TOPOLOGY PROCEEDINGS
Volume 38, 2011
Pages 237–251
http://topology.auburn.edu/tp/

Preserving the Lindelöf Property Under Forcing Extensions

by

Masaru Kada

Electronically published on October 1, 2010

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 38 (2011)
Pages 237-251

E-Published on October 1, 2010

PRESERVING THE LINDELÖF PROPERTY UNDER FORCING EXTENSIONS

MASARU KADA

Abstract. We investigate preservation of the Lindelöf property of topological spaces under forcing extensions. We give sufficient conditions for a forcing notion to preserve several strengthenings of the Lindelöf property, such as indestructible Lindelöf property, the Rothberger property, and being a Lindelöf P-space.

1. Introduction

One of several basic open problems about Lindelöf spaces asks what are possible cardinalities are for Lindelöf spaces in which each point is G_δ. A number of consistency results using forcing or large cardinal axioms have been obtained. A fundamental issue that emerged from that work is the question,

> *When does a forcing extension preserve the Lindelöf property?*

Surprisingly, little seems to be known about this question. Franklin D. Tall [19] introduced the notion of indestructibly Lindelöf spaces. A Lindelöf space is called an indestructibly Lindelöf space if it is still Lindelöf after forcing with any countably closed poset. Tall pointed out that a Lindelöf space is indestructibly Lindelöf if it

2010 *Mathematics Subject Classification.* Primary 54D20; Secondary 03E40, 54G10.

Key words and phrases. forcing, indestructibly Lindelöf space, infinite game, P-space, Rothberger property.

Supported by Grant-in-Aid for Young Scientists (B) 21740080, MEXT.

remains Lindelöf after forcing with the poset which adjoins a Cohen subset of ω_1 with countable conditions, which is just a particular instance of a countably closed poset.

The class of spaces with the Rothberger property is a natural and important subclass of the class of indestructibly Lindelöf spaces. It is of great interest to know which forcing notions preserve the Rothberger property. Marion Scheepers and Tall [18] showed that forcing with countably closed posets as well as the measure algebra preserve the Rothberger property.

Both the indestructible Lindelöf property and the Rothberger property are nicely characterized in terms of infinite games played on topological spaces. Janusz Pawlikowski [16] proved the Rothberger property is equivalent to the non-existence of a winning strategy for the first player in a certain game played on the space. Scheepers and Tall [18] proved that an indestructibly Lindelöf space is characterized as a space on which the first player has no winning strategy in a modification into transfinite length of the game which appears in Pawlikowski's theorem. Moreover, the existence of a winning strategy for the second player in the game for indestructibly Lindelöf spaces also determines a noteworthy class of spaces, for it is known that if a space in which each point is G_δ belongs to this class, then its cardinality is at most $2^{\aleph_0}$ [18, Theorem 2].

On the other hand, infinite games played on posets have been studied by many researchers, mainly in connection with Boolean-algebraic or forcing-theoretic properties. One of the most significant results of those studies is a game-theoretic characterization of proper forcing notions. Also, the relations between game-theoretic properties and various properties of forcing notions, such as countable closedness, semiproperness, α-properness, Axiom A, the Sacks property and the Laver property, have been studied by Matthew Foreman [4], Thomas J. Jech [10], Jech and Saharon Shelah [12], Boban Veličković [21], Jindřich Zapletal [22], Tetsuya Ishiu [8], Masaru Kada [13], and others.

In the present paper, we show that an indestructibly Lindelöf space remains Lindelöf after forcing with a poset in a natural class which is described using a game and larger than the class of countably closed posets. Also, we show that the Rothberger property is preserved under forcing with a poset in another natural class, which is again described using a game.

We also investigate preservation of being a Lindelöf space in the class of P-spaces. Forcing with a proper poset preserves being a P-space. We will show that a Lindelöf P-space remains Lindelöf after forcing with a poset in a large class of proper posets. It is an intriguing question if there is an example of a Lindelöf P-space which is no longer a Lindelöf space after forcing with some proper poset (see Question 6.4).

In section 4, we establish a general preservation theorem stated in terms of games, which yields all the preservation results mentioned above. We will prove this theorem by pursuing moves in two games played in parallel, one played on a topological space and the other on a poset.

The general investigation of preservation of the Lindelöf property and its strengthenings under a forcing extension has internal appeal, but the results may be useful in obtaining consistency results about a number of other basic open problems about Lindelöf spaces.

