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ASYMPTOTIC DIMENSION IN BĘDLEWO

G. BELL AND A. DRANISHNIKOV

Abstract. This survey was compiled from lectures and problem sessions at the International Conference on Geometric Topology at the Mathematical Research and Conference Center in Będlewo, Poland, in July 2005.

1. Introduction

These are the lecture notes from the Workshop in Asymptotic Dimension Theory given at the International Conference on Geometric Topology held in Będlewo, Poland, in July 2005. One workshop consisted of lectures on the basic theory by Alexander Dranishnikov and problem sessions intended for young researchers hosted by Greg Bell. The other workshop was given by Mladen Bestvina, with the assistance of Lars Louder and Henry Wilton, and concerned limit groups.

These notes seek to combine the lectures and problem sets into a single survey which is intended to serve as a basic introduction to the theory.

Section 2 gives the definition of asymptotic dimension and useful equivalent formulations of the definition. A detailed proof of the

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equivalences is given, as well as some basic examples of computations involving asymptotic dimension. Also in this section we prove that asymptotic dimension is a coarse invariant.

After proving a useful union theorem for asymptotic dimension, we connect the asymptotic dimension to Lebesgue covering dimension via the Higson corona. Also in this section we introduce various other notions of asymptotic dimension and give the known results on coincidence of these notions. The final part of section 2 is devoted to uniform embedding of metric spaces with finite asymptotic dimension into products of trees.

The third section of the paper contains the main result of the authors' paper [6]: a Hurewicz-type theorem for asymptotic dimension. Although the proof is omitted (it is quite technical), the idea of it is given in the proof of a much simpler version of the theorem that does not give a tight upper bound. The rest of this section is devoted to examples of applications of the Hurewicz theorem to geometric group theory. In particular, we relate the two workshops of the conference by applying the Hurewicz theorem to prove that limit groups have finite asymptotic dimension.

The fourth section provides motivation for considering the asymptotic dimension of a finitely generated group. In particular, we review results in [14], [20], [35], [36] concerning the Novikov higher signature conjecture for finitely generated groups with finite classifying spaces and finite asymptotic dimension. In this section we also give Nigel Higson and John Roe's proof that finitely generated groups with finite asymptotic dimension have Guoliang Yu's property A, i.e., are exact [25].

The final section of the paper is devoted to the dimension function of a finitely generated group. We give an example of a finitely generated group with infinite asymptotic dimension and show the quasi-isometry invariance of the growth of the dimension function for a finitely generated group.

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2. Definitions and basic results

In this section we give several equivalent definitions of asymptotic dimension (asdim) and prove that they are equivalent. We also show that asymptotic dimension is a coarse invariant and prove a useful union theorem.

2.1. Definitions of asymptotic dimension.

The asymptotic dimension can be defined for any coarse space (see [30]), but we will consider only metric spaces in this paper. We would like to view the asymptotic dimension of a space as somehow dual to Lebesgue covering dimension. Given a cover $\mathcal{V}$ of a topological space, we say that the cover $\mathcal{U}$ refines $\mathcal{V}$ if every $U \in \mathcal{U}$ is contained in some element $V \in \mathcal{V}$. Recall that the Lebesgue dimension of a metric space X ($\dim X$) can be defined as follows: $\dim X \leq n$ if and only if for every open cover $\mathcal{V}$ of X there is a cover $\mathcal{U}$ of X refining $\mathcal{V}$ with multiplicity $\leq n+1$. We will use the words "order" and "multiplicity" of a cover interchangeably to mean the largest number of elements of the cover meeting any point of the space. As usual, we define $\dim X = n$ if it is true that $\dim X \leq n$, but it is not true that $\dim X \leq n-1$. We will do the analogous thing to define $\operatorname{asdim} X = n$. In the following definition, we use open covers to give the definition the feel of an analog of covering dimension. Of course, in the large scale, whether the covers are open or not matters very little.

Definition 2.1.1. Let X be a metric space. We say that the *asymptotic dimension* of X does not exceed n and write $\operatorname{asdim} X \leq n$ provided for every uniformly bounded open cover $\mathcal{V}$ of X there is a uniformly bounded open cover $\mathcal{U}$ of X of multiplicity $\leq n+1$ so that $\mathcal{V}$ refines $\mathcal{U}$.

In practice this definition is rarely used; instead, one of the equivalent conditions described in the next theorem is used. In particular, for proving that the asymptotic dimension of a space is finite, it is easier to use condition (2) or (3). In order to obtain a tight upper bound for dimension, it is often better to work with condition (5), which is phrased in terms of maps to uniform complexes.

Before stating the theorem, we define the terminology used there. Often we will need to consider very large positive constants and we remind ourselves that they are large by writing $r < \infty$ instead of $r > 0$. On the other hand, writing $\epsilon > 0$ is supposed to mean that ϵ is a small positive constant.

Let $r < \infty$ be given and let X be a metric space. We will say that a family $\mathcal{U}$ of subsets of X is *r-disjoint* if $d(U, U') > r$ for every $U \neq U'$ in $\mathcal{U}$. Here, $d(U, U')$ is defined to be $\inf\{d(x, x') \mid x \in U, x' \in U'\}$. The *r-multiplicity* of a family $\mathcal{U}$ of subsets of X is defined to be the largest n so that there is an $x \in X$ so that $B_r(x)$ meets n of the sets from $\mathcal{U}$. Recall that the *Lebesgue number* of a cover $\mathcal{U}$ of X is the largest number λ so that if $A \subset X$ and $\operatorname{diam}(A) \leq \lambda$ then there is some $U \in \mathcal{U}$ so that $A \subset U$.

Let K be a countable simplicial complex. There are two natural metrics we can place on $|K|$, the geometric realization of K. We wish to consider the uniform metric on $|K|$. This is defined by embedding K into ℓ^2 by mapping each vertex to an element of an orthonormal basis for ℓ^2 and giving it the metric it inherits as a subspace. A map $\varphi : X \to Y$ between metric spaces is uniformly cobounded if for every $R > 0$, $\operatorname{diam}(\varphi^{-1}(B_R(y)))$ is uniformly bounded.

Theorem 2.1.2. *Let X be a metric space. The following conditions are equivalent.*

1. $\operatorname{asdim} X \leq n$;
2. *for every $r < \infty$, there exist uniformly bounded, r-disjoint families $\mathcal{U}^0, \ldots, \mathcal{U}^n$ of subsets of X such that $\cup_i \mathcal{U}^i$ is a cover of X;*
3. *for every $d < \infty$, there exists a uniformly bounded cover $\mathcal{V}$ of X with d-multiplicity $\leq n + 1$;*
4. *for every $\lambda < \infty$, there is a uniformly bounded cover $\mathcal{W}$ of X with Lebesgue number $> \lambda$ and multiplicity $\leq n + 1$; and*
5. *for every $\epsilon > 0$, there is a uniformly cobounded, ϵ-Lipschitz map $\varphi : X \to K$ to a uniform simplicial complex of dimension n.*

Proof: This proof is not the most efficient one. We prove (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (2), and then (2) $\Rightarrow$ (1) and (1) $\Rightarrow$ (4).

(2) $\Rightarrow$ (3): Let $d < \infty$ be given and take $r > 2d$. We can find uniformly bounded, r-disjoint families $\mathcal{U}^0, \ldots, \mathcal{U}^n$ of subsets of X covering X. Put $\mathcal{V} = \cup \mathcal{U}^i$. Suppose $x \in X$ and consider $B_d(x)$. If

$U \cap B_d(x)$ and $U' \cap B_d(x)$ are both nonempty, then $d(U, U') \leq 2d < r$, so U and U' must have come from distinct families $\mathcal{U}^i$ and $\mathcal{U}^j$ with $i \neq j$. Thus, there can be at most $n+1$ of the elements of $\mathcal{V}$ having non-trivial intersection with $B_d(x)$. So (3) is proved.

(3) $\Rightarrow$ (4): Let $\lambda < \infty$ be given and take a uniformly bounded cover $\mathcal{V}$ of X with 5λ-multiplicity $\leq n+1$. Define $\bar{V} = N_{2\lambda}(V)$. Then setting $\mathcal{W} = \{\bar{V} \mid V \in \mathcal{V}\}$, we obtain a uniformly bounded cover with Lebesgue number $> \lambda$. It remains to show that the multiplicity is bounded by $n+1$. To this end, take $x \in X$ and observe that if $x \in \bar{V}$, then $d(x, V) < 2\lambda$. So V is among the $n+1$ elements of $\mathcal{V}$ meeting $B_{2\lambda}(x)$. So the multiplicity of $\mathcal{W}$ is $\leq n+1$.

