TOPOLOGY PROCEEDINGS

Volume 38, 2011

Pages 193–207

http://topology.auburn.edu/tp/

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Electronically published on September 24, 2010

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/
TOPOLOGY PROCEEDINGS
Volume 38 (2011)
Pages 193-207
E-Published on September 24, 2010

ON MINIMAL R_1 AND MINIMAL REGULAR TOPOLOGIES

M. L. COLASANTE AND D. VAN DER ZYPEN

Abstract. By means of filters, minimal R_1 and minimal regular topologies are characterized on suitable intervals consisting of non-trivial R_0 topologies.

1. Introduction

The family $LT(X)$ of all topologies definable on a set X partially ordered by inclusion is a complete, atomic lattice in which the meet of a collection of topologies is their intersection, while the join is the topology with their union as a subbase. There has been a considerable amount of interest in topologies which are minimal in this lattice with respect to certain topological properties (see for instance [1], [2], [3], [4], [5], [8], [9], [11], [12], [13], [14], [16], [19]).

Given a topological property P (like a separation axiom) and given a family $\mathcal{S}$ of members of $LT(X)$, then $\tau \in \mathcal{S}$ is said to be *minimal* P in $\mathcal{S}$ if τ satisfies P but no member of $\mathcal{S}$ which is strictly weaker than τ satisfies P. It is well known that a T_2-topology on an infinite set X is minimal T_2 in $LT(X)$ if and only if every open filter on X with a unique adherent point is convergent [3]. Also, a regular T_1-topology is minimal regular in $LT(X)$ if and only if every regular filter on X with a unique adherent point is convergent [4].

2010 *Mathematics Subject Classification.* 54A10, 54D10, 54D25.
Key words and phrases. Alexandroff topology, filters, R_0, R_1, regular and presober topologies.

These are characterizations of minimal topologies satisfying separation axioms above T_1, and thus topologies in the lattice $\mathcal{L}_1 = \{\tau \in LT(X) : \mathcal{C} \leq \tau \leq 2^X\}$, where $\mathcal{C}$ denotes the cofinite topology (i.e., the minimal T_1-topology on X) and 2^X denotes the powerset of X. Some separation axioms independent of T_1 (even independent of T_0) are vacuously satisfied by the indiscrete topology; thus, the study of minimal topologies in $LT(X)$ satisfying such properties becomes trivial. This is the case of the R_1 and regularity (not necessarily T_1) separation axioms. The purpose of this paper is to show that, by restricting to suitable intervals $\mathcal{L}_\rho$ of $LT(X)$, associated each to a non-trivial R_0-topology ρ, then minimal regular and minimal R_1 topologies in $\mathcal{L}_\rho$ can be characterized in terms of filters. For instance, we prove in section 3 that an R_1-topology in $\mathcal{L}_\rho$ is minimal R_1 if and only if every open filter on X, for which the set of adherent points coincides with a point closure, is convergent, and that a regular topology in $\mathcal{L}_\rho$ is minimal regular if and only if every regular filter on X, for which the set of adherent points coincides with a point closure, is convergent. The characterizations for minimal T_2 and minimal regular topologies mentioned at the beginning of this paragraph are immediate corollaries of our results in case ρ is a T_1-topology. Additionally, in the last section, we consider another topological property independent of T_0, namely the presober property, and show that there are no minimal presober topologies in $\mathcal{L}_\rho$.

2. Preliminaries and notations

A topology $\tau \in LT(X)$ is said to be an *Alexandroff topology* if it is closed under arbitrary intersection. Juris Steprāns and Stephen Watson [17] attributed this notion to both P. S. Alexandroff and A. W. Tucker and called them AT topologies. This class of topologies is specially relevant for the study of non-T_1 topologies. Note that the only T_1 Alexandroff topology is the discrete topology. Among the characterizations known for AT topologies, we recall the one related with the specialization preorder: $\tau \in LT(X)$ is AT if and only if it is the finest topology on X consistent with the specialization preorder, i.e., the finest topology giving the preorder $\leq_\tau$ satisfying $x \leq_\tau y$ if and only if x belongs to the τ-closure of $\{y\}$. This preorder characterizes the T_0 property (for every two points there is an open set containing one and only one of the points) in the sense that a topology τ is T_0 if and only if the preorder $\leq_\tau$is a partial order.

By identifying a set with its characteristic function, 2^X can be endowed with the product topology of the Cantor cube $\{0,1\}^X$. It was proved in [18] that a topology τ on X is AT if and only if it is closed when viewed as a subset of 2^X. Moreover, it was proved there that the closure $\overline{\tau}$ of τ in 2^X is also a topology, and therefore it is the smallest AT topology containing τ.

By $cl_\tau(A)$, we denote the τ-closure of a set A. If $A = \{x\}$, we use $cl_\tau(x)$, instead of $cl_\tau(\{x\})$, and refer to it as a point closure. The τ-kernel of a set $A \subseteq X$, denoted by $\ker_\tau(A)$, is the intersection of all open sets containing A. For any $x \in X$, we denote $\ker_\tau(\{x\}) = \ker_\tau(x)$. It is obvious that $x \in cl_\tau(y)$ if and only if $y \in \ker_\tau(x)$. A set A is said to be *τ-kernelled* (or just kernelled) if $A = \ker_\tau(A)$. Equivalently, A is kernelled if and only if $A = \bigcup_{x\in A} \ker_\tau(x)$. Many authors have called sets which are intersections of open sets "saturated" (see, for example, [10]). The family of all kernelled subsets of X is closed under arbitrary unions and intersections, so it is an AT topology. Moreover, it coincides with $\overline{\tau}$. In fact, since every open set is kernelled and $\overline{\tau}$ is the smallest AT topology containing τ, then every member of $\overline{\tau}$ is kernelled. On the other hand, since $\overline{\tau}$ is closed under arbitrary intersections and it contains τ, then every kernelled set belongs to $\overline{\tau}$. Thus, $\overline{\tau}$ is the topology on X generated by the family $\{\ker_\tau(x) : x \in X\}$. In particular, $A \subseteq X$ is $\overline{\tau}$-closed if and only if $A = \bigcup_{x\in A} cl_\tau(x)$. Note that, since τ is T_1 if and only if every subset of X is kernelled, then τ is T_1 if and only if $\overline{\tau} = 2^X$.