2. Preliminaries

For a poset $\mathbb{P}$, an ordinal α, and a cardinal κ, the *cut-and-choose game* $CG^{<\alpha}(<\kappa)$ *on* $\mathbb{P}$ is defined as follows. The game is played by two players, One and Two, for α innings. In the beginning One chooses $p \in \mathbb{P}$. In each inning $\beta < \alpha$, One chooses a $\mathbb{P}$-name $\dot{\eta}_\beta$ for an ordinal, and then Two chooses a set C_β of ordinals with $|C_\beta| < \kappa$. Two wins in this game if, for every $\gamma < \alpha$, there is $q_\gamma \in \mathbb{P}$ such that $q_\gamma \leq p$ and $q_\gamma \Vdash_{\mathbb{P}}$ "$\forall \beta < \gamma\, (\dot{\eta}_\beta \in \check{C}_\beta)$." Note that for Two to win, it is not required to find $q \in \mathbb{P}$ such that $q \leq p$ and $q \Vdash_{\mathbb{P}}$ "$\forall \beta < \alpha\, (\dot{\eta}_\beta \in \check{C}_\beta)$." Sometimes we preliminarily fix One's beginning move $p \in \mathbb{P}$ and then start the innings; in such a case, we call it the *game* $CG^{<\alpha}(<\kappa)$ *on* $\mathbb{P}$ *below* p. If $\alpha = \gamma + 1$, we write $CG^{\gamma}(<\kappa)$ instead of $CG^{<\gamma+1}(<\kappa)$. Also, we write $CG^{<\alpha}(\lambda)$ instead of $CG^{<\alpha}(<\lambda^+)$.

The following theorem is well known.

Theorem 2.1 ([10]). *For a forcing notion* $\mathbb{P}$*, if* Two *has a winning strategy in* $CG^{\omega}(\aleph_0)$ *on* $\mathbb{P}$*, then* $\mathbb{P}$ *is proper.*

We say a forcing notion $\mathbb{P}$ is ω^ω*-bounding* if for $p \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{f}$ for an element of ω^ω, there are $q \in \mathbb{P}$ and $g \in \omega^\omega$ such that $q \leq p$ and $q \Vdash_{\mathbb{P}}$ "$\forall n < \omega\, (\dot{f}(n) \leq g(n))$." The following fact is easily checked.

Theorem 2.2. *For a forcing notion $\mathbb{P}$, if* Two *has a winning strategy in $CG^{\omega}(<\aleph_0)$ on $\mathbb{P}$, then $\mathbb{P}$ is ω^{ω}-bounding.*

Remark 2.3. Although the converse of Theorem 2.2 does not hold, most well-known proper ω^{ω}-bounding forcing notions, such as Sacks forcing, Silver forcing, and the measure algebra, are the ones on which Two has a winning strategy in $CG^{\omega}(<\aleph_0)$.

A poset $\mathbb{P}$ is *$<\alpha$-closed* if any descending sequence in $\mathbb{P}$ of length less than α has a lower bound in $\mathbb{P}$. If $\mathbb{P}$ is $<\alpha$-closed, then obviously Two has a winning strategy in $CG^{<\alpha}(1)$ on $\mathbb{P}$.

Remark 2.4. The game $CG^{<\alpha}(1)$ on a poset $\mathbb{P}$ is closely related to the strategic closure of $\mathbb{P}$. A poset $\mathbb{P}$ is *$<\alpha$-strategically closed* if the second player has a winning strategy in a *descending chain game* on $\mathbb{P}$ of length α, which is a generalization of a usual Banach–Mazur game into transfinite length, but the second player has the initiative in each limit inning (see [4] or [9] for a precise definition). For an ordinal α which is either a limit or the successor of a limit, $\mathbb{P}$ is $<\alpha$-strategically closed if and only if Two has a winning strategy in $CG^{<\alpha}(1)$ on $\mathbb{P}$ (it was proved in the case $\alpha = \omega + 1$ by Jech [11] and Veličković [21], and in a general case by Ishiu [7]).

It is unprovable in ZFC that if Two has a winning strategy in $CG^{<\omega_1}(1)$ on $\mathbb{P}$, then $\mathbb{P}$ is $<\omega_1$-closed, for the following reason: It is known that $\mathbb{P}$ is $<\omega_1$-strategically closed if and only if $\mathbb{P}$ is $<(\omega+1)$-strategically closed (proved independently by Foreman [4] and Veličković [20]; see also [9]), and it is known to be unprovable in ZFC that if $\mathbb{P}$ is $<(\omega+1)$-strategically closed, then $\mathbb{P}$ is $<\omega_1$-closed (due to Jech and Shelah [12]).