(4) $\Rightarrow$ (5): Let $\epsilon > 0$ be given and suppose that $\mathcal{W}$ is a uniformly bounded cover of X with multiplicity $\leq n+1$ and Lebesgue number greater than $\lambda = (2n+3)^2/\epsilon$. For each $W \in \mathcal{W}$, define the map $\varphi_W : X \to K$ by

$$\varphi_W(x) = \frac{d(x, X - W)}{\sum_{V \in \mathcal{W}} d(x, X - V)}.$$

The maps $\{\varphi_W\}_W$ define a map $\varphi : X \to Nerve(\mathcal{W})$ where the nerve is a n-dimensional complex in ℓ^2 with the uniform metric. It remains to check that φ is uniformly cobounded and ϵ-Lipschitz. If σ is a simplex in K and x and y both map to σ, then there exist sets $U, V \in \mathcal{W}$ so that $x \in U$ and $y \in V$ and $U \cap V \neq \emptyset$. Thus, $d(x, y) \leq 2B$, where B is a uniform bound on the diameter of the elements of $\mathcal{W}$.

Finally, we check that φ is ϵ-Lipschitz. Let $x, y \in X$ and $U \in \mathcal{W}$. Let $\bar{U}$ denote the complement $X - U$. The triangle inequality implies

$$|d(x, \bar{U}) - d(y, \bar{U})| \leq d(x, y).$$

Also, observe that for any $x \in X$, $\sum_{U \in \mathcal{W}} d(x, \bar{U}) \geq \lambda$ since λ is a Lebesgue number for $\mathcal{W}$. Thus, we have

$$|\varphi_U(x) - \varphi_U(y)| = \left| \frac{d(x, \bar{U})}{\sum_{V \in \mathcal{W}} d(x, \bar{V})} - \frac{d(y, \bar{U})}{\sum_{V \in \mathcal{W}} d(y, \bar{V})} \right|$$

$$\leq \frac{|d(x, \bar{U}) - d(y, \bar{U})|}{\sum_{V \in \mathcal{W}} d(x, \bar{V})} + \left| \frac{d(y, \bar{U})}{\sum_{V \in \mathcal{W}} d(x, \bar{V})} - \frac{d(y, \bar{U})}{\sum_{V \in \mathcal{W}} d(y, \bar{V})} \right|$$

$$\leq \frac{d(x, y)}{\sum_{V \in \mathcal{W}} d(x, \bar{V})} + \frac{d(y, \bar{U})}{\sum_{V \in \mathcal{W}} d(x, \bar{V}) \sum_{V \in \mathcal{W}} d(y, \bar{V})} \sum_{V \in \mathcal{W}} |d(x, \bar{V}) - d(y, \bar{V})|$$

$$\leq \frac{1}{\lambda}d(x,y) + \frac{1}{\lambda}\left(\sum_{V\in\mathcal{W}} \left|d(x,\bar{V}) - d(y,\bar{V})\right|\right)$$

$$\leq \frac{1}{\lambda}d(x,y) + \frac{2n+2}{\lambda}d(x,y) = \frac{(2n+3)}{\lambda}d(x,y).$$

Then we have

$$\|p(x)-p(y)\|_2 = \left(\sum_{U\in\mathcal{W}} |\varphi_U(x) - \varphi_U(y)|^2\right)^{\frac{1}{2}}$$

$$\leq \left((2n+2)\left(\frac{(2n+3)}{\lambda}d(x,y)\right)^2\right)^{\frac{1}{2}}$$

$$\leq \frac{(2n+3)^{3/2}}{\lambda}d(x,y) \leq \epsilon d(x,y).$$

(5) $\Rightarrow$ (2): Let r be given and take $\phi : X \to K$ to be a uniformly cobounded, c/r-Lipschitz map to a uniform complex of dimension n, where c is a constant depending only on n yet to be determined. For each $i = 0,\ldots,n$, put $\mathcal{V}^i = \{St(b_\sigma, \beta^2 K) \mid \sigma \subset K, \dim\sigma = i\}$, where b_σ is the barycenter of σ and $\beta^2 K$ denotes the second barycentric subdivision. Now, obviously there is some d so that $\operatorname{diam} V \leq d$ for all $V \in \mathcal{V}^i$ and there is a constant c depending only on the dimension of K so that the elements of $\mathcal{V}^i$ are c-disjoint for all i.

Define $\mathcal{U}^i = \{f^{-1}(V) \mid V \in \mathcal{V}^i\}$. Then $\operatorname{diam}(U)$ is uniformly bounded since f was uniformly cobounded. Next, since f is c/r-Lipschitz, $d(U,U') < r$ implies that $d(f(U), f(U')) < c$, so the families $\mathcal{U}^i$ are r-disjoint.

(2) $\Rightarrow$ (1): Let $\mathcal{V}$ be given with $\operatorname{diam}\mathcal{V} \leq \delta$. Take r-disjoint families $\mathcal{U}^0,\ldots,\mathcal{U}^n$ of uniformly bounded sets in X with $r > 2\delta$. Put $\bar{\mathcal{U}}^i = \{N_\delta(U) \mid U \in \mathcal{U}^i\}$. Put $\mathcal{U} = \cup_i \bar{\mathcal{U}}^i$. Then since the $\bar{\mathcal{U}}^i$ are disjoint, the multiplicity of $\mathcal{U}$ does not exceed $n+1$. Next, given $V \in \mathcal{V}$, V must intersect some $U \in \mathcal{U}^i$ for some i. Since $\operatorname{diam}(V) \leq \delta$, $V \subset N_\delta(U)$ which is an element of $\mathcal{U}$. Thus, $\mathcal{V}$ refines $\mathcal{U}$.

(1) $\Rightarrow$ (4): Let $\lambda < \infty$ be given. Let $\mathcal{V} = \{B_\lambda(x) \mid x \in X\}$. Clearly, $\mathcal{V}$ is a cover of X, so there is a uniformly bounded cover $\mathcal{U}$

of X with multiplicity $\leq n+1$ so that $\mathcal{V}$ refines $\mathcal{U}$. Thus, any set with diameter $\leq \lambda$ will be entirely contained within one element of $\mathcal{V}$ so it will be entirely contained within one element of $\mathcal{U}$, i.e., $L(\mathcal{U}) \geq \lambda$. □

We conclude this section with a computation.

Example 2.1.3. $\operatorname{asdim} T \leq 1$ *for all trees* T *in the edge-length metric.*

Proof: Fix some vertex x_0 to be the root of the tree. Let $r < \infty$ be given and take concentric annuli centered at x_0 of thickness r as follows: $A_k = \{x \in T \mid d(x, x_0) \in [kr, (k+1)r)\}$. Although alternating the annuli (odd k, even k) yields r-disjoint sets, these sets clearly do not have uniformly bounded diameter. We have to further subdivide each annulus.

Fix $k > 1$. Define $x \sim y$ in A_k if the geodesics $[x_0, x]$ and $[x_0, y]$ in T contain the same point z with $d(x_0, z) = r(k - \frac{1}{2})$. Clearly, in a tree this forms an equivalence relation. The equivalence classes are $3r$ bounded and elements from distinct classes are at least r apart. So, define $\mathcal{U}$ to be equivalence classes corresponding to even k (along with A_0 itself) and $\mathcal{V}$ to be equivalence classes corresponding to odd k. These two families cover T and consist of uniformly bounded, r-disjoint sets. Thus, $\operatorname{asdim} T \leq 1$. □

The proof that $\operatorname{asdim} T \leq 1$ can be modified (see [31]) to prove that δ-hyperbolic metric spaces with bounded growth have finite asymptotic dimension. Thus, finitely generated δ-hyperbolic groups have finite asymptotic dimension; see also [9], [24], [30].

2.2. Large-scale invariance of asymptotic dimension.

One of the goals of this section is to prove that asymptotic dimension is well defined for finitely generated groups given the word metric.

Let Γ be a finitely generated group with finite, symmetric generating set S. We can define a norm on the group Γ corresponding to S by setting $\|\gamma\|_S$ equal to the minimal number of S-letters necessary to present a word equal to γ. Here we adopt the convention that the identity is presented by the empty word. With this norm, we can define the (left-invariant) word metric on Γ by

$d_S(g,h) = \|g^{-1}h\|_S$. When S is understood, we will simply write $d(g,h)$; see also Corollary 2.2.2.