In what follows, $\mathcal{N}_\tau(x)$ denotes the filter base of τ-neighborhoods of $x \in X$. A filter $\mathcal{F}$ on X is said to be τ-convergent to a point $x \in X$ if $\mathcal{F} \supseteq \mathcal{N}_\tau(x)$. By $adh_\tau\mathcal{F}$, we denote the set of adherent points of $\mathcal{F}$ (i.e., $adh_\tau\mathcal{F} = \bigcap_{F\in\mathcal{F}} cl_\tau(F)$). Since $adh_\tau\mathcal{F}$ is a closed set, then it contains the τ-closure of all its points. It is immediate that if $\mathcal{F}$ is τ-convergent to x, then $\mathcal{F}$ is τ-convergent to every $y \in cl_\tau(x)$. A filter $\mathcal{F}$ is said to be a τ-open filter (τ-closed filter, respectively) if it has a base of τ-open sets (τ-closed sets, respectively), and $\mathcal{F}$ is said to be a τ-regular filter if it is τ-open and for every $F \in \mathcal{F}$ there exists $F' \in \mathcal{F}$ such that $cl_\tau(F') \subseteq F$. Thus, a τ-regular filter is equivalent to a τ-closed filter. A filter on X is said to be an ultrafilter if it is a maximal filter. If it is an ultrafilter, then for each $A \subseteq X$, either $A \in \mathcal{F}$ or $X \backslash A \in \mathcal{F}$, where $X \backslash A$ denotes the complement of A in X.

For definitions and notation not given here, we refer the reader to [20].

3. Minimal R_1 and minimal regular topologies in $\mathcal{L}_\rho$

In this section, we restrict our attention to suitable intervals consisting of R_0 topologies and give characterizations of minimal R_1 and minimal regular topologies on those intervals. Recall that a topology $\tau \in LT(X)$ is said to be

(R_0) if for all $x, y \in X$, $x \in cl_\tau(y)$if and only if $y \in cl_\tau(x)$; thus, τ is R_0 if and only if the point closures form a partition of X [15];

(R_1) if for all $x, y \in X$ with $cl_\tau(x) \neq cl_\tau(y)$, there are disjoint open sets separating $cl_\tau(x)$ and $cl_\tau(y)$ [7];

(*Regular*) if for each $V \in \tau$ and each $x \in V$, there exists $U \in \tau$ such that $x \in U \subseteq cl_\tau(U) \subseteq V$.

The separation axioms R_0 and R_1 are also denoted as S_1 and S_2, respectively [6]. We use in this paper the most common notation R_0 and R_1. It is easy to show that $Regular \Rightarrow R_1 \Rightarrow R_0$, and that none of the implications can be reversed. Moreover, τ is T_1 if and only if τ is R_0 and T_0, and τ is T_2 if and only if τ is R_1 and T_0.

Examples of topologies which are regular non-T_0 (thus, regular non-T_1) abound. For instance, if $\mathcal{P}$ denotes any non-trivial partition of a set X, then the associated partition topology $\tau_{\mathcal{P}}$, defined as the topology having as open sets the unions of elements of $\mathcal{P}$ together with the empty set, is a regular topology which is not T_0. On the other hand, if a topological space satisfies any of the properties R_0, R_1, or $Regular$, and one doubles the space by taking the product of X with the two point indiscrete space, then the resulting space is no longer T_0, but it satisfies the same property as did the original space.

The following characterizations, which are straightforward to prove, are used throughout the paper without explicitly mentioning them.

Lemma 3.1. *Let $\tau \in LT(X)$. Then*

(i) *τ is R_0 if and only if $cl_\tau(x) = \ker_\tau(x)$ for all $x \in X$, if and only if $cl_\tau(x) \subseteq V$ for all $V \in \tau$ and $x \in V$.*
(ii) *τ is R_1 if and only if τ is R_0, and for all $x, y \in X$ such that $y \notin cl_\tau(x)$, there are disjoint open sets separating x and y.*
(iii) *τ is R_1 if and only if τ is R_0 and $adh_\tau \mathcal{N}_\tau(x) = cl(x)$, for all $x \in X$.*

To each $\rho \in LT(X)$, we associate the interval

$$\mathcal{L}_\rho = \{\tau \in LT(X) : at(\rho) \leq \tau \leq \overline{\rho}\},$$

where $at(\rho)$ denotes the topology on X generated by the sets $\{X \setminus cl_\rho(H)$: H is a finite subset of $X\}$ and where $\overline{\rho}$ is the closure of ρ in 2^X.

Note that if ρ is any T_1-topology, then $at(\rho) = \mathcal{C}$ and $\overline{\rho} = 2^X$. In this case, $\mathcal{L}_\rho$ is precisely the lattice $\mathcal{L}_1$ of all T_1 topologies on X.

