\mathcal{K}$ such that $x \in K \subseteq U$. Setting $K^A = A \cap K$, we have $K^A \in \mathcal{K}_A$ and $x \in K^A \subseteq U^A$. $\square$

Proposition 2.5. *The product $\prod_{\lambda \in \Lambda} X_\lambda$ of a family $\{X_\lambda : \lambda \in \Lambda\}$ of $C(\tau)$-cosmic spaces, where $|\Lambda| \leq \tau$, is a $C(\tau)$-cosmic space.*

Proof: For every family of $C(\tau)$-cosmic spaces X_λ, $\lambda \in \Lambda$, there exists a collection $\mathcal{K}_\lambda$ of closed subsets of X_λ with the properties

(1) for every open subset U_λ of X_λ and every $x \in U_\lambda$, there exists $K_\lambda \in \mathcal{K}_\lambda$ such that $x \in K_\lambda \subseteq U_\lambda$, and
(2) $|\mathcal{K}_\lambda| \leq \tau$.

We consider the family $\mathcal{K}$ of all closed subsets of $\prod_{\lambda \in \Lambda} X_\lambda$ of the form $\bigcap_{i=1}^n \pi_{\lambda_i}^{-1}(K_{\lambda_i})$, where $\lambda_1, \ldots, \lambda_n \in \Lambda$, π_{λ_i} is the projection of

$\prod_{\lambda\in\Lambda} X_\lambda$ onto X_{λ_i}, and $K_{\lambda_i} \in \mathcal{K}_{\lambda_i}$, $i = 1, \ldots, n$. Obviously, the family $\mathcal{K}$ satisfies property (2) of Definition 2.1.

We prove property (1) of Definition 2.1. Let U be an open subset of $\prod_{\lambda\in\Lambda} X_\lambda$ and $x \in U$. Without loss of generality, we can suppose that $U \equiv \bigcap_{i=1}^{n} \pi_{\lambda_i}^{-1}(U_{\lambda_i})$, where $\lambda_1, \ldots, \lambda_n \in \Lambda$ and U_{λ_i} is open in X_{λ_i}, $i = 1, \ldots, n$. For $i = 1, \ldots, n$, there exists $K_{\lambda_i} \in \mathcal{K}_{\lambda_i}$ such that $\pi_{\lambda_i}(x) \in K_{\lambda_i} \subseteq U_{\lambda_i}$ and, therefore, $x \in \pi_{\lambda_i}^{-1}(K_{\lambda_i}) \subseteq \pi_{\lambda_i}^{-1}(U_{\lambda_i})$. Setting $K = \bigcap_{i=1}^{n} \pi_{\lambda_i}^{-1}(K_{\lambda_i})$, we have $K \in \mathcal{K}$ and $x \in K \subseteq U$. □

Proposition 2.6. *Let* $\mathbf{S} = \{X_\lambda, f_\lambda^\mu, \Lambda\}$ *be an inverse system where* $|\Lambda| \leq \tau$*. If the spaces* X_λ*,* $\lambda \in \Lambda$*, are* $C(\tau)$*-cosmic, then the inverse limit* $\varprojlim \mathbf{S}$ *is a* $C(\tau)$*-cosmic space.*

Proof: Since the spaces X_λ, $\lambda \in \Lambda$, are $C(\tau)$-cosmic, by Proposition 2.5, the product $\prod_{\lambda\in\Lambda} X_\lambda$ is a $C(\tau)$-cosmic space. Moreover, by Proposition 2.4, every subspace of a $C(\tau)$-cosmic space is a $C(\tau)$-cosmic space. Therefore, the subspace $\varprojlim \mathbf{S}$ of $\prod_{\lambda\in\Lambda} X_\lambda$ is a $C(\tau)$-cosmic space. □

Proposition 2.7. *Let* f *be a continuous and closed map from a* $C(\tau)$*-cosmic space* X *onto a space* Y*. Then* Y *is a* $C(\tau)$*-cosmic space.*

Proof: There exists a collection $\mathcal{K}$ of closed subsets of X with the properties

(1) for every open subset U of X and every $x \in U$, there exists $K \in \mathcal{K}$ such that $x \in K \subseteq U$, and
(2) $|\mathcal{K}| \leq \tau$.

We consider the family $f(\mathcal{K}) \equiv \{f(K) : K \in \mathcal{K}\}$. The family $\mathcal{K}$ satisfies property (2) of Definition 2.1. Also, since f is closed, every element of the family $f(\mathcal{K})$ is a closed subset of Y.

