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by

J. C. GÓMEZ-LARRAÑAGA, F. GONZÁLEZ-ACUÑA, AND
WOLFGANG HEIL

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Department of Mathematics & Statistics

Auburn University, Alabama 36849, USA

E-mail: topolog@auburn.edu

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LINKS IN S^3 OF S^1 -CATEGORY 2

J. C. GÓMEZ-LARRAÑAGA, F. GONZÁLEZ-ACUÑA,
AND WOLFGANG HEIL

ABSTRACT. A non-splittable link of S^1 -category 2 is a Burde-Murasugi link.

1. INTRODUCTION

A subset W in an n -manifold M is S^1 -contractible if there are maps $f : W \rightarrow S^1$ and $\alpha : S^1 \rightarrow M$ such that the inclusion map $i : W \rightarrow M$ is homotopic to $\alpha \cdot f$. The S^1 -category $cat_{S^1} M$ of M is the smallest number of sets, open and S^1 -contractible needed to cover M . Note that if M is closed, $2 \leq cat_{S^1} M \leq n + 1$.

It was shown in [3], for dimension 3, that a closed 3-manifold M^3 has $cat_{S^1} M^3 = 2$ if and only if $\pi_1(M^3)$ is cyclic. By results of Paul Olum [7] and Grigori Perelman (see [6]), this implies that $cat_{S^1} M^3 = 2$ if and only if M^3 is a lens space or M^3 is the non-orientable S^2 -bundle over S^1 .

In this paper, we consider the question of S^1 -category for knot spaces and more generally for compact irreducible 3-manifolds with boundary. Note that if $\partial M \neq \emptyset$ and $cat_{S^1} M^3 = 1$, then $\pi_1(M)$ is trivial or cyclic and it follows from [6] that M is a ball, a solid torus, or a solid Klein bottle. Our main result is that an orientable and irreducible 3-manifold M with $cat_{S^1} M^3 = 2$ is a Seifert fiber space with handles and at most two exceptional fibers. In particular, M

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can then be obtained from two solid tori by gluing their boundaries along incompressible annuli and disks. As a corollary, we obtain that the space of a non-splittable link L in S^3 has S^1 -category 2 if and only if L is a Burde-Murasugi link different from the trivial knot. (For the definition of Burde-Murasugi link, see section 3 and [1].)

2. IRREDUCIBLE AND INCOMPRESSIBLE
 S^1 -CONTRACTIBLE SUBMANIFOLDS

When $cat_{S^1} M = 2$, there are two open subsets W_0 and W_1 of M such that $M = W_0 \cup W_1$ and, for $i = 0, 1$, there are maps f_i and α_i such that the inclusion $W_i \hookrightarrow M$ is homotopic to $\alpha_i \cdot f_i$. Note that a compact 3-submanifold of an S^1 -contractible subset is S^1 -contractible.

In [3, Corollary 1], it was shown that the open sets W_i can be replaced by compact submanifolds meeting only along their boundaries (the hypothesis that M is closed is not used in the proof).

Proposition 1. *Let M be an n -manifold with $cat_{S^1} M = 2$. Then M can be expressed as a union of two compact S^1 -contractible n -submanifolds W_0 and W_1 such that $W_0 \cap W_1 = \partial W_0 \cap \partial W_1$ is a properly embedded $(n - 1)$ -submanifold F .*

From now on we assume that $M = W_0 \cup W_1$ is a compact 3-manifold, where W_0 and W_1 are S^1 -contractible 3-submanifolds as in Proposition 1, so there are maps f_i and α_i such that the diagram (*) is homotopy commutative.

$$\begin{array}{ccc}
 W_i & \xrightarrow{\quad} & M \\
 \searrow f_i & & \nearrow \alpha_i \\
 & S^1 &
 \end{array}
 \quad (*)$$

Proposition 2. *For $i = 0, 1$, we can take α_i so that $\alpha_i(S^1) \cap F = \emptyset$.*

Proof: This is Proposition 1 of [3]. (The hypothesis that M is closed is not used in the proof.) □

Lemma 1. *Suppose that M is compact, orientable, irreducible, and $\partial M \neq \emptyset$. Then there is a decomposition $M = W_0^* \cup W_1^*$ where W_0^* and W_1^* are S^1 -contractible 3-submanifolds as in Proposition 1 such*

that every component F^* of $W_0^* \cap W_1^*$ is incompressible in W_0^* and W_1^* or a 2-sphere.