Lemma 3.2. *Let $\rho \in LT(X)$. Then $cl_{at(\rho)}(x) = cl_\rho(x) = cl_{\overline{\rho}}(x)$, for every $x \in X$.*

Proof: Let $x \in X$. Since a set is $\overline{\rho}$-closed if and only if it is a union of ρ-closed sets, then $cl_\rho(x) \subseteq cl_{\overline{\rho}}(x)$. On the other hand, $cl_\rho(x)$ is an $at(\rho)$-closed set, and thus $cl_{at(\rho)}(x) \subseteq cl_\rho(x)$. Since $at(\rho) \subseteq \rho \subseteq \overline{\rho}$, then $cl_{\overline{\rho}}(x) \subseteq cl_\rho(x) \subseteq cl_{at(\rho)}(x)$. From this we have the result. □

Corollary 3.3. *Let $\tau, \rho \in LT(X)$. Then $\tau \in \mathcal{L}_\rho$ if and only if $cl_\rho(x) = cl_\tau(x)$, for every $x \in X$.*

Proof: If $\tau \in \mathcal{L}_\rho$ and $x \in X$, then Lemma 3.2 implies that $cl_\tau(x) = cl_\rho(x)$. Conversely, suppose $cl_\rho(x) = cl_\tau(x)$, for every $x \in X$. It is immediate that $at(\rho) \leq \tau$. Note that $\ker_\rho(x) = \ker_\tau(x)$; thus, if $V \in \tau$, then $V = \bigcup_{x \in V} \ker_\rho(x) = \bigcup_{x \in V} \ker_\tau(x)$ is a $\overline{\rho}$-open set. Therefore, $at(\rho) \leq \tau \leq \overline{\rho}$. □

Corollary 3.3 can be stated as follows: $\tau \in \mathcal{L}_\rho$ if and only if τ has the same preorder of specialization as ρ. Thus, when one refers to the τ-closure of $x \in X$, for any $\tau \in \mathcal{L}_\rho$, there is no need to specify the topology. We will often write $cl(x)$ without further comment. It is clear that the topologies on $\mathcal{L}_\rho$ share the topological properties defined in terms of point closures. In particular, $\tau \in \mathcal{L}_\rho$ is R_0 if and only if ρ is R_0. Note that the property R_1 is expansive in $\mathcal{L}_\rho$ (i.e., if $\tau \in \mathcal{L}_\rho$ is R_1, then τ' is R_1 for all $\tau' \in \mathcal{L}_\rho$ finer than τ).

In [6], it was proved that the properties R_0, R_1, and regularity coincide for AT topologies. Thus, $\overline{\rho}$ is R_0 if and only if $\overline{\rho}$ is R_1 if and only if $\overline{\rho}$ is regular. If we start with an R_0-topology ρ on X, it is immediate that there exists at least a regular topology (so at least an R_1-topology) in $\mathcal{L}_\rho$. Our goal is to characterize minimal R_1 and minimal regular topologies in $\mathcal{L}_\rho$. Note that if ρ is R_0 and X can be written as a finite union of disjoint point closures, then, for each $x \in X$, the set $cl(x)$ is the complement of a finite union of point closures; thus, $cl(x) \in at(\rho)$. It follows that $at(\rho) = \rho = \overline{\rho}$, and therefore $\mathcal{L}_\rho = \{\rho\}$. To avoid triviality, from now on we assume that $\rho \in LT(X)$ is an R_0-topology such that X can be written as an infinite union of disjoint point closures (in particular, this is the case for any T_1-topology on an infinite set). It is worth noticing that $at(\rho)$ cannot be R_1; thus, it cannot be regular, since any pair of non-empty $at(\rho)$-open sets intersects. We give an example of an R_0 (not T_0) topology satisfying the above conditions.

Example 3.4. Let X be the set of all positive integers N, and let ρ be the topology generated by the subbase $\{\emptyset, N\backslash\{1\}\}\cup$ $\{N\backslash\{2n, 2n+1\}, n \geq 1\}$. It is easy to see that ρ is an R_0-topology which is not T_0, and that N can be written as the infinite disjoint union of the odd integers point closure. Note that $at(\rho) = \rho$, and $\overline{\rho}$ is the topology generated by the sets $\{1\}$ and $\{2n, 2n+1\}, n \geq 1$.

For $x \in X$, let $\mathcal{E}(x)$ denote the family of all the subsets of X not containing x. If $\mathcal{F}$ is any filter on X, then $\mathcal{E}(x) \cup \mathcal{F}$ is a topology on X. Given $\tau \in LT(X)$, we consider the topology $\beta = \tau \cap (\mathcal{E}(x) \cup \mathcal{F})$. Note that $\beta \leq \tau$ and $\beta = \tau$ if and only if $\mathcal{F} = \mathcal{N}_\tau(x)$.

Now, if ρ is R_0 and $\tau \in \mathcal{L}_\rho$, a local base for the topology β can be described as follows:

$$\mathcal{N}_\beta(y) = \mathcal{N}_\tau(y) \cap \mathcal{E}(x), \text{ for every } y \notin cl(x);$$
$$\mathcal{N}_\beta(y) = \mathcal{N}_\tau(x) \cap \mathcal{F}, \text{ for every } y \in cl(x).$$

A set $A \subseteq X$ is β-closed if and only if A is τ-closed and either $x \in A$ or $X\backslash A \in \mathcal{F}$. Thus, $cl_\tau(A) \subseteq cl_\beta(A) \subseteq cl_\tau(A) \cup cl(x)$ for all $A \subseteq X$. In particular, $cl_\beta(x) = cl(x)$.