So, for the family $\mathcal{K}$, it suffices to prove property (1) of Definition 2.1. Let W be an open subset of Y and $y \in W$. Since f is onto, there exists $x \in X$ such that $f(x) = y$. Moreover, since the map f is continuous, the subset $f^{-1}(W)$ of Y is an open neighborhood of x in Y. There exists $K \in \mathcal{K}$ such that $x \in K \subseteq f^{-1}(W)$ and, therefore, $y \in f(K) \subseteq W$. For the subset $F = f(K) \in f(\mathcal{K})$, we have that $y \in F \subseteq W$. Thus, the space Y is a $C(\tau)$-cosmic space. □

Corollary 2.8. *The class of all $C(\tau)$-cosmic spaces is topological. That is, every space homeomorphic to a $C(\tau)$-cosmic space is a $C(\tau)$-cosmic.*

Definition 2.9. Let $C(Y,Z)$ be the set of all continuous maps from a space Y to a space Z. The *pointwise topology* on $C(Y,Z)$ (see [3]) is the topology for which the family of all sets of the form $(y,U) = \{f \in C(Y,Z) : f(y) \in U\}$, where $y \in Y$ and U is open in Z, is a subbasis.

Proposition 2.10. *Let Y be a space with $|Y| \leq \tau$ and Z be a $C(\tau)$-cosmic space. The space $C(Y,Z)$ with the pointwise topology is a $C(\tau)$-cosmic space.*

Proof: The pointwise topology on $C(Y,Z)$ coincides with the topology of a subspace of $\prod_{y\in Y} Z_y$, where $Z_y = Z$ for every $y \in Y$ (see [3, Proposition 2.6.3]). Since Z is a $C(\tau)$-cosmic space and $|Y| \leq \tau$, by Proposition 2.5, the product $\prod_{y\in Y} Z_y$ is a $C(\tau)$-cosmic space. By Proposition 2.4, the space $C(Y,Z)$ is a $C(\tau)$-cosmic space. □

Remark 2.11. Let $Y = Z = \mathbb{R}$. Then $|Y| = \mathbf{c} > \omega$, Z is cosmic, and the space $C_p(Y,Z)$ is Tychonoff and cosmic. Now, let Y be a discrete space with $|Y| = \mathbf{c}$ and $Z = \mathbb{R}$. Then Z is cosmic and $C_p(Y,Z) = \mathbb{R}^Y$ has the network weight $\mathbf{c}$ and, therefore, it is not cosmic.

Proposition 2.12. *Every $C(\omega)$-cosmic space X is a perfect space; that is, each closed subset of X is a G_δ-set.*

Proof: There exists a collection $\mathcal{K}$ of closed subsets of X with the properties

(1) for every open subset U of X and every $x \in U$, there exists $K \in \mathcal{K}$ such that $x \in K \subseteq U$, and
(2) $|\mathcal{K}| \leq \omega$.

Let F be an arbitrary closed subset of X. We prove that F is a G_δ-set. It suffices to show that $X \setminus F$ is an F_σ-set. For every $x \in X \setminus F$, there exists $K_x \in \mathcal{K}$ such that $x \in K_x \subseteq X \setminus F$. Therefore, $X \setminus F = \bigcup_{x\in X} K_x$. Since $|\mathcal{K}| \leq \omega$, there exist $x_i \in X$ and $i \in \omega$ such that $X \setminus F = \bigcup_{i\in\omega} K_{x_i}$. Hence, $X \setminus F$ is an F_σ-set. □

Corollary 2.13 ([3]). *Every regular space with a countable base is a perfect space.*

Proof: Follows immediately by the fact that every regular space with a countable base is $C(\omega)$-cosmic. □

Proposition 2.12 can be generalized as follows.

Proposition 2.14. *Every closed set of a $C(\tau)$-cosmic space is the intersection of τ many open sets.*

Proof: It is similar to the proof of Proposition 2.12. □

Proposition 2.15. *Any two distinct points x and y of a $C(\tau)$-cosmic T_0-space can be separated by closed sets; that is, there exist two closed sets F and K such that $x \in F$, $y \in K$, and $F \cap K = \emptyset$.*

Proof: Let x and y be two distinct points of a $C(\tau)$-cosmic T_0-space X. We prove that there exist two closed sets F and K such that $x \in F$, $y \in K$, and $F \cap K = \emptyset$. Since X is a T_0-space, there exists an open set U such that either $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$. Without loss of generality, we can suppose that $x \notin U$ and $y \in U$. Then $\mathrm{Cl}_X(\{x\}) \cap U = \emptyset$. Since X is $C(\tau)$-cosmic, there exists a closed set K such that $y \in K \subseteq U$. Setting $F = \mathrm{Cl}_X(\{x\})$, we have that $x \in F$, $y \in K$, and $F \cap K = \emptyset$. □

Definition 2.16. The *Lindelöf number* $L(X)$ of a space X (see [3]) is the least infinite cardinal number τ such that every open cover of X has an open refinement of cardinality $\leq \tau$.