Let X and Y be metric spaces. A map $f : X \to Y$ is a (λ, ϵ)-*quasi-isometry* if $d(f(x), f(x')) \leq \lambda d(x, x') + \epsilon$ for every pair of points $x, x' \in X$. A map between metric spaces is a quasi-isometry if it is a (λ, ϵ)-quasi-isometry for some $\lambda > 0$ and some $\epsilon > 0$. The two spaces X and Y are quasi-isometric if there is a quasi-isometry $f : X \to Y$ and a constant C so that $Y \subset N_C(f(X))$.

Coarse equivalence is a weaker notion of equivalence. A map $f : X \to Y$ between metric spaces is a *coarse embedding* if there exist non-decreasing functions ρ_1 and ρ_2, $\rho_i : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ such that $\rho_i \to \infty$ and for every $x, x' \in X$

$$\rho_1(d_X(x,x')) \leq d_Y(f(x), f(x')) \leq \rho_2(d_X(x,x')).$$

Such a map is often called a *coarsely uniform embedding* or just a *uniform embedding*. The metric spaces X and Y are *coarsely equivalent* if there is a coarse embedding $f : X \to Y$ so that there is some R such that $Y \subset N_R(f(X))$.

Observe that quasi-isometric spaces are coarsely equivalent with linear ρ_i. (This is not entirely obvious from the definition. One way to see this is to construct a so-called quasi-inverse from the quasi-isometry.) Also, the function ρ_2 can always be assumed to be linear.

Proposition 2.2.1. *Let $f : X \to Y$ be a coarse equivalence. Then* $\operatorname{asdim} X = \operatorname{asdim} Y$.

Proof: If $\mathcal{U}^0, \ldots, \mathcal{U}^n$ are r-disjoint, D-bounded families covering X, then the families $f(\mathcal{U}^i)$ are $\rho_1(r)$-disjoint and $\rho_2(D)$-bounded. Since $N_R(f(X))$ contains Y, we see that taking families $N_R(f(\mathcal{U}^i))$ will cover Y and be $(2R+\rho_2(D))$-bounded and $(\rho_1(r)-2R)$-disjoint. Since $\rho_i \to \infty$, r can be chosen large enough for $\rho_1(r) - 2R$ to be as large as one likes. Therefore, $\operatorname{asdim} Y \leq \operatorname{asdim} X$.

The same proof applied to a coarse inverse for f proves that $\operatorname{asdim} X \leq \operatorname{asdim} Y$. □

Corollary 2.2.2. *Let Γ be a finitely generated group. Then* $\operatorname{asdim} \Gamma$ *is an invariant of the choice of generating set; i.e., it is a group property.*

Proof: Let S and S' be finite generating sets for Γ. We have to show that (Γ, d_S) and $(\Gamma, d_{S'})$ are coarsely equivalent. In fact, they are Lipschitz equivalent, as we now show.

Let $\lambda_1 = \max\{\|s\|_{S'} \mid s \in S\}$ and $\lambda_2 = \max\{\|s'\|_S \mid s' \in S'\}$. It follows that $\lambda_2^{-1}\|\gamma\|_{S'} \leq \|\gamma\|_S \leq \lambda_2\|\gamma\|_{S'}$. Take $\lambda = \max\{\lambda_1, \lambda_2\}$. Then $\lambda^{-1} d_{S'}(g,h) \leq d_S(g,h) \leq \lambda d_{S'}(g,h)$. □

Example 2.2.3. $\operatorname{asdim} \mathbb{R} = \operatorname{asdim} \mathbb{Z} = 1$.

Proof: First, we show that $\mathbb{R}$ and $\mathbb{Z}$ are coarsely isometric. To this end, let $f : \mathbb{Z} \to \mathbb{R}$ be the identity map. Since the metric on $\mathbb{R}$ restricted to $\mathbb{Z}$ is the same as the metric on the image of f, this map is a coarse equivalence. Observe, too, that $N_1(f(\mathbb{Z})) = \mathbb{R}$.

We will show that $\operatorname{asdim} \mathbb{Z} \leq 1$ and $\operatorname{asdim} \mathbb{Z} \geq 1$. First, given an $R < \infty$, consider the sets $A_k = [2Rk, 2R(k+1)]$, where $k \in \mathbb{Z}$. Let $\mathcal{U} = \{A_{2k} \mid k \in \mathbb{Z}\}$ and $\mathcal{V} = \{A_{2k-1} \mid k \in \mathbb{Z}\}$. Clearly, the elements of each of these families have diameter bounded by $2R$. It is also easy to check that any two elements from the same family are R-disjoint as required.

On the other hand, if $\operatorname{asdim} \mathbb{Z} \leq 0$, then for any $R < \infty$, there would be a cover of $\mathbb{Z}$ by an R-disjoint family of subsets of $\mathbb{Z}$ with uniformly bounded diameter. Let $R > 1$ and let U be the element of the family containing 0. Since $d(n, n+1) = 1$, no consecutive integers can belong to different elements of $\mathcal{U}$. Thus, $\mathbb{Z} \subset U$, so $\operatorname{diam}(U) = \infty$. Thus, $\operatorname{asdim} \mathbb{Z} > 0$. □

The technique used in the proof that $\operatorname{asdim} \mathbb{Z} > 0$ can also be used to show the following result.

Example 2.2.4. *Let Γ be a finitely generated group. Then* $\operatorname{asdim} \Gamma = 0$ *if and only if Γ is finite.*

J. Smith [33] has classified all countable (not just finitely generated) groups with a left-invariant, proper metric, and asymptotic dimension 0 and so this last example can be deduced as a corollary.

Theorem 2.2.5. *Let G be a countable group. Then* $\operatorname{asdim} G = 0$ *if and only if every finitely generated subgroup of G is finite.*

Proposition 2.2.6. *Let X be a metric space and $Y \subset X$. Then* $\operatorname{asdim} Y \leq \operatorname{asdim} X$.

Proof: Let $R < \infty$ be given and take a cover $\mathcal{U}$ of X by uniformly bounded sets with R-multiplicity $\leq n+1$. Clearly, the restriction of this cover to Y yields a cover whose elements have uniformly bounded diameter, and at most $n+1$ of them can meet any ball of radius R in Y. Thus, $\operatorname{asdim} Y \leq \operatorname{asdim} X$. □

Proposition 2.2.7. $\operatorname{asdim} \mathbb{R}^n = n$.

Proof: We will see in the next section that $\operatorname{asdim}(X \times Y) \leq \operatorname{asdim} X + \operatorname{asdim} Y$, or you could convince yourself of the upper bound by drawing pictures for the plane and imagining their extensions to higher dimensions.

For the lower bound, we use the technique of [21]. We could also use homological methods; see [30]. First, we assume $\dim[0,1]^n = n$ to be known. If $\operatorname{asdim} \mathbb{R}^n \leq k$, we can take $k+1$ R-disjoint families of uniformly bounded subsets of $\mathbb{R}^n$ which cover $\mathbb{R}^n$. But contracting the covers and taking closures gives an ϵ-cover of $[0,1]^n$ for small ϵ with multiplicity $\leq k+1$. Thus, $k \geq n$. □

2.3. A union theorem for asymptotic dimension.

In this section we establish a union theorem for asymptotic dimension. It should be noted that here asymptotic dimension varies slightly from covering dimension. For example, the finite union theorem for covering dimension says $\dim(X \cup Y) \leq \dim X + \dim Y + 1$ and that inequality is sharp. Also, a countable union theorem for covering dimension is $\dim(\cup_i C_i) \leq \max_i\{\dim C_i\}$ where the C_i are closed subsets of X. Notice that there can be no direct analog of this theorem for asymptotic dimension since every finitely generated group is a countable set of points, and as we shall see, finitely generated groups can have arbitrary (even infinite) asymptotic dimension.

Let $\mathcal{U}$ and $\mathcal{V}$ be families of subsets of X. Define the r-saturated union of $\mathcal{V}$ with $\mathcal{U}$ by

$$\mathcal{V} \cup_r \mathcal{U} = \{N_r(V;\mathcal{U}) \mid V \in \mathcal{V}\} \cup \{U \in \mathcal{U} \mid d(U,\mathcal{V}) > r\},$$

where $N_r(V;\mathcal{U}) = V \cup \bigcup_{d(U,V)\leq r} U$.