Lemma 3.5. *Let $\tau \in \mathcal{L}_\rho$. Given $x \in X$ and a filter $\mathcal{F}$ on X, let $\beta = \tau \cap (\mathcal{E}(x) \cup \mathcal{F})$. Then*

(i) *β is R_0 if and only if $\mathcal{F} \supseteq \mathcal{N}_{at(\rho)}(x)$ if and only if $\beta \in \mathcal{L}_\rho$.*

(ii) *If $adh_\tau \mathcal{F} = cl(x)$, then $\beta \in \mathcal{L}_\rho$.*

Proof: (i) It is immediate that $\mathcal{F} \supseteq \mathcal{N}_{at(\rho)}(x)$ if and only if $\beta \in \mathcal{L}_\rho$, and that if $\beta \in \mathcal{L}_\rho$, then β is R_0. On the other hand, if β is R_0 and $y \notin cl(x) = cl_\beta(x)$, then $x \notin cl_\beta(y)$. Thus, $X \setminus cl_\beta(y) \in \mathcal{F}$, and this implies that $X \setminus cl(y) \in \mathcal{F}$. Since this holds for every $y \notin cl(x)$, it follows that $\mathcal{F} \supseteq \mathcal{N}_{at(\rho)}(x)$.

(ii) If $adh_\tau \mathcal{F} = cl(x)$ and $y \notin cl(x)$, then $y \notin adh_\tau \mathcal{F}$, and thus, there exist $F \in \mathcal{F}$ and $V \in \mathcal{N}_\tau(y)$ such that $V \cap F = \emptyset$. Since $cl(y) \subseteq V$, then $F \subseteq X \setminus cl(y)$, and thus $X \setminus cl(y) \in \mathcal{F}$. Hence, $\mathcal{F} \supseteq \mathcal{N}_{at(\rho)}(x)$. □

Proposition 3.6. *Let $\tau \in \mathcal{L}_\rho$ be R_1. Given $x \in X$ and a filter $\mathcal{F}$ on X, then the topology $\beta = \tau \cap (\mathcal{E}(x) \cup \mathcal{F})$ is R_1 if and only if there exists a τ-open filter $\mathcal{F}_0 \subseteq \mathcal{F}$ such that $adh_\tau \mathcal{F}_0 = cl(x)$.*

Proof: ($\Rightarrow$) If $\beta = \tau \cap (\mathcal{E}(x) \cup \mathcal{F})$ is R_1, then $adh_\beta \mathcal{N}_\beta(x) = cl(x)$. By Lemma 3.5(i), $\beta \in \mathcal{L}_\rho$. Now, since $\beta \leq \tau$, then $cl(x) \subseteq adh_\tau \mathcal{N}_\beta(x) \subseteq adh_\beta \mathcal{N}_\beta(x) = cl(x)$. Let $\mathcal{F}_0 = \mathcal{N}_\beta(x) = \mathcal{N}_\tau(x) \cap \mathcal{F}$. It is clear that $\mathcal{F}_0$ is a τ-open filter contained in $\mathcal{F}$ such that $adh_\tau \mathcal{F}_0 = cl(x)$.

($\Leftarrow$) Suppose there exists a τ-open filter $\mathcal{F}_0 \subseteq \mathcal{F}$ such that $adh_\tau \mathcal{F}_0 = cl(x)$. By Lemma 3.5(i), $\beta \in \mathcal{L}_\rho$. To prove that $\beta = \tau \cap (\mathcal{E}(x) \cup \mathcal{F})$ is R_1, let $y, z \in X$ such that $y \notin cl(z)$. We will show that y and z can be separated by β-open sets. Since τ is R_1, there exist $W_y \in \mathcal{N}_\tau(y)$ and $W_z \in \mathcal{N}_\tau(z)$ such that $W_y \cap W_z = \emptyset$. We consider two possible cases.

Case 1: If $x \notin cl(y)$ and $x \notin cl(z)$, then $y, z \notin cl(x)$. Choose $V_y \in \mathcal{N}_\tau(y)$ and $V_z \in \mathcal{N}_\tau(z)$ such that $x \notin V_y$ and $x \notin V_z$. Let $O_y = W_y \cap V_y$ and $O_z = W_z \cap V_z$. Then $O_y, O_z \in \tau \cap \mathcal{E}(x) \leq \beta$ and $O_y \cap O_z = \emptyset$.

Case 2: If $x \in cl(y)$, then $cl(y) = cl(x) = adh_\tau \mathcal{F}_0$. Since $z \notin cl(y)$, there exists $U \in \mathcal{N}_\tau(z)$ and $F \in \mathcal{F}_0$ such that $U \cap F = \emptyset$. Take $O_y = W_y \cup F$ and $O_z = W_z \cap U$. Then it is immediate that $O_y \in \tau \cap \mathcal{F}$ and $O_z \in \tau \cap \mathcal{E}(x)$. Thus, O_y and O_z are disjoint β-neighborhoods of y and z, respectively. □

Remark 3.7. For any $x \in X$, the open filter $\mathcal{F} = \mathcal{N}_{at(\rho)}(x)$ satisfies $adh_{\overline{\rho}} \mathcal{F} = cl(x)$. In fact, $cl(y) = \ker(y) \in \mathcal{N}_{\overline{\rho}}(y)$ for each $y \in X$. Then $y \in cl(x)$ implies that $x \in V$, for all $V \in \mathcal{N}_{\overline{\rho}}(y)$, and thus $y \in adh_{\overline{\rho}} \mathcal{N}_{at(\rho)}(x)$. On the other hand, if $y \notin cl(x)$, then the disjoint sets $cl(y) \in \mathcal{N}_{\overline{\rho}}(y)$ and $X \setminus cl(y) \in \mathcal{N}_{at(\rho)}(x)$ witness that