Proposition 2.17. *For every $C(\tau)$-cosmic space X, we have that $L(X) \leq \tau$.*

Proof: There exists a collection $\mathcal{K}$ of subsets of X with the properties

(1) for every open subset U of X and every $x \in U$, there exists $K \in \mathcal{K}$ such that $x \in K \subseteq U$, and
(2) $|\mathcal{K}| \leq \tau$.

Let $\{U_\lambda : \lambda \in \Lambda\}$ be an open cover of X. We prove that $L(X) \leq \tau$. For this, it suffices to prove that there exists a subset Λ_0 of Λ such that $\bigcup_{\lambda \in \Lambda_0} U_\lambda = \bigcup_{\lambda \in \Lambda} U_\lambda$ and $|\Lambda_0| \leq \tau$. We consider the family

$$\mathcal{K}_0 = \{K \in \mathcal{K} : \text{there exists } \lambda \in \Lambda \text{ such that } K \subseteq U_\lambda\}.$$

For every $K \in \mathcal{K}_0$, we choose an element $\lambda(K)$ of Λ with $K \subseteq U_{\lambda(K)}$. Consider the function $f : \mathcal{K}_0 \to \Lambda$ given by the form $f(K) = \lambda(K)$. Setting $\Lambda_0 \equiv f(\mathcal{K}_0) \subseteq \Lambda$, we have

$$|\Lambda_0| = |f(\mathcal{K}_0)| \leq |\mathcal{K}_0| \leq |\mathcal{K}| \leq \tau.$$

Also, if $x \in \bigcup_{\lambda\in\Lambda} U_\lambda$, then there exists $\lambda_0 \in \Lambda$ such that $x \in U_{\lambda_0}$. By property (1), there exists $K_0 \in \mathcal{K}$ such that $x \in K_0 \subseteq U_{\lambda_0}$. Therefore, $K_0 \in \mathcal{K}_0$ and, hence, $x \in K_0 \subseteq U_{\lambda(K_0)} \subseteq \bigcup_{\lambda\in\Lambda_0} U_\lambda$. Thus, $\bigcup_{\lambda\in\Lambda} U_\lambda \subseteq \bigcup_{\lambda\in\Lambda_0} U_\lambda$ and, therefore, $\bigcup_{\lambda\in\Lambda_0} U_\lambda = \bigcup_{\lambda\in\Lambda} U_\lambda$. □

Definition 2.18. The *density* $d(X)$ of a space X (see [3]) is the least cardinal number τ such that X has a dense subset of cardinality τ.

Proposition 2.19. *For every $C(\tau)$-cosmic space X, we have that $d(X) \leq \tau$.*

Proof: There exists a collection $\mathcal{K}$ of subsets of X with the properties

(1) for every open subset U of X and every $x \in U$, there exists $K \in \mathcal{K}$ such that $x \in K \subseteq U$, and
(2) $|\mathcal{K}| \leq \tau$.

For every $K \in \mathcal{K}$, we choose a point $\alpha_K \in K$. Setting

$$D \equiv \{\alpha_K : K \in \mathcal{K}\},$$

we have that D is dense in X and since $|\mathcal{K}| \leq \tau$, $d(X) \leq \tau$. □

Notation 2.20. Let X be a topological space. By $\mathcal{F}(X)$, we denote the set of nonempty finite subsets of X.

For every finite set $\{A_1, \ldots, A_n\}$ of subsets of X, we denote by $\langle A_1, \ldots, A_n\rangle$ the family

$$\{F \in \mathcal{F}(X) : F \subset \cup_{i=1}^{n} A_i \text{ and } F \cap A_i \neq \emptyset \text{ for } i = 1, \ldots, n\}.$$

Definition 2.21. The *Vietoris topology* τ_V on $\mathcal{F}(X)$ is the topology for which the family

$$\{\langle U_1, \ldots, U_n\rangle : U_i \text{ is open in X for i} = 1, \ldots, \text{n}\}$$

is a base.

Proposition 2.22. *If X is a $C(\tau)$-cosmic space, then the space $(\mathcal{F}(X), \tau_V)$ is $C(\tau)$-cosmic.*

Proof: There exists a collection $\mathcal{K}$ of subsets of X with the properties

(1) for every open subset U of X and every $x \in U$, there exists $K \in \mathcal{K}$ such that $x \in K \subseteq U$, and
(2) $|\mathcal{K}| \leq \tau$.

We consider the family

$$\mathcal{K}_0 = \{\langle K_1, K_2, \ldots, K_m \rangle : K_i \in \mathcal{K} \text{ for i} = 1, 2, \ldots, \text{m}\}.$$

Obviously, the family $\mathcal{K}_0$ satisfies property (2) of Definition 2.1.