Proof: Suppose there is a compressing disk $D \subset W_0$, $\partial D = D \cap \partial W_0$ not $\simeq 0$ on ∂W_0 . For a regular neighborhood $U(D)$ in W_0 , let $W'_0 = \overline{W_0 - N(D)}$ and $W'_1 = W_1 \cup N(D)$. Then $W'_0 \subset W_0$ is S^1 -contractible and we claim that W'_1 is S^1 -contractible as well.

Consider the diagram (*) for W_1 with a homotopy $H : W_1 \times I \rightarrow M$ from $\alpha_1 \cdot f_1$ to the inclusion. Extend H to $D \times 1$ by inclusion. For a loop t representing a generator of $\pi_1(S^1)$, we have $f_1(\partial D) \simeq t^m$ for some integer m .

If $f_1(\partial D) \simeq 0$, then f_1 extends to $f'_1 : W'_1 \rightarrow S^1$, and we define H on $D \times 0$ to be $\alpha_1 f'_1$. Now H is defined on $W_1 \times I \cup (D \times \partial I) \cup (\partial D \times I)$, and since $\pi_2(M) = 0$, H can be extended to $H' : W'_1 \times I \rightarrow M$, which is a homotopy from $\alpha_1 f'_1$ to the inclusion.

If $f_1(\partial D)$ is not homotopic to 0, then since $\alpha_1 f_1(\partial D) \simeq 0$, it follows that $\alpha_1(t^m) \simeq 0$. Since M is aspherical, $\pi_1(M)$ is torsion free. Hence, $\alpha_1(t) \simeq 0$ and we can replace α_1 and hence f_1 in diagram (*) by a constant map. Then extending f_1 and H by the constant map on $D \times 0$, we can again extend H to H' since $\pi_2(M) = 0$.

For the new decomposition $M = W'_0 \cup W'_1$ into S^1 -contractible subspaces, the Euler characteristic $-\kappa(W'_0 \cap W'_1) < -\kappa(W_0 \cap W_1)$. Hence, the lemma follows. \square

Lemma 2. *Suppose that M is compact, orientable, irreducible, and $\partial M \neq \emptyset$. Then there is a decomposition $M = W_0 \cup W_1$ as in Lemma 1 and such that each component of W_0 and W_1 is irreducible.*

Proof: Let $M = W_0 \cup W_1$ be as in Lemma 1 such that the sum $c(W_0, W_1)$ of the number of components of W_0 and W_1 is minimal.

Suppose there is a 2-sphere Σ in a component W_0^i of W_0 that does not bound a ball in W_0^i . Let B be the ball in M bounded by Σ , let $W' = \overline{W_0^i - W_0^i \cap B}$, and let $W'' = W' \cup B$. Then W' is S^1 -contractible and we have maps f' and α' as in (*) with a homotopy $H : W' \times I \rightarrow M$ from $\alpha' \cdot f'$ to the inclusion. f' can be extended to $f'' : W'' \rightarrow S^1$ and we get a homotopy $H : W' \times I \cup (B \times \partial I) \cup (\partial B \times I) \rightarrow M$. Since $\pi_3(M) = 0$, H can be extended to $H : W'' \times I \rightarrow M$. Now let W'_0 be obtained from W_0 by replacing the component W_0^i with W'' and let W'_1 be obtained from

W_1 by deleting all components that lie in B . Note that $M = W'_0 \cup W'_1$ is as in Lemma 1 and $c(W'_0, W'_1) < c(W_0, W_1)$, contradicting minimality. \square

Corollary 1. *Suppose that M is compact, orientable, irreducible, and $\partial M \neq \emptyset$. If $\text{cat}_{S^1}(M) = 2$, then there is a decomposition $M = W_0 \cup W_1$ such that $W_0 \cap W_1 = \partial W_0 \cap \partial W_1$ and every component of W_0 and W_1 is a ball or a solid torus. Furthermore, each component of $W_0^i \cap W_1^j$ is incompressible in both W_0^i and W_1^j .*