$y \notin adh_{\overline{\rho}}\mathcal{N}_{at(\rho)}(x)$. Since $\overline{\rho}$ is R_1, the above proposition implies that $\beta = \overline{\rho} \cap (\mathcal{E}(x) \cup \mathcal{N}_{at(\rho)}(x))$ is R_1, and hence $\beta \in \mathcal{L}_\rho$. Moreover, β is strictly weaker than $\overline{\rho}$ since $cl(x) \in \overline{\rho}$, but $cl(x) \notin \mathcal{N}_{at(\rho)}(x)$. Therefore, $\overline{\rho}$ is not the minimal R_1 topology in $\mathcal{L}_\rho$.

We are now ready to prove a characterization of minimal R_1 in $\mathcal{L}_\rho$.

Theorem 3.8. *Let $\tau \in \mathcal{L}_\rho$ be R_1. Then τ is minimal R_1 if and only if, given any open filter $\mathcal{F}$ on X such that $adh_\tau\mathcal{F} = cl(x)$ for some $x \in X$, then $\mathcal{F}$ is convergent (necessarily to every point of $cl(x)$).*

Proof: Suppose τ is minimal R_1, and let $\mathcal{F}$ be an open filter on X such that $adh_\tau\mathcal{F} = cl(x)$ for some $x \in X$. Let $\beta = \tau \cap (\mathcal{E}(x) \cup \mathcal{F})$. By Lemma 3.5(i), $\beta \in \mathcal{L}_\rho$, and by Proposition 3.6, β is R_1. Since τ is minimal R_1 in $\mathcal{L}_\rho$, we have that $\beta = \tau$, and thus $\mathcal{F} \supseteq \mathcal{N}_\tau(x)$.

Conversely, suppose every open filter $\mathcal{F}$ on X such that $adh_\tau\mathcal{F} = cl(x)$ for some $x \in X$, is τ-convergent, and let $\tau' \in \mathcal{L}_\rho$ be an R_1-topology such that $\tau' \leq \tau$. Let $V \in \tau$ and $x \in V$. Since $adh_{\tau'}\mathcal{N}_{\tau'}(x) = cl(x)$, the hypothesis implies that the τ-open filter $\mathcal{N}_{\tau'}(x)$ is τ-convergent to x. Thus, $\mathcal{N}_{\tau'}(x) \supseteq \mathcal{N}_\tau(x)$, and hence $V \in \mathcal{N}_{\tau'}(x)$. Since this happens for all $x \in V$, then $V \in \tau'$. Therefore, $\tau = \tau'$, and this implies that τ is minimal R_1. □

Since τ is minimal T_2 if and only if $\tau \in \mathcal{L}_1$ and is minimal R_1, then Theorem 3.8 applied to any T_1-topology ρ yields the following well-known result on minimal T_2.

Corollary 3.9. *Let X be an infinite set and let $\tau \in LT(X)$ be T_2. Then τ is minimal T_2 if and only if every open filter on X with a unique adherent point is convergent (to that point).*

Recall that $\tau \in LT(X)$ is said to be compact if every open cover of X has a finite subcover. Equivalently, τ is compact if and only if every filter on X has an adherent point if and only if every ultrafilter on X converges [20]. It is known that if τ is minimal T_2, then τ is regular if and only if it is compact [20]. We will show that this last equivalence holds for minimal R_1 topologies in $\mathcal{L}_\rho$. The results given in the following lemma are well known. For the sake of completeness, we include the proofs.

Lemma 3.10. *Let $\tau \in LT(X)$.*

(i) *If τ is R_1 and compact, then τ is regular.*
(ii) *If τ is regular and every open filter on X has an adherent point, then τ is compact.*

Proof: (i) Let τ be R_1 and compact and let $x \in V \in \tau$. Then for each $y \in X \backslash V$, there exist $U^y \in \mathcal{N}_\tau(x)$ and $V_y \in \mathcal{N}_\tau(y)$ such that $U^y \cap V_y = \emptyset$. Now the family $\{V_y\}_{y \in X\backslash V}$ is an open cover of $X\backslash V$, a closed set, and hence a compact set. Thus, $X\backslash V \subseteq \bigcup_{i=1}^n V_{y_i}$ for some finite collection $\{y_1, ..., y_n\}$ of points in $X\backslash V$. Let $U = \bigcap_{i=1}^n U^{y_i}$. It is immediate that $U \in \mathcal{N}_\tau(x)$ and $cl(U) \subseteq V$, which show that τ is regular.