We prove property (1) of Definition 2.1. Let W be an open subset of $\mathcal{F}(X)$ and $F \in W$. Clearly, F is a finite subset of X. Without loss of generality, we can suppose that

$$W \equiv \langle U_1, \ldots, U_n \rangle,$$

where U_i is open in X for every $i = 1, \ldots, n$. Then $F \subset \cup_{i=1}^{n} U_i$ and $F \cap U_i \neq \emptyset$, $i = 1, \ldots, n$. Let

$$F \cap U_i = \{x_{i,1}, \ldots, x_{i,m_i}\},\ i = 1, \ldots, n.$$

For every $i = 1, \ldots, n$, there exists $K_{i,j} \in \mathcal{K}$ such that

$$x_{i,j} \in K_{i,j} \subseteq U_i,\ j = 1, \ldots, m_i.$$

We consider the element

$$K = \langle K_{1,1}, \ldots, K_{1,m_1}, K_{2,1}, \ldots, K_{2,m_2}, \ldots, K_{n,1}, \ldots, K_{n,m_n} \rangle$$

of the family $\mathcal{K}_0$. By the above, we have $F \in K \subseteq W$. Thus, the space $(\mathcal{F}(X), \tau_V)$ is $C(\tau)$-cosmic. □

Remark 2.23. We observe that Proposition 2.22 is also true if we replace the notion of $C(\tau)$-cosmic space by the notion of cosmic space.

Notation 2.24. Let X be a topological space and $\sim$ be an equivalence relation on X. We denote by $\mathrm{C}(\sim)$ the set of all equivalence classes of X, by $[x]$ the equivalence class of X with $x \in X$, and by $q : X \to \mathrm{C}(\sim)$ the map for which $q(x) = [x]$ for every $x \in X$. The topology $\tau = \{U \subseteq \mathrm{C}(\sim) : q^{-1}(U) \text{ is open in } X\}$ on $\mathrm{C}(\sim)$ is called the quotient topology and the set $\mathrm{C}(\sim)$ equipped with it is called the quotient space.

We say that an equivalence relation $\sim$ on a space X is closed if the map q is closed. We note that the equivalence relation $\sim$ on

X is closed if and only if for every closed subset F of X the set $\cup\{[x] : x \in F\}$ is closed in X.

By Proposition 2.7 the following proposition is true.

Proposition 2.25. *Let X be a $C(\tau)$-cosmic space and $\sim$ be a closed equivalence relation on X. Then the space $\mathrm{C}(\sim)$ is $C(\tau)$-cosmic.*

3. $C(\tau)$-cosmic spaces and universality

In this section we use notions and notation from [4]. For this reason, we begin with some of them.

In what follows, all spaces are considered to be T_0-spaces of weight $\leq \tau$, where τ is a fixed infinite cardinal.

We shall use the symbol "$\equiv$" in order to introduce new notations without mention of this fact. If "$\sim$" is an equivalence relation on a non-empty set X, then the set of all equivalence classes of $\sim$ is denoted by $\mathrm{C}(\sim)$.

Let $\mathbf{S}$ be an indexed collection of spaces. An indexed collection

$$\mathbf{M} \equiv \{\{U_\delta^X : \delta \in \tau\} : X \in \mathbf{S}\}, \tag{1}$$

where $\{U_\delta^X : \delta \in \tau\}$ is an indexed base for X, is called a *co-mark of* $\mathbf{S}$. The co-mark $\mathbf{M}$ of $\mathbf{S}$ is said to be a *co-extension* of a co-mark

$$\mathbf{M}^+ \equiv \{\{V_\delta^X : \delta \in \tau\} : X \in \mathbf{S}\}$$

of $\mathbf{S}$ if there exists a one-to-one mapping θ of τ into itself such that for every $X \in \mathbf{S}$ and for every $\delta \in \tau$, $V_\delta^X = U_{\theta(\delta)}^X$. The corresponding mapping θ is called an *indicial mapping from* $\mathbf{M}^+$ *to* $\mathbf{M}$.

Let

$$\mathrm{R}_1 \equiv \{\sim_1^s : s \in \mathcal{F}\}$$

and

$$\mathrm{R}_0 \equiv \{\sim_0^s : s \in \mathcal{F}\}$$

be two indexed families of equivalence relations on $\mathbf{S}$. It is said that R_1 is a *final refinement of* R_0 if, for every $s \in \mathcal{F}$, there exists $t \in \mathcal{F}$ such that $\sim_1^t \subseteq \sim_0^s$.