Here we list properties of a forcing notion $\mathbb{P}$ which are relevant to the results in this paper. $\mathrm{Fn}(\omega_1, 2, \omega_1)$ denotes the poset which adjoins a Cohen subset of ω_1 with countable conditions [14]. The list is ordered *stronger to weaker*.

(1) $\mathbb{P} = \mathrm{Fn}(\omega_1, 2, \omega_1)$.
(2) $\mathbb{P}$ is $<\omega_1$-closed.
(3) Two has a winning strategy in $CG^{<\omega_1}(1)$ on $\mathbb{P}$.
(4) Two has a winning strategy in $CG^{<\omega_1}(<\aleph_0)$ on $\mathbb{P}$.
(5) Two has a winning strategy in $CG^{\omega}(<\aleph_0)$ on $\mathbb{P}$.
(6) Two has a winning strategy in $CG^{\omega}(\aleph_0)$ on $\mathbb{P}$.
(7) $\mathbb{P}$ is proper.

Now we turn to the games played on topological spaces.

For a topological space (X, τ) and ordinal α, the game $G_1^{<\alpha}(\mathcal{O}, \mathcal{O})$ on (X, τ) is played by two players, ONE and TWO, for α innings as follows. In the inning $\beta < \alpha$, ONE chooses an open cover $\mathcal{U}_\beta$ of X and then TWO chooses $H_\beta \in \mathcal{U}_\beta$. TWO wins in this game if there is $\gamma < \alpha$ such that $\{H_\beta : \beta < \gamma\}$ covers X. Note that TWO does not win if just $\{H_\beta : \beta < \alpha\}$ covers X. If $\alpha = \gamma + 1$, we write $G_1^{\gamma}(\mathcal{O}, \mathcal{O})$ instead of $G_1^{<\gamma+1}(\mathcal{O}, \mathcal{O})$.

We say a space (X, τ) has the *Rothberger property* if, for every sequence $\langle \mathcal{U}_n : n < \omega \rangle$ of open covers of X, there is an open cover $\{U_n : n < \omega\}$ of X such that $U_n \in \mathcal{U}_n$ for all $n < \omega$. It is easy to see that if ONE does not have a winning strategy in the game $G_1^{\omega}(\mathcal{O}, \mathcal{O})$ on (X, τ), then (X, τ) has the Rothberger property. The following theorem tells us that the converse also holds.

Theorem 2.5 (Pawlikowski [16]). *A space (X, τ) has the Rothberger property if and only if* ONE *does not have a winning strategy in the game $G_1^{\omega}(\mathcal{O}, \mathcal{O})$ on (X, τ).*

For a space (X, τ) and a forcing notion $\mathbb{P}$, we let $\tau^{\mathbb{P}}$ denote a $\mathbb{P}$-name representing the topology on X generated by τ in a generic extension by $\mathbb{P}$.

We say a forcing notion $\mathbb{P}$ *destroys* a Lindelöf space (X, τ) if we have

$$\Vdash_{\mathbb{P}} \text{“}(\check{X}, \tau^{\mathbb{P}}) \text{ is not Lindelöf.”}$$

A Lindelöf space (X, τ) is called an *indestructibly Lindelöf* space if (X, τ) is not destroyed by any $<\omega_1$-closed poset.

The equivalence (1) $\Leftrightarrow$ (2) in the following theorem is due to Scheepers and Tall [18, Theorem 1]. The equivalence (2) $\Leftrightarrow$ (3) is easily checked.

Theorem 2.6. *For a space (X, τ), the following are equivalent.*

(1) *(X, τ) is an indestructibly Lindelöf space.*
(2) *(X, τ) is a Lindelöf space and* ONE *does not have a winning strategy in $G_1^{\omega_1}(\mathcal{O}, \mathcal{O})$ on (X, τ).*
(3) ONE *does not have a winning strategy in $G_1^{<\omega_1}(\mathcal{O}, \mathcal{O})$ on (X, τ).*

As a consequence of Theorem 2.5 and Theorem 2.6, we see the following fact.

Corollary 2.7 ([18, Corollary 10])**.** *A space with the Rothberger property is an indestructibly Lindelöf space.*

We say X is a *P-space* if every G_δ-set in X is an open set. It is known that a Lindelöf P-space has the Rothberger property (due to Galvin; see the following remark).