(ii) Let τ be regular and such that every open filter on X has an adherent point. Given an ultrafilter $\mathcal{R}$ on X, consider the open filter $\mathcal{F} = \mathcal{R} \cap \tau$. Then $\mathcal{F}$ has an adherent point $x \in X$. Now, if $\mathcal{R}$ does not converge to x, there exists $V \in \mathcal{N}_\tau(x)$ such that $V \notin \mathcal{R}$, and hence $X\backslash V \in \mathcal{R}$ since $\mathcal{R}$ is ultrafilter. By regularity of τ, one can choose $U \in \mathcal{N}_\tau(x)$ with $cl_\tau(U) \subseteq V$. Then $X\backslash cl_\tau(U) \supseteq X\backslash V$, and thus $X\backslash cl_\tau(U) \in \mathcal{R} \cap \tau = \mathcal{F}$. But since $x \in adh_\tau \mathcal{F}$, it must be that $U \cap X\backslash cl_\tau(U) \neq \emptyset$, a contradiction. Thus, $\mathcal{R}$ converges to x, and therefore τ is compact. □

Proposition 3.11. *Let $\tau \in \mathcal{L}_\rho$. If τ is minimal R_1, then every open filter on X has an adherent point.*

Proof: Suppose there is an open filter $\mathcal{F}$ on X such that $adh_\tau \mathcal{F} = \emptyset$. For each $x \in X$, there exist $V \in \mathcal{N}_\tau(x)$ and $F \in \mathcal{F}$ such that $V \cap F = \emptyset$. In particular, $V \notin \mathcal{F}$. On the other hand, since $cl(x) \subseteq V$, $X\backslash cl(x) \supseteq X\backslash V \supseteq F$, and thus $X\backslash cl(x) \in \mathcal{F}$. This shows that $\mathcal{F} \supseteq \mathcal{N}_{at(\rho)}(x)$ and $\mathcal{F} \not\supseteq \mathcal{N}_\tau(x)$ for each $x \in X$. Now, fix $x \in X$ and let $\beta = \tau \cap (\mathcal{E}(x) \cup \mathcal{F})$. Then β is a topology in $\mathcal{L}_\rho$ which is strictly weaker than τ. We will prove that β is R_1, and thus τ is not minimal R_1.

By Proposition 3.6, it suffices to show that $\mathcal{F}$ contains an open filter $\mathcal{F}_0$ such that $adh_\tau \mathcal{F}_0 = cl(x)$. Let $\mathcal{F}_0 = \{F \in \mathcal{F} : F \cap V \neq \emptyset$, for all $V \in \mathcal{N}_\tau(x)\}$. It is clear that $\mathcal{F}_0$ is an open non-empty proper sub-filter of $\mathcal{F}$ and that $cl(x) \subseteq adh_\tau \mathcal{F}_0$. Now, let $y \notin cl(x)$. Since τ is R_1, there exist $V \in \mathcal{N}_\tau(x)$ and $W \in \mathcal{N}_\tau(y)$ such that $V \cap W = \emptyset$. On the other hand, since $y \notin adh_\tau \mathcal{F}$, there exist $U \in \mathcal{N}_\tau(y)$ and $F \in \mathcal{F}$ such that $U \cap F = \emptyset$. If $O = W \cap U$ and $G = V \cup F$, then

$O \in \mathcal{N}_\tau(y)$, $G \in \mathcal{F}_0$, and $O \cap G = \emptyset$. Thus, $y \notin adh_\tau \mathcal{F}_0$, and therefore $adh_\tau \mathcal{F}_0 = cl(x)$. □

The next result follows immediately from Lemma 3.10 and Proposition 3.11.

Theorem 3.12. *Let $\tau \in \mathcal{L}_\rho$ be minimal R_1. Then τ is compact if and only if it is regular.*

We end this section with the characterization of minimal regular topologies in $\mathcal{L}_\rho$, announced in the introduction of this paper.

Theorem 3.13. *Let $\tau \in \mathcal{L}_\rho$ be regular. Then τ is minimal regular if and only if every regular filter $\mathcal{F}$ on X, such that $adh_\tau \mathcal{F} = cl(x)$, for some $x \in X$, is convergent (necessarily to every point of $cl(x)$).*

Proof: ($\Rightarrow$) Let $\mathcal{F}$ be a τ-regular filter on X such that $adh_\tau \mathcal{F} = cl(x)$ for some $x \in X$, and suppose $\mathcal{F}$ does not converge. Then there exists $U \in \mathcal{N}_\tau(x)$ such that $U \notin \mathcal{F}$, and hence $\beta = \tau \cap ((\mathcal{E}(x) \cup \mathcal{F}) \in \mathcal{L}_\rho$ is strictly weaker than τ. Note that $x \in F$ for all $F \in \mathcal{F}$. Otherwise, $x \notin cl_\tau(F')$ for some $F' \in \mathcal{F}$, and hence $x \notin adh_\tau \mathcal{F}$, which contradicts the hypothesis that $adh_\tau \mathcal{F} = cl(x)$. We prove that β is regular, and therefore τ is not minimal regular.

Let $V \in \beta$ and $y \in V$. If $y \in cl(x)$, then $V \in \mathcal{N}_\tau(x) \cap \mathcal{F} = \mathcal{F}$, a regular filter, and thus there exists $U \in \mathcal{F}$ such that $cl_\tau(U) \subseteq V$. Since $x \in U$, then $cl_\beta(U) = cl_\tau(U) \subseteq V$. Now, if $y \notin cl(x) = adh_\tau \mathcal{F}$, there exist $U' \in \mathcal{N}_\tau(y)$ and $F \in \mathcal{F}$ such that $U' \cap F = \emptyset$. Choose $U \in \mathcal{N}_\tau(y)$ such that $cl_\tau(U) \subseteq V$ (this is possible since τ is regular). If $W = U \cap U'$, then $cl_\tau(W) \cap F = \emptyset$, and thus $X \setminus cl_\tau(W) \in \mathcal{F}$. It follows that $cl_\beta(W) = cl_\tau(W) \subseteq cl_\tau(U) \subseteq V$.