An indexed family $\mathrm{R} \equiv \{\sim^s : s \in \mathcal{F}\}$ of equivalence relations on $\mathbf{S}$ is said to be *admissible* if the following conditions are satisfied: (a) $\sim^\emptyset = \mathbf{S} \times \mathbf{S}$, (b) for every $s \in \mathcal{F}$, the number of $\sim^s$-equivalence classes is finite, and (c) $\sim^s \subseteq \sim^t$, if $t \subseteq s$. We denote by $\mathrm{C}(\mathrm{R})$ the set $\cup\{\mathrm{C}(\sim^s) : s \in \mathcal{F}\}$. The minimal ring of subsets of $\mathbf{S}$ containing $\mathrm{C(R)}$ is denoted by $\mathrm{C}^{\diamondsuit}(\mathrm{R})$.

Consider the co-mark (1) of $\mathbf{S}$. We denote by

$$\mathrm{R}_{\mathbf{M}} \equiv \{\sim^s_{\mathbf{M}} : s \in \mathcal{F}\}$$

the indexed family of equivalence relations $\sim^s_{\mathbf{M}}$ on $\mathbf{S}$ defined as follows: For every X, $Y \in \mathbf{S}$, we set $X \sim^s_{\mathbf{M}} Y$ if and only if there exists an isomorphism i of the algebra of subsets of X generated by the set $\{U^X_\delta \ : \ \delta \in s\}$ onto the algebra of subsets of Y generated by the set $\{U^Y_\delta \ : \ \delta \in s\}$ such that $i(U^X_\delta) = U^Y_\delta$ for every $\delta \in s$. Also, we set $\sim^{\emptyset}_{\mathbf{M}} = \mathbf{S} \times \mathbf{S}$. An admissible family R of equivalence relations on $\mathbf{S}$ is said to be $\mathbf{M}$-*admissible* if R is a final refinement of $\mathrm{R}_{\mathbf{M}}$.

Let $\mathrm{R} \equiv \{\sim^s : s \in \mathcal{F}\}$ be an $\mathbf{M}$-admissible family of equivalence relations on $\mathbf{S}$. On the set of all pairs (x, X), where $X \in \mathbf{S}$ and $x \in X$, we consider an equivalence relation, denoted by $\sim^{\mathbf{M}}_{\mathrm{R}}$, as follows: $(x, X) \sim^{\mathbf{M}}_{\mathrm{R}} (y, Y)$ if and only if $X \sim^s Y$ for every $s \in \mathcal{F}$, and either $x \in U^X_\delta$ and $y \in U^Y_\delta$ or $x \notin U^X_\delta$ and $y \notin U^Y_\delta$ for every $\delta \in \tau$. The set of all equivalence classes of the relation $\sim^{\mathbf{M}}_{\mathrm{R}}$ is denoted by $\mathrm{T}(\mathbf{M}, \mathrm{R})$ or simply by T.

For every $\mathbf{H} \in \mathrm{C}^{\diamondsuit}(\mathrm{R})$, the set of all $\mathbf{a} \in \mathrm{T}(\mathbf{M}, \mathrm{R})$ for which there exists an element $(x, X) \in \mathbf{a}$ such that $X \in \mathbf{H}$ is denoted by $\mathrm{T}(\mathbf{H})$. For every $\delta \in \tau$ and $\mathbf{H} \in \mathrm{C}^{\diamondsuit}(\mathrm{R})$, we denote by $U^{\mathrm{T}}_\delta(\mathbf{H})$ the set of all $\mathbf{a} \in \mathrm{T}(\mathbf{M}, \mathrm{R})$ for which there exists an element $(x, X) \in \mathbf{a}$ such that $X \in \mathbf{H}$ and $x \in U^X_\delta$.

For every subset κ of τ and $\mathbf{L} \in \mathrm{C}^{\diamondsuit}(\mathrm{R})$, we set

(1) $\mathrm{B}^{\mathrm{T}}_{\diamondsuit} \equiv \{U^{\mathrm{T}}_\delta(\mathbf{H}) : \delta \in \tau \text{ and } \mathbf{H} \in \mathrm{C}^{\diamondsuit}(\mathrm{R})\}$,
(2) $\mathrm{B}^{\mathrm{T}}_{\diamondsuit,\kappa} \equiv \{U^{\mathrm{T}}_\delta(\mathbf{H}) : \delta \in \kappa \text{ and } \mathbf{H} \in \mathrm{C}^{\diamondsuit}(\mathrm{R})\}$,
(3) $\mathrm{B}^{\mathbf{L}}_{\diamondsuit,\kappa} \equiv \{U^{\mathrm{T}}_\delta(\mathbf{H}) \in B^T_{\diamondsuit,\kappa} : \mathbf{H} \subseteq \mathbf{L}\}$.