Proposition 2.3.1. *Let $\mathcal{U}$ be an r-disjoint, R-bounded family of subsets of X with $R \geq r$. Let $\mathcal{V}$ be a $5R$-disjoint, D-bounded family of subsets of X. Then $\mathcal{V} \cup_r \mathcal{U}$ is r-disjoint and $(D+2(r+R))$-bounded.*

Proof: The uniform bound on the diameters of elements of $\mathcal{V}\cup_r\mathcal{U}$ is clear. To see that the disjointness condition holds, we consider the types of elements in $\mathcal{V}\cup_r\mathcal{U}$, those of the form U and those of the form $N_r(V;\mathcal{U})$. Obviously, if $U \neq U'$, then U and U' are r-disjoint by the definition of $\mathcal{U}$. A set of the form U and a set $N_r(V;\mathcal{U})$ are r-disjoint by definition of $N_r(V;\mathcal{U})$. Finally, we consider $N_r(V;\mathcal{U})$ and $N_r(V';\mathcal{U})$, where $V \neq V'$. Clearly, these sets are contained in $N_{r+R}(V)$ and $N_{r+R}(V')$, respectively. Since $d(V,V') \geq 5R$ and $R \geq r$, we find that $d(N_r(V;\mathcal{U}), N_r(V';\mathcal{U})) \geq r$. □

Let X be a metric space. We will say that the family $\{X_\alpha\}$ of subsets of X satisfies the inequality $\operatorname{asdim} X_\alpha \leq n$ *uniformly* if for every $r < \infty$, a constant R can be found so that for every α, there exist r-disjoint families $\mathcal{U}_\alpha^0, \dots, \mathcal{U}_\alpha^n$ of R-bounded subsets of X_α covering X_α. A typical example of such a family is a family of isometric subsets of a metric space. Another example is any family containing finitely many sets.

Theorem 2.3.2 (Union Theorem)**.** *Let $X = \cup_\alpha X_\alpha$ be a metric space where the family $\{X_\alpha\}$ satisfies the inequality $\operatorname{asdim} X_\alpha \leq n$ uniformly. Suppose further that for every r there is a $Y_r \subset X$ with $\operatorname{asdim} Y_r \leq n$ so that $d(X_\alpha - Y_r, X_{\alpha'} - Y_r) \geq r$ whenever $X_\alpha \neq X_{\alpha'}$. Then $\operatorname{asdim} X \leq n$.*

Before proving this theorem, we state a corollary: the finite union theorem for asymptotic dimension.

Corollary 2.3.3. *Let X be a metric space with $A, B \subset X$. Then* $\operatorname{asdim}(A \cup B) \leq \max\{\operatorname{asdim} A, \operatorname{asdim} B\}$.

Proof of Corollary: Apply the union theorem to the family A and B with $B = Y_r$ for every r. □

Proof of Union Theorem: Let $r < \infty$ be given and take r-disjoint, R-bounded families $\mathcal{U}_\alpha^i$ $(i = 0, \dots, n)$ of subsets of X_α so that $\cup_i \mathcal{U}_\alpha^i$ covers X_α. We may assume $R \geq r$. Take $Y = Y_{5R}$ as in the statement of the theorem and cover Y by families $\mathcal{V}^0, \dots, \mathcal{V}^n$ which are D-bounded and $5R$-disjoint. Let $\bar{\mathcal{U}}_\alpha^i$ denote the restriction of $\mathcal{U}_\alpha^i$ to the set $X_\alpha - Y$. For each i, take $\mathcal{W}_\alpha^i = \mathcal{V}^i \cup_r \bar{\mathcal{U}}_\alpha^i$. By the proposition, $\mathcal{W}^i$ consists of uniformly bounded sets and is r-disjoint. Finally, put $\mathcal{W}^i = \{W \in \mathcal{W}_\alpha^i \mid \alpha\}$. Observe that each $\mathcal{W}^i$ is r-disjoint and uniformly bounded. Also, it is easy to check that $\cup_i \mathcal{W}^i$ covers X. □

2.4. Connection to the classical dimension.

Let $\varphi\colon X \to \mathbb{R}$ be a function defined on a metric space X. For every $x \in X$ and every $r > 0$, let $V_r(x) = \sup\{|\varphi(y) - \varphi(x)| \mid y \in N_r(x)\}$. A function φ is called *slowly oscillating* whenever for every $r > 0$, we have $V_r(x) \to 0$ as $x \to \infty$ (the latter means that for every $\varepsilon > 0$, there exists a compact subspace $K \subset X$ such that $|V_r(x)| < \varepsilon$ for all $x \in X \setminus K$). Let $\bar{X}$ be the compactification of X that corresponds to the family of all continuous bounded slowly oscillating functions. The *Higson corona* of X is the remainder $\nu X = \bar{X} \setminus X$ of this compactification.

It is known that the Higson corona is a functor from the category of proper metric space and coarse maps into the category of compact Hausdorff spaces. In particular, if $X \subset Y$, then $\nu X \subset \nu Y$.

For any subset A of X, we denote by A' its trace on νX, i.e., the intersection of the closure of A in $\bar{X}$ with νX. Obviously, the set A' coincides with the Higson corona νA.

Dranishnikov, J. Keesling, and V. V. Uspenskij [21] proved the inequality

$$\dim \nu X \leq \operatorname{asdim} X.$$

It was shown there that $\dim \nu X \geq \operatorname{asdim} X$ for a large class of spaces, in particular for $X = \mathbb{R}^n$. This gives another approach to Proposition 2.2.7. Later, Dranishnikov [16] proved that the equality $\dim \nu X = \operatorname{asdim} X$ holds, provided $\operatorname{asdim} X < \infty$. The question of whether there is a metric space X with $\operatorname{asdim} X = \infty$ and $\dim \nu X < \infty$ is still open.

2.5. Asymptotic inductive dimension.

The notion of asymptotic inductive dimension (asInd) was introduced in [17].

Recall that a closed subset C of a topological space X is a *separator* between disjoint subsets $A, B \subset X$ if $X \setminus C = U \cup V$, where U and V are open subsets in X, $U \cap V = \emptyset$, $A \subset U$, and $V \subset B$. A closed subset C of a topological space X is a *cut* between disjoint subsets $A, B \subset X$ if every continuum (compact connected space) $T \subset X$ that intersects both A and B also intersects C.

Let X be a proper metric space. A subset $W \subset X$ is called an *asymptotic neighborhood* of a subset $A \subset X$ if $\lim_{r\to\infty} d(X \setminus N_r(x_0), X \setminus W) = \infty$. Two sets A and B in a metric space are *asymptotically disjoint* if $\lim_{r\to\infty} d(A \setminus N_r(x_0), B \setminus N_r(x_0)) = \infty$. In other words, two sets are asymptotically disjoint if the traces A' and B' on νX are disjoint.

A subset C of a metric space X is an *asymptotic separator* between asymptotically disjoint subsets $A_1, A_2 \subset X$ if the trace C' is a separator in νX between A_1' and A_2'.

We recall the definition of the asymptotic Dimensiongrad in the sense of Brouwer (asInd_b) from [23].

Let X be a metric space and $\lambda > 0$. A finite sequence $x_1, \dots, x_k$ in X is a λ-*chain* between subsets $A_1, A_2 \subset X$ if $x_1 \in A_1$, $x_k \in A_2$, and $d(x_i, x_{i+1}) < \lambda$ for every $i = 1, \dots, k-1$. We say that a subset C of a metric space X is an *asymptotic cut* between asymptotically disjoint subsets $A_1, A_2 \subset X$ if for every $D > 0$, there is $\lambda > 0$ such that every λ-chain between A_1 and A_2 intersects $N_D(C)$.

By definition, $\operatorname{asInd} X = \operatorname{asInd}_b X = -1$ if and only if X is bounded. Suppose we have defined the class of all proper metric spaces Y with $\operatorname{asInd} Y \le n-1$ (with $\operatorname{asInd}_b Y \le n-1$, respectively). Then $\operatorname{asInd} X \le n$ ($\operatorname{asInd}_b Y \le n$, respectively) if and only if for every pair of asymptotically disjoint subsets $A_1, A_2 \subset X$, there exists an asymptotic separator (asymptotic cut, respectively) C between A_1 and A_2 with $\operatorname{asInd} C \le n-1$ ($\operatorname{asInd}_b C \le n-1$, respectively). The dimension functions asInd and asInd_b are called the *asymptotic inductive dimension* and *asymptotic Brouwer inductive dimension*, respectively.