($\Leftarrow$) Suppose that every τ-regular filter on X for which the set of adherent points coincides with a point closure is τ-convergent. Let $\tau' \in \mathcal{L}_\rho$ be a regular topology such that $\tau' \leq \tau$. Fix $V \in \tau$ and $x \in V$. It is clear that $cl(x) = adh_\tau \mathcal{N}_\tau(x) \subseteq adh_\tau \mathcal{N}_{\tau'}(x) \subseteq adh_{\tau'} \mathcal{N}_{\tau'}(x) = cl(x)$. Since $\mathcal{N}_{\tau'}(x)$ is a τ'-regular filter, then $\mathcal{N}_{\tau'}(x)$ is a τ-regular filter. By hypothesis, $\mathcal{N}_{\tau'}(x)$ is τ-convergent, i.e., $\mathcal{N}_\tau(x) \subseteq \mathcal{N}_{\tau'}(x)$. Since this holds for every $x \in V$, then $V \in \tau'$, and thus $\tau' = \tau$. Therefore, τ is minimal regular in $\mathcal{L}_\rho$. □

Corollary 3.14. *A regular and T_1-topology on X is minimal regular if and only if every regular filter on X with a unique adherent point is convergent.*

Proof: Apply Theorem 3.13 to any T_1-topology ρ. □

4. Presober topologies in $\mathcal{L}_\rho$

In this last section, we consider a topological property known as presobriety, which is strictly weaker than R_1, and show that there are no minimal presober topologies in $\mathcal{L}_\rho$. As in the previous section, we assume $\rho \in LT(X)$ is any R_0-topology such that X can be written as an infinite union of disjoint point closures.

Definition 4.1. A non-empty closed subset C of X is said to be *reducible* if there are non-empty proper closed subsets C_1 and C_2 of C, such that $C = C_1 \cup C_2$. Otherwise, C is *irreducible.* By convention, $\emptyset$ is neither reducible nor irreducible.

Every point closure is irreducible. If C is an irreducible closed set, then it may be the case that it is the point closure of some point x. If so, x is called a generic point of C.

Definition 4.2. A topology is said to be *presober* if and only if each irreducible closed set has at least one generic point.

In case that every irreducible closed subset of a space has a unique generic point, the topology is said to be *sober*. Sobriety is thus a combination of two properties: the existence of generic points and their uniqueness. It is straightforward to see that the generic points in a topological space are unique if and only if the space satisfies the T_0 separation axiom. Thus, a topology is sober precisely when it is T_0 and presober.

In any T_2-topology, the irreducible closed sets are the singleton, so T_2 implies sobriety. The cofinite topology on an infinite set is an example of a T_1-topology which is not sober, so it is also an example of a T_0 and not presober topology.

Proposition 4.3. *Every R_1-topology $\tau \in LT(X)$ is presober.*

Proof: Let $\tau \in LT(X)$ be R_1 and let $C \subseteq X$ be closed. Let $x, y \in C$ with $x \neq y$. Then $cl_\tau(x)$ and $cl_\tau(y) \subseteq C$. If $y \notin cl_\tau(x)$, there exist disjoint open sets $U \in \mathcal{N}_\tau(x)$ and $V \in \mathcal{N}_\tau(y)$ such that $cl_\tau(x) \subseteq U$ and $cl_\tau(y) \subseteq V$. Let $C_1 = C \cap X \backslash U$ and $C_2 = C \cap X \backslash V$. Then C_1 and C_2 are non-empty proper closed subsets of C such that $C_1 \cup C_2 = C$, and thus C is reducible. It follows that an irreducible closed set must be a point closure, and hence τ is presober. □

Presobriety does not imply R_1, as the next example shows.

Example 4.4. Let X be a set with cardinality ≥ 3, and let $a, b \in X$ with $a \neq b$. Let τ be the topology $\{G \subseteq X : \{a,b\} \subseteq G\} \cup \{\emptyset\}$. Then a set C is closed if and only if $C \cap \{a,b\} = \emptyset$ or $C = X$. It is clear that for every $x \notin \{a,b\}$, the set $\{x\}$ is closed. If C is a non-empty closed proper subset of X, then C is irreducible if and only if it is a singleton $\{x\}$, with $x \notin \{a,b\}$, since otherwise, $C = \{y\} \cup (C\backslash\{y\})$ for any $y \in C$, and both $\{y\}$ and $C\backslash\{y\}$ are closed and non-empty. Also, X is itself irreducible since it is a point closure, $X = cl_\tau(a) = cl_\tau(b)$. Thus, the irreducible closed sets are the point closures, and so τ is presober. But τ is not R_1 since given any $x \notin \{a,b\}$, then $cl_\tau(x) \neq cl_\tau(a)$, but $cl_\tau(x)$ and $cl_\tau(a)$ can not be separated by disjoint open sets. Note that τ is an Alexandroff topology on X which is not T_0.