The set $\mathrm{B}^{\mathrm{T}}_{\diamondsuit}$ is a base for a topology on the set $\mathrm{T}(\mathbf{M}, \mathrm{R})$ such that the corresponding space is a T_0-space of weight $\leq \tau$. Moreover, if, for every $X \in \mathbf{S}$, the set $\{U^X_\delta : \delta \in \kappa\}$ is a base for X, then the set $\mathrm{B}^{\mathrm{T}}_{\diamondsuit,\kappa}$ is a base for the same topology on $\mathrm{T}(\mathbf{M}, \mathrm{R})$. Therefore, the family $\mathrm{B}^{\mathbf{L}}_{\diamondsuit,\kappa}$ is a base for $\mathrm{T}(\mathbf{L})$.

For every element X of $\mathbf{S}$, there exists a natural embedding i^X_{T} of X into the space $\mathrm{T}(\mathbf{M}, \mathrm{R})$ defined as follows: For every $x \in X$, $i^X_{\mathrm{T}}(x) = \mathbf{a}$, where $\mathbf{a}$ is the element of $\mathrm{T}(\mathbf{M}, \mathrm{R})$ containing the pair

(x, X). Thus, we have constructed a containing space $\mathrm{T}(\mathbf{M}, \mathrm{R})$ for $\mathbf{S}$ of weight $\leq \tau$.

A class $\mathbb{P}$ of spaces is said to be *saturated* if, for every indexed collection $\mathbf{S}$ of spaces belonging to $\mathbb{P}$, there exists a co-mark $\mathbf{M}^+$ of $\mathbf{S}$ satisfying the following condition: For every co-extension $\mathbf{M}$ of $\mathbf{M}^+$, there exists an $\mathbf{M}$-admissible family R^+ of equivalence relations on $\mathbf{S}$ such that for every admissible family R of equivalence relations on $\mathbf{S}$, which is a final refinement of R^+, and for every $\mathbf{L} \in \mathrm{C}^{\diamondsuit}(\mathrm{R})$, the space $\mathrm{T}(\mathbf{L})$ belongs to $\mathbb{P}$. The co-mark $\mathbf{M}^+$ is said to be an *initial co-mark of* $\mathbf{S}$ *corresponding to the class* $\mathbb{P}$ and the family R is said to be an *initial family of* $\mathbf{S}$ *corresponding to the co-mark* $\mathbf{M}$ *and the class* $\mathbb{P}$.

We recall that

(a) the class of all T_0-spaces of weight $\leq \tau$,
(b) the class of all T_0 countable-dimensional spaces of weight $\leq \tau$,
(c) the class of all T_0 strongly countable-dimensional spaces of weight $\leq \tau$,
(d) the class of all T_0 locally finite-dimensional spaces of weight $\leq \tau$, and
(e) the class of all T_0-spaces X of weight $\leq \tau$ such that $\mathrm{ind}(X) \leq \alpha \in \tau^+$

are saturated.

It is known that if $\mathbb{P}$ is a saturated class of spaces, then in $\mathbb{P}$ there exists a universal element (see [4, Proposition 2.1.4]).

Proposition 3.1. *The class* $\mathbb{P}$ *of all* $C(\tau)$*-cosmic* T_0*-spaces of weight* $\leq \tau$*, where* τ *is a fixed infinite cardinal, is saturated.*

Proof: Let $\mathbf{S}$ be an indexed collection of elements of $\mathbb{P}$ and

$$\mathbf{M}^+ \equiv \{\{V_\delta^X : \delta \in \tau\} : X \in \mathbf{S}\}$$

a co-mark of $\mathbf{S}$. For every $X \in \mathbf{S}$, we denote by

$$\mathcal{K}^X \equiv \{K_\varepsilon^X : \varepsilon \in \tau\}$$

a collection of closed subsets of X satisfying properties (1) and (2) of Definition 2.1. Consider an arbitrary co-mark

$$\mathbf{M} \equiv \{\{U_\delta^X : \delta \in \tau\} : X \in \mathbf{S}\}$$

of $\mathbf{S}$, which is a co-extension of $\mathbf{M}^+$. Denote by R^+ any $\mathbf{M}$-admissible family of equivalence relations on $\mathbf{S}$. We prove that $\mathbf{M}^+$ is an initial co-mark of $\mathbf{S}$ corresponding to the class $\mathbb{P}$, and the family R^+ is an initial family of equivalence relations on $\mathbf{S}$ corresponding to the co-mark $\mathbf{M}$ and the class $\mathbb{P}$.