It is easy to prove that $\operatorname{asInd}_b X \le \operatorname{asInd} X$ for every X. It is unknown if $\operatorname{asInd} = \operatorname{asInd}_b$ for proper metric spaces.

Theorem 2.5.1. *For all proper metric spaces X with $0 < \operatorname{asdim} X < \infty$, we have*

$$\operatorname{asdim} X = \operatorname{asInd} X.$$

This theorem is a very important step in the existing proof of the exact formula of the asymptotic dimension of the free product $\operatorname{asdim} A * B$ of groups [7]; see section 3.4.

Notice that there is a small problem with coincidence of asymptotic dimension and asymptotic inductive dimension in dimension 0. This leads to philosophical discussions of whether bounded metric spaces should be defined to have asdim $= -1$ or 0. Observe that in the world of finitely generated groups, $\operatorname{asdim}\Gamma = 0$ if and only if

Γ is finite. On the other hand, there are metric spaces, for instance $2^n \subset \mathbb{R}$, that are unbounded yet have asymptotic dimension 0.

2.6. Embeddings into trees.

In [18], Dranishnikov showed that every proper metric space X with $\operatorname{asdim} X \leq n$ admits a uniform embedding into a product of $n+1$ locally finite regular $\mathbb{R}$-trees. Using this result, Dranishnikov and M. Zarichnyi [23] constructed a metric space M_n with $\operatorname{asdim} M_n = n$ that is universal for the class of proper metric spaces with $\operatorname{asdim} X \leq n$. The space M_n plays a crucial role in the proof of the above theorem.

We recall the notion of Assouad-Nagata dimension (N-dim) defined in [1]: For a metric space X, we have $\text{N-dim}\, X \leq n$ if there is a constant C such that for each $r > 0$ there is an open cover $\mathcal{U}(r)$ of X by sets of diameter $\leq Cr$ such that each open ball of radius r meets at most $n+1$ members of $\mathcal{U}(r)$.

Urs Lang and Thilo Schlichenmaier gave the following refinement of Dranishnikov's embedding [26].

Theorem 2.6.1. *If for a metric space* $N\text{-}\mathrm{dim}(X, d) \leq n$*, then for sufficiently small* ϵ*,* (X, d^ϵ) *admits a bi-Lipschitz embedding in the product of* $n+1$ *locally finite trees.*

Dranishnikov and Viktor Schroeder [22] proved that the hyperbolic plane admits a bi-Lipschitz embedding into the product of two binary trees.

Later Sergei Buyalo, Dranishnikov, and Schroeder [13], using techniques of [22] and some results of Buyalo [12] showed that a finitely generated hyperbolic group Γ can be quasi-isometrically embedded into a product of n binary trees where $\dim \partial\Gamma = n$ and that this result is optimal.

Combining this result with work of Jacek Świątkowski [34] saying that $\operatorname{asdim} \Gamma \geq \dim \partial\Gamma + 1$ for finitely generated hyperbolic groups, one sees that $\operatorname{asdim} \Gamma = \dim \partial\Gamma + 1$. Notice, too, that this is not the case for hyperbolic spaces. The Comb Space C of Panos Papasoglu and Thanos Gentimis [29] has $\dim \partial C = 0$, whereas $\operatorname{asdim} C = 2$. This example can be easily modified to give arbitrarily high asymptotic dimension without changing $\dim \partial C$.

3. Hurewicz-type theorem for asymptotic dimension

In this section we state (without proof) a Hurewicz theorem for asymptotic dimension. This theorem allows us to compute asymptotic dimension in a myriad of situations including direct products, free products of groups, and group extensions.

3.1. Groups acting on metric spaces.

Before stating the asymptotic version of the Hurewicz theorem, we state with proof an easier special case of the theorem for groups acting on finite dimensional metric spaces. This gives the flavor of the proof of the Hurewicz theorem. In order to state the result, we need the idea of an R-stabilizer. It is a metric subspace of the group Γ, not a subgroup.

Let Γ act on the metric space X by isometries. Let $R > 0$ be given. Let $x_0 \in X$. Define the R-stabilizer of x_0 by $W_R(x_0) = \{\gamma \in \Gamma \mid d(\gamma.x_0, x_0) \leq R\}$.

Theorem 3.1.1. *Assume that the finitely generated group* Γ *acts by isometries on the metric space* X *with* $x_0 \in X$ *and* $\operatorname{asdim} X \leq k$. *Suppose further that* $\operatorname{asdim} W_R(x_0) \leq n$ *for all* R. *Then* $\operatorname{asdim} \Gamma < \infty$.

The coarseness of the upper bound is a consequence of the use of covers. As mentioned following the definition of asymptotic dimension, our methods of proving a tight upper bound require the use of maps to uniform polyhedra.

Proof: We will show that $\operatorname{asdim} \Gamma \leq (n+1)(k+1) - 1$.

We define a map $\pi : \Gamma \to X$ by the formula $\pi(g) = g(x_0)$. Then $W_R(x_0) = \pi^{-1}(B_R(x_0))$. Let $\lambda = \max\{d_X(s(x_0), x_0) \mid s \in S\}$. We show now that π is λ-Lipschitz. Since the metric d_S on Γ is induced from the geodesic metric on the Cayley graph, it suffices to check that $d_X(\pi(g), \pi(g')) \leq \lambda$ for all $g, g' \in \Gamma$ with $d_S(g, g') = 1$. Without loss of generality, we assume that $g' = gs$ where $s \in S$. Then $d_X(\pi(g), \pi(g')) = d_X(g(x_0), gs(x_0)) = d_X(x_0, s(x_0)) \leq \lambda$.

Note that, for all $\gamma \in \Gamma$, $x \in X$, and all R, $\gamma B_R(x) = B_R(\gamma(x))$ and $\gamma(\pi^{-1}(B_R(x))) = \pi^{-1}(B_R(\gamma(x)))$.

Given $r > 0$, there are λr-disjoint, R-bounded families $\mathcal{F}^0, \ldots, \mathcal{F}^k$ on the orbit Γx_0. Let $\mathcal{V}^0, \ldots, \mathcal{V}^n$ on $W_{2R}(x_0)$ be r-disjoint uniformly bounded families given by the definition of the inequality

$\operatorname{asdim} W_R(x_0) \leq n$. For every element $F \in \mathcal{F}^i$, we choose an element $g_F \in \Gamma$ such that $g_F(x_0) \in F$. We define $(k+1)(n+1)$ families of subsets of Γ as

$$\mathcal{W}^{ij} = \{g_F(C) \cap \pi^{-1}(F) \mid F \in \mathcal{F}^i, C \in \mathcal{V}^j\}.$$

Since multiplication by g_F from the left is an isometry, every two distinct sets $g_F(C)$ and $g_F(C')$ are r-disjoint. Note that $\pi(g_F(C) \cap \pi^{-1}(F))$ and $\pi(g_{F'}(C) \cap \pi^{-1}(F'))$ are λr-disjoint for $F \neq F'$. Since π is λ-Lipschitz, the sets $g_F(C) \cap \pi^{-1}(F)$ and $g_{F'}(C') \cap \pi^{-1}(F')$ are r-disjoint. The families $\mathcal{W}^{ij}$ are uniformly bounded since the families $\mathcal{V}^j$ are, and multiplication by g from the left is an isometry on Γ. We check that the union of the families $\mathcal{W}^{ij}$ forms a cover of Γ. Let $g \in \Gamma$ and let $\pi(g) = F$, i.e., $g(x_0) \in F$. Since $\operatorname{diam} F \leq R$, $x_0 \in g_F^{-1}(F) \leq R$, and g_F^{-1} acts as an isometry, we have $g_F^{-1}(F) \subset B_R(x_0)$. Therefore, $g_F^{-1}g(x_0) \in B_R(x_0)$, i.e., $g_F^{-1}g \in W_R(x_0)$. Hence, $g_F^{-1}g$ lies in some set $C \in \mathcal{V}^j$ for some j. Therefore, $g \in g_F(C)$. Thus, $g \in g_F(C) \cap \pi^{-1}(F)$. □

3.2. Hurewicz-type theorem and applications.

In general, it is not possible to say anything about the asymptotic dimension of a surjective image based on the asymptotic dimension of the domain space. Indeed, a countable set can be metrized as $\mathbb{Z}^n$ for any n and so can be made to have arbitrary (even infinite) asymptotic dimension. On the other hand, when we have a map from a space X and we know the asymptotic dimension of the codomain, we can often estimate $\operatorname{asdim} X$. This is the situation of the following theorem.