Proposition 4.5. *The presober property is expansive in $\mathcal{L}_\rho$ (i.e., if $\tau \in \mathcal{L}_\rho$ is presober, then τ' is presober for all $\tau' \in \mathcal{L}_\rho$ finer than τ).*

Proof: Let $\tau \in \mathcal{L}_\rho$ be presober and let $\tau' \in \mathcal{L}_\rho$ with $\tau \leq \tau'$. Let A be a non-empty, τ'-irreducible, τ'-closed subset of X, and let $B = cl_\tau(A)$. Then B is τ-irreducible. In fact, if B is τ-reducible and F and G are two non-empty, τ-closed, proper subsets of B such that $B = F \cup G$, then $F_1 = (A \cap F)$ and $G_1 = (A \cap G)$ are two non-empty, τ'-closed, proper subsets of A such that $A = F_1 \cup G_1$. Hence, A would be τ'-reducible, which contradicts the hypothesis. However, if B is τ-irreducible, then since τ is presober, there is some $b \in B$ such that $cl_\tau(b) = B$. Suppose that no point $a \in A$ is such that $cl_\tau(a) = B$, then since τ is R_0, $cl_\tau(a) \cap cl_\tau(b) = \emptyset$ for all $a \in A$, which is a contradiction since $A \subseteq B$. Thus, we may assume that $b \in A$. However, since $\tau' \in \mathcal{L}_\rho$, it follows that $cl_{\tau'}(b) = cl_\tau(b) = B \supseteq A$, and hence A is a point closure in the topology τ'. Therefore, τ' is presober. □

Since ρ is R_0, then $\overline{\rho}$ is R_1, and thus it is presober. Therefore, there exists at least a presober member of $\mathcal{L}_\rho$. On the other hand, $at(\rho)$ is not presober since a proper subset of X is $at(\rho)$-closed if and only if it is a finite union of disjoint point closure sets, and hence X is $at(\rho)$-irreducible, but X is not a point closure. Thus,

$at(\rho)$ is an example of an R_0-topology which is not presober. We will prove that there are no minimal presober topologies in $\mathcal{L}_\rho$.

Given $\tau \in \mathcal{L}_\rho$, $x \in X$, and $\mathcal{F}$ a filter on X, consider the topology $\beta = \tau \cap (\mathcal{E}(x) \cup \mathcal{F})$.

Lemma 4.6. *Let $\tau \in \mathcal{L}_\rho$ be presober and let $A \subseteq X$ be β-closed. If A is τ-reducible, then it is also β-reducible.*

Proof: Let $A \subseteq X$ be β-closed and τ-reducible, and let F and G be non-empty, τ-closed, proper subsets of A such that $A = F \cup G$. Then either $x \in A$ or $X \backslash A \in \mathcal{F}$. If $X \backslash A \in \mathcal{F}$ or $x \in F \cap G$, then F and G are β-closed, and therefore A is β-reducible. Thus, we just need to consider the case when x belongs to only one of the sets F or G.

Suppose $x \in F \backslash G$ (the case $x \in G \backslash F$ is similar). Then it is clear that F is β-closed. Moreover, since $x \notin G$ and since τ is R_0, it must be that $cl(x) \cap G = \emptyset$ (if $y \in cl(x) \cap G$, then $x \in cl(y) \subseteq G$). Write $A = F \cup \{cl(x) \cup G\}$. If $F \backslash \{cl(x) \cup G\} \neq \emptyset$, then F and $cl(x) \cup G$ are non-empty, β-closed, proper subsets of A, and thus A is β-reducible. If $F \backslash \{cl(x) \cup G\} = \emptyset$, we distinguish the following cases.

Case 1: G is τ-irreducible. In this case, $G = cl(g)$ for some $g \in G$, since τ is presober. Thus, $A = cl(x) \cup cl(g)$, and therefore A is β-reducible.

Case 2: G is τ-reducible. Then there exist G_1 and G_2, non-empty, τ-closed, proper subsets of G, such that $G = G_1 \cup G_2$. Write $A = (cl(x) \cup G_1) \cup (cl(x) \cup G_2)$. It is clear that A is β-reducible. □

The following result is an immediate consequence of Lemma 4.6.

Corollary 4.7. *Let $\tau \in \mathcal{L}_\rho$ be presober. Then every β-irreducible subset of X is also τ-irreducible.*

Proposition 4.8. *Let $\tau \in \mathcal{L}_\rho$ be presober, $x \in X$, and $\mathcal{F}$ a filter on X. If $\mathcal{F} \supseteq N_{at(\rho)}(x)$, then $\beta = \tau \cap (\mathcal{E}(x) \cup \mathcal{F})$ is presober.*

Proof: If $\mathcal{F} \supseteq N_{at(\rho)}(x)$, then $\beta \in \mathcal{L}_\rho$ (Lemma 3.5(i)). Given a β-irreducible set A, then A is τ-irreducible (Corollary 4.7), and hence A is the τ-closure of a point, and thus the β-closure of a point. Therefore, β is presober. □

Proposition 4.9. *There are no minimal presober members of $\mathcal{L}_\rho$.*

Proof: Let $\tau \in \mathcal{L}_\rho$ be a presober topology. Since $at(\rho)$ can not be presober, there is $V \in \tau \setminus at(\rho)$. Let $y \in V$ and let $\beta = \tau \cap (\mathcal{E}(y) \cup N_{at(\rho)}(y))$. By Proposition 4.8, β is a presober topology which is obviously strictly weaker than τ. Therefore, τ is not minimal presober. □

Corollary 4.10. *There are no minimal (sober and T_1) topologies on an infinite set.*

Proof: Follows from Proposition 4.9 with ρ any T_1-topology. □

Acknowledgment. The authors are very grateful to the referee for the comments and suggestions which improved the quality of the paper. In particular, the final proof of Proposition 4.5 is his/hers. The first author would like to thank Professors Carlos Uzcátegui and Jorge Vielma for helpful discussions related to this research.

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(Colasante) Department of Mathematics, Faculty of Sciences; University of Los Andes; Merida 5101, Venezuela

E-mail address: marucola@ula.ve

(van der Zypen) Allianz Suisse Insurance; Bleicherweg 19; CH-8022 Zurich, Switzerland

E-mail address: dominic.zypen@gmail.com