Let $\mathrm{R} \equiv \{\sim^s : s \in \mathcal{F}\}$ be an arbitrary admissible family of equivalence relations on $\mathbf{S}$, which is a final refinement of R^+. We need to prove that $\mathrm{T} \equiv \mathrm{T}(\mathbf{M}, \mathrm{R})$ is a $C(\tau)$-cosmic space. For this, we consider a point $\mathbf{a}$ of T and an open neighborhood U of $\mathbf{a}$. Without loss of generality, we can suppose that $U \equiv U^{\mathrm{T}}_{\delta}(\mathbf{H}) \in \mathrm{B}^{\mathrm{T}}_{\diamondsuit,\delta}$ for some $\delta \in \tau$ and some $\mathbf{H} \in C(\sim^t)$, $t \in \mathcal{F}$. Let $(x, X) \in \mathbf{a}$. Then $x \in U^X_{\delta}$ and $X \in \mathbf{H}$. There exists $\varepsilon(\delta) \in \tau$ such that $x \in K^X_{\varepsilon(\delta)} \subseteq U^X_{\delta}$.

We consider the collection $\mathcal{K}^T$ of all sets of the form

$$K^T_{\eta}(\mathbf{E}) \equiv \{\mathbf{b} \in \mathrm{T} : \text{there exists } (y, Y) \in \mathbf{b} \text{ with } y \in K^Y_{\varepsilon(\eta)},\ Y \in \mathbf{E}\},$$

where $\eta \in \tau$ and $\mathbf{E} \in \mathrm{C}^{\diamondsuit}(\mathrm{R})$. Then, we have

(1) $\mathbf{a} \in K^T_{\delta}(\mathbf{H}) \subseteq U^{\mathrm{T}}_{\delta}(\mathbf{H})$,
(2) $|\mathcal{K}^T| \leq \tau$.

By the above, it suffices to prove that the set $K^T_{\delta}(\mathbf{H})$ is closed in T. Indeed, let $\mathbf{b} \in \mathrm{T} \setminus K^T_{\delta}(\mathbf{H})$. We need to prove that there exists $U^{\mathrm{T}}_{\eta}(\mathbf{L}) \in \mathrm{B}^{\mathrm{T}}_{\diamondsuit,\delta}$ such that $\mathbf{b} \in U^{\mathrm{T}}_{\eta}(\mathbf{L}) \subseteq \mathrm{T} \setminus K^T_{\delta}(\mathbf{H})$.

Since $\mathbf{b} \notin K^T_{\delta}(\mathbf{H})$, for every $(y, Y) \in \mathbf{b}$, we have $y \notin K^Y_{\varepsilon(\delta)}$ or $Y \notin \mathbf{H}$. Let $(y, Y) \in \mathbf{b}$. There exists $\eta \in \tau$ such that $y \in U^Y_{\eta}$. For the two cases $Y \notin \mathbf{H}$ or $y \notin K^Y_{\varepsilon(\delta)}$, we have the following:

If $Y \notin \mathbf{H}$, then there exists $\mathbf{L} \in C(\sim^t)$ such that $Y \in \mathbf{L}$ and, therefore, $\mathbf{b} \in U^{\mathrm{T}}_{\eta}(\mathbf{L})$. We note that $\mathbf{H} \cap \mathbf{L} = \emptyset$. We prove that $U^{\mathrm{T}}_{\eta}(\mathbf{L}) \subseteq \mathrm{T} \setminus K^T_{\delta}(\mathbf{H})$. Let $\mathbf{c} \in U^{\mathrm{T}}_{\eta}(\mathbf{L})$. For every $(z, Z) \in \mathbf{c}$, we have $Z \in \mathbf{L}$ and, hence, $Z \notin \mathbf{H}$. Thus, by the definition of the set $K^T_{\delta}(\mathbf{H})$, $\mathbf{c} \in \mathrm{T} \setminus K^T_{\delta}(\mathbf{H})$.

Now, we suppose that $y \notin K^Y_{\varepsilon(\delta)}$ and $Y \in \mathbf{H}$. Then $y \in Y \setminus K^Y_{\varepsilon(\delta)}$. Since the subset $Y \setminus K^Y_{\varepsilon(\delta)}$ of Y is open and $y \in U^Y_{\eta}$, there exists $\eta' \in \tau$ such that

$$y \in U^Y_{\eta'} \subseteq U^Y_{\eta} \cap (Y \setminus K^Y_{\varepsilon(\delta)}).$$

By the fact that $y \in U^Y_{\eta'}$ and $Y \in \mathbf{H}$, we have $\mathbf{b} \in U^{\mathrm{T}}_{\eta'}(\mathbf{H})$. Also, since $y \in Y \setminus K^Y_{\varepsilon(\delta)}$, by the definition of the set $K^T_{\delta}(\mathbf{H})$, we have