Theorem 3.2.1 (Hurewicz-type theorem)**.** *Let $f : X \to Y$ be a Lipschitz map from a geodesic metric space to a metric space. Suppose that for every $R < \infty$ the set* $\operatorname{asdim} f^{-1}(B_R(y)) \leq n$ *uniformly (in $y \in Y$). Then* $\operatorname{asdim} X \leq n + \operatorname{asdim} Y$.

The proof of this theorem uses the definition of asymptotic dimension in terms of Lipschitz maps to uniform complexes. A complete proof can be found in [6]. Another simpler version of this result is the main theorem in [5] involving the case where Y is a tree.

Recently, N. Brodskiy, J. Dydak, M. Levin, and A. Mitra [11] proved a stronger result using covers. In their version of the theorem, they do not require the space to be geodesic, and they are able to replace the requirement that the map be Lipschitz, with the requirement that the map be large-scale uniform.

The Hurewicz-type theorem allows us to estimate the asymptotic dimension of a direct product of metric spaces.

Corollary 3.2.2. *Let X and Y be metric spaces. Then* $\operatorname{asdim} X \times Y \leq \operatorname{asdim} X + \operatorname{asdim} Y$.

Although it is not difficult to prove that the product of two spaces with finite asymptotic dimension has finite asymptotic dimension, getting a sharp upper bound from definitions involving covers is difficult; compare trying to pass from a cover of $\mathbb{R}$ to one for $\mathbb{R}^2$. However, with a little work, we can prove this sharp upper bound for the asymptotic dimension of a product from the definition of asymptotic dimension involving uniformly co-bounded Lipschitz maps to uniform polyhedra. Instead, we apply the Hurewicz-type theorem.

Proof: The map $f : X \times Y \to Y$ given by $f(x, y) = y$ is clearly Lipschitz since $d(x \times y, x' \times y') = \left(d_X(x, x')^2 + d_Y(y, y')^2\right)^{\frac{1}{2}} \leq d_X(x, x')$. It remains only to show that $\operatorname{asdim} f^{-1}(B_R(y)) \leq \operatorname{asdim} X$ uniformly for some n. To see this, for any $r < \infty$, take $n + 1$ families of r-disjoint, B-bounded sets $\mathcal{U}^0, \ldots, \mathcal{U}^n$ whose union covers X, and define $\mathcal{V}_y^i = \{U \times B_R(y) \mid U \in \mathcal{V}^i\}$. For each $y \in Y$, the $\mathcal{V}_y^i$ are r-disjoint and $\sqrt{2}\max\{B, R\}$-bounded. Obviously, the union of the $\mathcal{V}_y^i$ forms a cover of $f^{-1}(B_R(y))$ for each y. Thus, the family satisfies the inequality $\operatorname{asdim} f^{-1}(B_R(y)) \leq \operatorname{asdim} X$ uniformly. By the Hurewicz theorem, $\operatorname{asdim}(X \times Y) \leq \operatorname{asdim} X + \operatorname{asdim} Y$. $\square$

As another corollary of the Hurewicz-type theorem, we arrive at the case of interest to geometric group theorists: a finitely generated group acting by isometries on a metric space.

Corollary 3.2.3. *Let Γ be a finitely generated group acting by isometries on a metric space X. Fix some x_0 in X and suppose that* $\operatorname{asdim} W_R(x_0) \leq k$ *for all R. Then* $\operatorname{asdim} \Gamma \leq k + \operatorname{asdim} X$.

Proof: Fix a symmetric generating set S for Γ. Let $\lambda = \max\{d_X(s.x_0, x_0) \mid s \in S\}$. Define $\pi : \Gamma \to X$ by $\pi(\gamma) = \gamma.x_0$.

We claim that π is λ-Lipschitz and that $\operatorname{asdim}\pi^{-1}(B_R(x)) \le k$ uniformly in $x \in \Gamma.x_0$.

Since Γ is a finitely generated group with the word metric, it is a discrete geodesic space, so it suffices to check the Lipschitz condition on pairs at distance 1 from each other. It is easy to see that such a pair must be of the form $(\gamma, \gamma s)$, where $s \in S$. We compute

$$d_X(\pi(\gamma), \pi(\gamma s)) = d_X(\gamma.x_0, \gamma s.x_0) = d_X(x_0, s.x_0) \le \lambda,$$

so π is λ-Lipschitz. Finally, it is easy to check that $\pi^{-1}(B_R(g.x_0)) = gW_R(x_0)$. Since left multiplication is an isometry, the sets $\pi^{-1}(B_R(g.x_0))$ are all isometric to $W_R(x_0)$, so $\operatorname{asdim}\pi^{-1}(B_R(g.x_0)) \le k$ uniformly. Finally, since $\operatorname{asdim}\Gamma.x_0 \le \operatorname{asdim} X$, we get the desired inequality. $\square$

Another application of the Hurewicz theorem is to group extensions, i.e., groups G arising in exact sequences of the form

$$1 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 1.$$

Corollary 3.2.4. *Let $f : G \to H$ be a surjective homomorphism of finitely generated groups with* $\ker f = K$. *Then* $\operatorname{asdim} G \le \operatorname{asdim} H + \operatorname{asdim} K$.

Proof: Let S be a symmetric generating set for G and take $\bar{S} = f(S)$ to be a generating set for H. If $x^{-1}y = s_{i_1} \cdots s_{i_k}$ is a shortest presentation in terms of generators, then $d(x, y) = k$ and $d_H(f(x), f(y)) = \|f(x^{-1}y)\| = \|\bar{s}_{i_1} \cdots \bar{s}_{i_k}\| \le k$. Thus, f is 1-Lipschitz.

Next, we claim that $W_R(e) = N_R(K)$. Since $N_R(K)$ is quasi-isometric to K, this will say that $\operatorname{asdim} W_R(e) = \operatorname{asdim} K$ for all R so that we may apply the Hurewicz theorem to get the desired result.

First, suppose that $g \in W_R(e)$. Then $\|f(g)\| \le R$. Thus, $f(g) = \bar{s}_{i_1} \cdots \bar{s}_{i_k}$ with $k \le R$. If $u = s_{i_1} \cdots s_{i_k}$, then $gu^{-1} \in K$ and $d_G(g, gu^{-1}) = k \le R$. On the other hand, if $d_G(x, K) \le R$, then, since f is 1-Lipschitz, $d_H(f(x), e) \le R$. $\square$

Corollary 3.2.5. *Let Γ be a finitely generated polycyclic group with Hirsch length $h(\Gamma) = n$. Then* $\operatorname{asdim}\Gamma \le n$.

Proof: Since Γ is polycyclic, there exists a chain

$$1 = \Gamma_0 \lhd \Gamma_1 \lhd \cdots \lhd \Gamma_n = \Gamma$$

where each Γ_{i+1}/Γ_i is cyclic. The Hirsch length is $h(\Gamma) = \sum \mathrm{rk}(\Gamma_{i+1}/\Gamma_i)$. Applying the extension theorem, we see that $\operatorname{asdim}\Gamma \leq h(\Gamma)$. □

Since every finitely generated nilpotent group is polycyclic, we immediately obtain the following result.

Corollary 3.2.6. *Let Γ be a finitely generated nilpotent group. Then* $\operatorname{asdim}\Gamma \leq h(\Gamma)$.

Example 3.2.7. Let H denote the 3×3 integral Heisenberg group. Then $\operatorname{asdim} H \leq 3$.

Corollary 3.2.6 can be extended to nilpotent Lie groups N if we define the Hirsch length $h(N)$ as the sum of the number of factors in Γ_{i+1}/Γ_i isomorphic to $\mathbb{R}$ for the central series $\{\Gamma_i\}$ of N. We take an equivariant metric on N and on the quotients. Then the projection $\Gamma_{i+1} \to \Gamma_{i+1}/\Gamma_i$ is 1-Lipschitz and Γ_{i+1}/Γ_i is coarsely isomorphic to $\mathbb{R}^{n_i}$. Then we have the following corollary.