Proof: By lemmas 1 and 2, we may assume that every component of W_0 and W_1 is irreducible and every component of $W_0 \cap W_1$ is incompressible in W_0 and W_1 . It follows that for each component W_l^k of W_k , the inclusion induces an injection $\pi_1(W_l^k) \rightarrow \pi_1(M)$. Since each W_l^k is S^1 -contractible, $\text{im}(\pi_1(W_l^k) \rightarrow \pi_1(M))$ is cyclic. Hence, W_l^k is irreducible with trivial or infinite cyclic fundamental group and the corollary follows. \square

3. MAIN THEOREM

By a *Seifert fiber space with handles*, we mean a 3-manifold that is a Seifert fiber space or is obtained from a Seifert fiber space by attaching 1-handles along the boundary.

Theorem 1. (a) *A compact, orientable, irreducible 3-manifold with S^1 -category 2 is a Seifert fiber space with handles and with at most two exceptional fibers.*

(b) *A Seifert fiber space with handles and with at most two exceptional fibers has S^1 -category ≤ 2 .*

Proof: (a) If M is closed, then by [3], $\pi_1(M)$ is cyclic and by Perelman (see [6]), M is a lens space. Thus, assume $\partial M \neq \emptyset$. By Corollary 1, $M = W_0 \cup W_1$ such that $W_0 \cap W_1 = \partial W_0 \cap \partial W_1$ and every component of W_0 and W_1 is a ball or solid torus and each component of $W_0^i \cap W_1^j$ is a disk or an annulus, incompressible in both W_0^i and W_1^j . Let N be a regular neighborhood of the disk components of $W_0 \cap W_1$ and let B be the union of the ball components of W_0 and W_1 . We choose a Seifert fibration of the solid torus components of W_0 and W_1 such that $M' = \overline{M - (N \cup B)}$ is a Seifert fiber space (not necessarily connected) and M is obtained from M' by attaching 1-handles. Hence, M is a “Seifert web” (see

[4]) of the form $M = S_1 \cup \cdots \cup S_n \cup H$, where the S_i 's are disjoint Seifert fiber spaces obtained from solid torus components of W_0 and W_1 by identifying along essential annuli; H is a collection of 1-handles; and $H \cap (S_1 \cup \cdots \cup S_n)$ is a collection of disks.

We first show that M is a Seifert fiber space with handles.

If every S_i is a solid torus, then M is a handlebody and we can think of M as being a Seifert fiber space S_1 with handles. Thus, assume at least one S_i is not a solid torus. Then we may assume that no S_i is a solid torus; otherwise, we put it together with H . Hence, each S_i contains solid torus components of both W_0 and W_1 . We show that $n = 1$.

Assume $n > 1$ and let $\beta = \alpha_0(S^1) \subset W_i^j$ ($i = 0$ or 1) and let $W_i^j \subset \overline{M - S_1}$, say. Let $W_0^k \subset S_1$ be a solid torus component of W_0 and let γ represent a generator of $\pi_1(W_0^k)$. Since $S_1 \cap \overline{M - S_1}$ consists of disks, $H_1(S_1 \cap \overline{M - S_1}) \rightarrow H_1(S_1)$ is not onto and so γ is not 0 in $H_1(S_1, S_1 \cap \overline{M - S_1})$. Now $\gamma \simeq \alpha_0 f_0 \gamma \simeq \beta^m$ for some $m \in \mathbb{Z}$ and so γ is 0 in $H_1(M, \overline{M - S_1})$. By excision, inclusion induces an isomorphism $H_1(S_1, S_1 \cap \overline{M - S_1}) \rightarrow H_1(M, \overline{M - S_1})$, a contradiction.

We complete the proof by showing that at most one solid torus component of W_0 and at most one solid torus component of W_1 are exceptionally fibered.

First, assume $\beta = \alpha_0(S^1) \subset W_0^j \subset W_0$. Suppose some W_0^i for $i \neq j$ is exceptionally fibered with exceptional fiber γ of multiplicity $q > 1$. It follows that $H_1(W_0^i \cap \overline{M - W_0^i}) \rightarrow H_1(W_0^i)$ is not onto and so γ is not 0 in $H_1(W_0^i, W_0^i \cap \overline{M - W_0^i})$. As before, $\gamma \simeq \beta^m$ is 0 in $H_1(M, \overline{M - W_0^i})$ and inclusion induces an isomorphism $H_1(W_0^i, W_0^i \cap \overline{M - W_0^i}) \rightarrow H_1(M, \overline{M - W_0^i})$, a contradiction.