$U^{\mathrm{T}}_{\eta'}(\mathbf{H}) \subseteq \mathrm{T} \setminus K^{T}_{\delta}(\mathbf{H})$. Thus,

$$\mathbf{b} \in U^{\mathrm{T}}_{\eta'}(\mathbf{H}) \subseteq \mathrm{T} \setminus K^{T}_{\delta}(\mathbf{H})$$

and, therefore, the set $K^{T}_{\delta}(\mathbf{H})$ is closed in T. □

Corollary 3.2. *In the class of all $C(\tau)$-cosmic T_0-spaces of weight $\leq \tau$, where τ is a fixed infinite cardinal, there exists a universal element.*

Using the fact that the intersection of saturated classes of spaces is saturated (see [4, Proposition 2.1.3]), we have the following corollary.

Corollary 3.3. *Let $\mathbb{P}_\tau$ be the class of all $C(\tau)$-cosmic T_0-spaces of weight $\leq \tau$, where τ is a fixed infinite cardinal, and let $\mathbb{P}$ be an arbitrary saturated class of spaces. Then, in the class $\mathbb{P}_\tau \cap \mathbb{P}$, there exist universal elements.*

Remark 3.4. Let $S = \{0, 1\}$ with the topology $\{\emptyset, \{0\}, \{0, 1\}\}$ (Sierpiński space). It is known that the Alexandroff cube S^τ is a universal space for all T_0-spaces of weight $\leq \tau$, where $\tau \geq \omega$. However, the Alexandroff cube S^τ is not $C(\tau)$-cosmic.

Let Λ be a set with $|\Lambda| = \tau$, $X_\lambda = S$ for every $\lambda \in \Lambda$, and $\lambda_0 \in \Lambda$. Obviously, $S^\tau = \prod_{\lambda\in\Lambda} X_\lambda$. We consider the point $x = \{x_\lambda\}_{\lambda\in\Lambda}$ of S^τ, where $x_\lambda = 0$ for every $\lambda \in \Lambda$. Then $x \in \{0\} \times \prod_{\lambda\in\Lambda\setminus\{\lambda_0\}} X_\lambda$, but there is not a closed subset F of S^τ such that

$$x \in F \subseteq \{0\} \times \prod_{\lambda\in\Lambda\setminus\{\lambda_0\}} X_\lambda.$$

Indeed, let F be a closed subset of S^τ such that

$$x \in F \subseteq \{0\} \times \prod_{\lambda\in\Lambda\setminus\{\lambda_0\}} X_\lambda.$$

Then

$$\prod_{\lambda\in\Lambda} X_\lambda \setminus \Big(\{0\} \times \prod_{\lambda\in\Lambda\setminus\{\lambda_0\}} X_\lambda\Big) \subseteq \prod_{\lambda\in\Lambda} X_\lambda \setminus F$$

and

$$\{y_\lambda\}_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} X_\lambda \setminus \Big(\{0\} \times \prod_{\lambda\in\Lambda\setminus\{\lambda_0\}} X_\lambda\Big),$$

where $y_\lambda = 1$ for every $\lambda \in \Lambda$. Therefore, $\{y_\lambda\}_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} X_\lambda \setminus F$. Since the subset $\prod_{\lambda\in\Lambda} X_\lambda \setminus F$ of $\prod_{\lambda\in\Lambda} X_\lambda$ is open, we have

$$\prod_{\lambda\in\Lambda} X_\lambda \setminus F = \prod_{\lambda\in\Lambda} X_\lambda$$

and, hence, $F = \emptyset$, which is a contradiction.

Therefore, the Alexandroff cube S^τ is containing space for all $C(\tau)$-cosmic spaces of weight $\leq \tau$.

Remark 3.5. In the class of all cosmic spaces, there does not exist a universal element. This is a well-known fact, apparently due to V. V. Uspenskiĭ (see [6]). There are $2^{\mathbf{c}}$ non-homeomorphic regular topologies on a countable set – for example, of the form $\omega \cup \{f\}$, where $f \in \beta\omega \setminus \omega$. On the other hand, every cosmic space has at most $\mathbf{c}$ countable subspaces. By [4], this implies that the class of all cosmic spaces is not saturated.

A similar argument (for spaces of cardinality τ rather than countable), shows that there is no universal element in the class of $C(\tau)$-cosmic spaces for any infinite τ.

4. Questions

1. Does there exist a T_1-space X of weight τ, where τ is an infinite cardinal, which is not $C(\tau)$-cosmic?

2. Does there exist a T_2-space X of weight τ, where τ is an infinite cardinal, which is not $C(\tau)$-cosmic?

3. Is Proposition **??** true for the compact open topology on the space $C(Y, Z)$?

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