Corollary 3.2.8. *Let N be a nilpotent Lie group endowed with an equivariant metric. Then* $\operatorname{asdim} N \leq h(N)$.

Since $h(N) = \dim N$ for simply connected N, we obtain the following.

Corollary 3.2.9 ([14, Theorem 3.5])**.** *For a simply connected nilpotent Lie group N endowed with an equivariant metric,* $\operatorname{asdim} N \leq \dim N$.

Actually, in view of [24, Corollary 1.F1], the inequalities in Corollary 3.2.8 and Corollary 3.2.9 are equalities.

Corollary 3.2.9 is the main step in the proof of the following theorem.

Theorem 3.2.10 ([14])**.** *For a connected Lie group G and its maximal compact subgroup K, there is a formula* $\operatorname{asdim} G/K = \dim G/K$ *where G/K is endowed with a G-invariant metric.*

3.3. Asymptotic dimension of hyperbolic space.

This last theorem in particular allows us to show that the asymptotic dimension of the hyperbolic space $\mathbb{H}^n$ is n.

Corollary 3.3.1. $\operatorname{asdim}\mathbb{H}^n = n$.

Proof: Take $G = O(n, 1)_+$ and $K = O(n)$. □

This computation can be generalized in the spirit of [30].

Let (X, d) be a metric space. By $\mathcal{H}(X)$, we denote the space of balls in X endowed with the following metric

$$\rho(B_t(x), B_s(y)) = 2\ln\left(\frac{d(x,y) + \max\{t,s\}}{\sqrt{ts}}\right).$$

We note that $\mathcal{H}(\mathbb{R}^n)$ is coarsely equivalent to $\mathbb{H}^{n+1}$ [30, Example 2.60].

We recall that a metric space X with $\operatorname{asdim} X \leq n$ is said to satisfy the *Higson property* [23] if there exists $C > 0$ such that for every $D > 0$, there exists a cover $\mathcal{U}$ of X with $\operatorname{mesh}(\mathcal{U}) < CD$ and such that $\mathcal{U} = \mathcal{U}^0 \cup \cdots \cup \mathcal{U}^n$, where $\mathcal{U}^0, \ldots, \mathcal{U}^n$ are D-disjoint. In [30], spaces X satisfying this condition are said to have asymptotic dimension $\leq n$ *of linear type*. Note that this condition is equivalent to the asymptotic inequality $N{-}\dim X \leq n$ for the Assouad-Nagata dimension. It is shown in [23] that every metric space of bounded geometry with $\operatorname{asdim} X \leq n$ admits a coarsely equivalent metric with the Higson property. Unfortunately, the coarse type of $\mathcal{H}(X)$ depends on a metric on X not only the coarse class of metrics.

Theorem 3.3.2. *Suppose that the metric space* (X, d) *possesses the Higson property. Then* $\operatorname{asdim}\mathcal{H}(X) = \operatorname{asdim} X + 1$.

Proof: Consider the projection $\pi : \mathcal{H}(X) \to \mathbb{R}$ defined by $\pi(B_t(x)) = \ln t$ and apply the Hurewicz-type theorem to it (see [30, Corollary 9.21]). □

3.4. Groups acting on trees and Bass-Serre theory.

For the basics of Bass-Serre theory, the reader is referred to [32] or (for generalizations of Bass-Serre theory) to [10]. Recall that a tree T has $\operatorname{asdim} T \leq 1$.

The Bass-Serre theory allows us to consider various constructions with groups simultaneously. In particular, given two finitely generated groups A and B, we would like to be able to estimate the asymptotic dimension of their free product $A * B$, an amalgamated free product $A *_C B$, or an HNN-extension formed from one of these groups. The Bass-Serre theory tells us that these are all examples of groups which act co-compactly by isometries on trees.

Corollary 3.4.1. *Let* Γ *be a finitely generated group acting cocompactly by isometries on a tree* T*. Suppose that for all vertices* v, $\operatorname{asdim}\Gamma_v \leq n$*. Then* $\operatorname{asdim}\Gamma \leq n+1$.

Applying Corollary 3.2.3, we see that the only thing we need to show is that for all R, $\operatorname{asdim} W_R(x) \leq n$. This is not obvious. For the proof, the reader is referred to [5, Lemma 3]. Instead, we offer a simpler case, that of the free product $A * B$.

Corollary 3.4.2. *Let* A *and* B *be finitely generated groups with* $\operatorname{asdim} A \leq n$ *and* $\operatorname{asdim} B \leq n$*. Then* $\operatorname{asdim} A * B \leq n+1$.

Proof: The free product $A * B$ is the fundamental group of the graph of groups with two vertices, labeled A and B, and one edge, labeled $\{e\}$. This group acts on a tree by isometries so that the quotient consists of two vertices and one edge. The vertices of the tree consist of formal cosets of either A or B in the group. The vertices xA and yB are connected by an edge in the tree when there is a $z \in A * B$ such that $zA = xA$ and $zB = xB$.

It is not difficult to show in this case that $W_R(eA)$ consists of alternating products of the form $AB \cdots ABA$ of length $R+1$ when R is even and $AB \cdots BAB$ when R is odd.

To see that the asymptotic dimension of such products does not exceed n, one applies the union theorem for asymptotic dimension and induction.

In the case $R = 0$, we have $W_0(eA) = A$, and $\operatorname{asdim} A \leq n$ by assumption. For the case $R > 0$, we assume R is even (if it is odd, the proof is essentially the same). Write $AB \cdots ABA$ as $\cup_{x \in AB \cdots AB} xA$. By assumption, $\operatorname{asdim} AB \cdots AB \leq n$, and since xA is isometric to A, we know that $\operatorname{asdim} xA \leq n$ uniformly.

To apply the union theorem, it remains only to find Y_r so that $\operatorname{asdim} Y_r \leq n$ and so that the sets $xA - Y_r$ and $x'A - Y_r$ are r-disjoint when $xA \neq xA'$. To this end, set $Y_r = AB \cdots ABB_r(e)$ where $B_r(e)$ denotes the ball of radius R around e taken in A. This is coarsely isometric to $AB \cdots AB$, which, by the inductive hypothesis, has asymptotic dimension not exceeding n. Finally, if xa and $x'a'$ are in distinct $xA - Y_r$ and $x'A - Y_r$, then $d(xa, x'a') = \|a^{-1}x^{-1}x'a'\| \geq \|a\| + \|a'\| \geq r$. □

Using the asymptotic inductive dimension, in [7], the authors and Keesling were able to give an exact formula for the asymptotic dimension of such a free product: $\operatorname{asdim} A * B = \max\{n, 1\}$. An exact formula in the case of amalgamated free products still does not exist.

Applying the Bass-Serre theory, one can prove this result for the more general situation of graphs of groups; see [5]. This situation includes amalgamated free products and HNN extensions as special cases.

Theorem 3.4.3. *Let π be the fundamental group of a finite graph of groups where all vertex groups satisfy* $\operatorname{asdim} \Gamma_v \leq n$. *Then* $\operatorname{asdim} \pi \leq n + 1$.

Also, one can extend this result to complexes of groups in a natural way; see [3].

Corollary 3.4.4. *Let Γ be a finitely generated group with one defining relator. Then* $\operatorname{asdim} \Gamma < \infty$.

Proof: A result of D. I. Moldavanskii from 1967 states that a finitely generated one-relator group is an HNN extension of a finitely presented group with shorter defining relator or is cyclic; see [27]. If A is a finitely generated group with $\operatorname{asdim} A = n$ and $A*_C$ is an HNN extension of A, then since $A*_C$ is the fundamental group of a loop of groups, the Hurewicz-type theorem tells us that $\operatorname{asdim} A*_C \leq n + 1$. Iterating this procedure finitely many times gives the desired result. □

The class of limit groups consists of those groups which naturally arise in the study of solutions to equations in finitely generated groups. One definition is the following: A finitely presented group L is a *limit group* if for each finite subset $L_0 \subset L$, there is a homomorphism to a free group which is injective on L_0. For more information, the reader is referred to [8] and the references therein.

The next result does not (to our knowledge) appear in the literature. It was pointed out to the first author by Bestvina to be an easy consequence of the Hurewicz-type theorem and deep results in the theory of limit groups.

Proposition 3.4.5. *Let L be a limit group. Then* $\operatorname{asdim} L < \infty$.