Hence, W_0^j is the only component of W_0 that could be exceptionally fibered.

Now assume $\beta = \alpha_0(S^1) \subset W_1$. The above argument shows that in this case no component of W_0 is exceptionally fibered. The same proof applies to the components of W_1 .

(b) Let $M = S \cup H$ be a Seifert fiber space with handles and with at most two exceptional fibers. Decompose the orbit surface of S into two disks D_0 and D_1 , each with at most one exceptional point and $D_0 \cap D_1 = \partial D_0 \cap \partial D_1$ (see, e.g., [5]). Then $S = V_0 \cap V_1$, where

V_i is the solid torus corresponding to D_i . For each handle $H_k \approx D^2 \times [-1, 1]$ whose ends $D^2 \times \{-1\}$ and $D^2 \times \{1\}$ are attached to V_i and V_{1-i} , respectively, replace V_i by the solid torus $V_i \cup D^2 \times [-1, 0]$ and V_{1-i} by $V_{1-i} \cup D^2 \times [0, 1]$. Then let $W_i = V_i \cup$ all handles for which both ends are attached to V_{1-i} . Now $M = W_0 \cup W_1$, and each W_i is S^1 -contractible.

This completes the proof of the theorem. \square

Corollary 2. *Let M be a compact, orientable, irreducible 3-manifold with $H_1(M) = \mathbb{Z}$. If $\text{cat}_{S^1}(M) = 2$, then M is a Seifert fiber space with orbit surface a disk and at most two exceptional fibers.*

Proof: M is not closed since otherwise M would be a lens space. Since $H_1(M) = \mathbb{Z}$, the boundary of M is a torus. By Theorem 1, M is a Seifert fiber space with at most two exceptional fibers. The projection of M to the orbit surface \bar{M} induces an epimorphism $H_1(M) \rightarrow H_1(\bar{M})$ and it follows that \bar{M} is a disk. \square

We say that a link L in S^3 which is not the unlink of more than one component is a *Burde-Murasugi link* if the components of L can be chosen to be fibers of some Seifert fibration of S^3 , including the singular fibration and its unknotted circle (see [1]). Such a link is non-splittable and the components are components of a torus link on the boundary of an unknotted solid torus V in S^3 , possibly together with the core curve of V and a curve isotopic to a meridian curve of V .

We say that a link $L \subset S^3$ is of S^1 -category m if its space $\overline{S^3 - N(L)}$ has S^1 -category m .

Corollary 3. (a) *A non-splittable link L of at least two components in S^3 has S^1 -category 2 if and only if L is a Burde-Murasugi link.*
 (b) *A knot in S^3 has S^1 -category 2 if and only if it is a non-trivial torus knot.*

Proof: The trivial knot has $\overline{S^1}$ -category 1. In any other case, since L is non-splittable, $M = \overline{S^3 - N(L)}$ is irreducible. If L is of S^1 -category 2, then by Theorem 1, M is a Seifert fiber space and the result follows from [1] and [2]. Conversely, the complements of these links are Seifert fiber spaces with at most two exceptional fibers. \square

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(Gómez-Larrañaga) CENTRO DE INVESTIGACIÓN EN MATEMÁTICAS; A.P. 402; GUANAJUATO 36000, GTO. MÉXICO
E-mail address: jcarlos@cimat.mx

(González-Acuña) INSTITUTO DE MATEMÁTICAS, UNAM; CIUDAD UNIVERSITARIA; 04510 MÉXICO, D.F. MÉXICO AND
CENTRO DE INVESTIGACIÓN EN MATEMÁTICAS; A.P. 402; GUANAJUATO 36000, GTO. MÉXICO
E-mail address: fico@math.unam.mx

(Heil) DEPARTMENT OF MATHEMATICS; FLORIDA STATE UNIVERSITY; TALLAHASSEE, FL 32306 USA
E-mail address: heil@math.fsu.edu