Proof: There are two ways to see this. The first way is to construct L (say with height h) via fundamental groups of graphs of groups where vertices have height $(h-1)$ and height 0 groups are free groups, free abelian groups, and surface groups.

Another is to combine D. Osin's theorem [28] that groups that are hyperbolic relative to a collection of subgroups with finite asymptotic dimension themselves have finite asymptotic dimension with a result of François Dahmani [15] stating that limit groups are relatively hyperbolic with respect to their maximal abelian non-cyclic subgroups. □

4. Motivation

Although asymptotic dimension was introduced in 1993, it did not garner much attention until a paper of Yu in 1998 [35] in which he proved that the Novikov higher signature conjecture holds for groups with finite asymptotic dimension. Below we list some related results.

Theorem 4.1 ([35]). *Let Γ be a finitely generated group with finite $B\Gamma$ and* $\operatorname{asdim}\Gamma < \infty$. *Then the Novikov Conjecture holds for Γ.*

Later, Yu [36] would generalize this theorem to the following.

Theorem 4.2 ([36]). *Let Γ be a discrete metric space with bounded geometry admitting a uniform embedding into Hilbert space. Then the coarse Baum-Connes conjecture holds for Γ.*

An easy way to verify that a discrete metric space admits a uniform embedding into Hilbert space is to verify that it has *Property A* [36]. A discrete metric space Z has property A if there exist maps $\{a_n\}_{n\in\mathbb{N}}$, $a^n : Z \to \operatorname{Prob}(Z)$ such that the following two conditions hold:

(1) for every n, there is an R so that for every $z \in Z$, $\operatorname{supp}(a^n_z) \subset B_R(z)$; and
(2) for every $K > 0$,

$$\lim_{n\to\infty} \sup_{d(z,w)<K} \|a^n_z - a^n_w\|_1 = 0.$$

Here, by way of notation, $a^n_z(x) = (a^n(z))(x)$.

Before moving on to groups with finite asymptotic dimension, we give an example.

Example 4.3. *Let T be a tree, then T has property A.*

Proof: Fix some geodesic ray γ in T. It is easy to see that for any point $x \in T$, there is exactly one geodesic ray from x whose intersection with γ is also a geodesic ray. Let γ_x be this unique ray for x. We abuse notation by writing $\gamma_x : \mathbb{N} \to T$ for the function yielding γ_x. For each n, define $a_x^n = \frac{1}{n+1}\sum_{i=0}^{n} \gamma_x(i)\delta_{\gamma_x(i)}$.

Clearly the support of a_x^n is contained in the n neighborhood of x and it is easy to check that for any K, $\|a_z^n - a_w^n\| \leq \frac{n-(n-k)}{n+1}$ which goes to 0 as $n \to \infty$. □

To see that finitely generated groups with finite asymptotic dimension admit a uniform embedding into Hilbert space, we check that they have property A. This result and its proof are due to Higson and Roe [25]; see also Theorem 5.3.

Theorem 4.4. *Let Γ be a finitely generated group with finite asymptotic dimension. Then Γ has property A.*

Proof: Suppose $\operatorname{asdim}\Gamma = n$ and $R < \infty$ is given. Let $\mathcal{U}$ be a cover of Γ by uniformly bounded sets with multiplicity $\leq n+1$ and Lebesgue number $> R$. There is a partition of unity $\{\phi\}$ subordinate to this cover which has the following properties:

(1) each ϕ is Lipschitz with Lipschitz constant $< 2/R$;
(2) $\sup_\phi(\operatorname{diam}(\operatorname{supp}\phi)) < \infty$; and
(3) for any γ, at most $n+1$ of the $\phi(\gamma) \neq 0$.

Now, for any $\gamma \in \Gamma$, take γ_ϕ to be a non-zero value of ϕ. Then put $a_\gamma^R = \sum_\phi \phi(\gamma)\delta_{\gamma_\phi}$.

Each a_γ^R is in $\operatorname{Prob}(\Gamma)$; condition (2) implies that for each R, there is a D so that $\operatorname{supp}(a_\gamma^R) \subset B_D(\gamma)$; and conditions (1) and (3) say that for all K,

$$\lim_{R\to\infty} \sup_{d(\gamma,\gamma')<K} \|a_\gamma^R - a_{\gamma'}^R\|_1 = 0$$ □

Other implications of finite asymptotic dimension are the following theorems.

Theorem 4.5 ([17])**.** *If Γ is the fundamental group of an aspherical manifold M and* $\operatorname{asdim}\Gamma < \infty$*, then the universal cover of M is hyperspherical.*

Theorem 4.6 ([20]). *If Γ is a finitely generated group with finite $B\Gamma$ and* $\operatorname{asdim}\Gamma < \infty$*, then the integral Novikov conjecture holds for* Γ.

Theorem 4.7 ([2], [14]). *If Γ is a finitely generated group with finite $B\Gamma$ and* $\operatorname{asdim}\Gamma < \infty$*, then the integral K-theoretic Novikov conjecture holds for* Γ.

5. Dimension growth

Using the fact that $\operatorname{asdim}\mathbb{Z}^n = n$, it is not difficult to construct an example of a finitely generated group with infinite asymptotic dimension. In fact, the reduced wreath product of $\mathbb{Z}$ by $\mathbb{Z}$ (denoted $\mathbb{Z}\wr\mathbb{Z}$) is finitely generated and contains a copy of $\mathbb{Z}^n$ for every n (see [30] for details). Therefore, $\operatorname{asdim}\mathbb{Z}\wr\mathbb{Z} = \infty$.

Note that this result implies that even the finiteness of asymptotic dimension is not preserved under quotients, as this group is a quotient of $\mathbb{F}_2$ and $\operatorname{asdim}\mathbb{F}_2 = 1$.

The invariant we associate to groups with infinite asymptotic dimension is the *dimension growth* of the space. We define the dimension function of the metric space X as

$$d_X(\lambda) = \min\{m(\mathcal{U}) - 1 \mid L(\mathcal{U}) \geq \lambda, \sup_{U\in\mathcal{U}} \operatorname{diam}(U) < \infty, \mathcal{U} \text{ covers } X\},$$

where $m(\mathcal{U})$ denotes the multiplicity of the cover $\mathcal{U}$ and $L(\mathcal{U})$ denotes a Lebesgue number for the cover.

Observe that $\lim_{\lambda\to\infty} d_X(\lambda) = \operatorname{asdim} X$ and that the function is monotonic.

Obviously, changing the metric on X could drastically alter the function d_X; however, the growth of the function d_X is an invariant of quasi-isometry.

Proposition 5.1 ([4]). *Let X and Y be discrete metric spaces with bounded geometry. Suppose that X and Y are quasi-isometric. Then there is some positive k so that $d_X(\lambda) \leq k d_Y(k\lambda + k) + k$. In particular, the growth rate of d_Γ is well defined for finitely generated group Γ.*

The proof is technical so it is omitted.

It is easy to see that the dimension function for a group cannot grow arbitrarily fast.

Proposition 5.2 ([19])**.** *For a finitely generated group* Γ*, we have* $d_\Gamma(\lambda) \le e^{\alpha\lambda}$.

Proof: There is some a so that $\mathrm{vol}(B_\lambda(x)) < e^{a\lambda}$. Take $\mathcal{U}_\lambda = \{B_\lambda(x) \mid x \in \Gamma\}$. Then $L(\mathcal{U}_\lambda) = \lambda$ and $m(U_\lambda) \le e^{a\lambda}$. □

Dranishnikov [16] was able to generalize a theorem of [25] to spaces with bounded geometry where the growth of the dimension function is linear (see Theorem 4.4). The best result along these lines is the following theorem.

Theorem 5.3 ([19])**.** *Let* Γ *be a finitely generated group with* $d_\Gamma(\lambda) \le \lambda^m$*. Then* Γ *has property A; in particular, the Novikov higher signature conjecture holds for* Γ.

We saw at the beginning of this section that $\mathbb{Z} \wr \mathbb{Z}$ has infinite asymptotic dimension. The next result tells us about the growth of its dimension function. In particular, this gives an example of a group with infinite asymptotic dimension whose dimension function grows at most polynomially.

Proposition 5.4. *Let* N *be a finitely generated nilpotent group. Suppose that* $\operatorname{asdim} G < \infty$ *where* G *is a finitely generated group. Then* $d_{G \wr N}(\lambda) \le \lambda^n$ *for some* n*. Here, as before,* $G \wr N$ *denotes the reduced wreath product.*

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