Remark 2.8. An open cover $\mathcal{U}$ of a space X is an *ω-cover* if $X \notin \mathcal{U}$, and for every finite set $F \subseteq X$, there is a $U \in \mathcal{U}$ with $F \subseteq U$. An open cover $\mathcal{U}$ of X is a *γ-cover* if $\mathcal{U}$ is infinite and any infinite subset of $\mathcal{U}$ covers X. A space X is called a *γ-space* if, for every sequence $\langle \mathcal{U}_n : n < \omega \rangle$ of ω-covers of X, there is a γ-cover $\{U_n : n < \omega\}$ of X such that $U_n \in \mathcal{U}_n$ for all $n < \omega$. It is known that a γ-space has the Rothberger property. Fred Galvin proved that a Lindelöf P-space is a γ-space (see [5] and [18, Theorem 47]).

Here we list properties of a topological space $X = (X, \tau)$ which are relevant to the results in this paper. The list is ordered from *weaker to stronger*.

(1) X is a Lindelöf space.
(2) X is an indestructibly Lindelöf space (equivalently, One does not have a winning strategy in $G_1^{<\omega_1}(\mathcal{O}, \mathcal{O})$ on X).
(3) X has the Rothberger property (equivalently, One does not have a winning strategy in $G_1^{\omega}(\mathcal{O}, \mathcal{O})$ on X).
(4) X is a Lindelöf P-space.

Remark 2.9. Here we state facts about the topological property "Two has a winning strategy in $G_1^{<\omega_1}(\mathcal{O}, \mathcal{O})$ on X," which does not fit in the above list. Clearly, if Two has a winning strategy in $G_1^{<\omega_1}(\mathcal{O}, \mathcal{O})$ on X, then X is an indestructibly Lindelöf space. Peg Daniels and Gary Gruenhage [3] showed that if X is a hereditarily Lindelöf space, then Two has a winning strategy in $G_1^{<\omega_1}(\mathcal{O}, \mathcal{O})$ on X. The real line $\mathbb{R}$ is a hereditarily Lindelöf space and so Two has a winning strategy in $G_1^{<\omega_1}(\mathcal{O}, \mathcal{O})$ on $\mathbb{R}$, whereas $\mathbb{R}$ does not have the Rothberger property. On the other hand, using results due to Scheepers and Tall [18, Theorem 2 and Example 3], we can see that it is consistent with ZFC that there is a space with the Rothberger property on which Two does not have a winning strategy in $G_1^{<\omega_1}(\mathcal{O}, \mathcal{O})$.

3. Proper forcing preserves P-spaces

We prove that a P-space remains a P-space after forcing with a proper forcing notion.

Proposition 3.1. *Suppose that a space* (X, τ) *is a P-space and* $\mathbb{P}$ *is a proper forcing notion. Then* $\Vdash_{\mathbb{P}}$ *"*$(\check{X}, \tau^{\mathbb{P}})$ *is a P-space."*

Proof: Fix a countable set $\{\dot{G}_n : n < \omega\}$ of $\mathbb{P}$-names such that $\Vdash_{\mathbb{P}}$ "$\dot{G}_n \in \tau^{\mathbb{P}}$" for all $n < \omega$. For each n, take a $\mathbb{P}$-name $\dot{\mathcal{T}}_n$ such that

$$\Vdash_{\mathbb{P}} \text{"}\dot{\mathcal{T}}_n \subseteq \check{\tau} \text{ and } \dot{G}_n = \bigcup \dot{\mathcal{T}}_n.\text{"}$$

We are going to prove the following sentence.

$$\Vdash_{\mathbb{P}} \text{"}\forall x \in \check{X} \left[x \in \bigcap_{n<\omega} \dot{G}_n \to \exists H \in \check{\tau}\, (x \in H \text{ and } H \subseteq \bigcap_{n<\omega} \dot{G}_n) \right],\text{"}$$

which implies that $\Vdash_{\mathbb{P}}$ "$\bigcap_{n<\omega} \dot{G}_n \in \tau^{\mathbb{P}}$." It suffices to show that, for $x \in X$ and $p \in \mathbb{P}$, if $p \Vdash_{\mathbb{P}}$ "$\check{x} \in \bigcap_{n<\omega} \dot{G}_n$," then there are $q \leq p$ and $H \in \tau$ such that $x \in H$ and $q \Vdash_{\mathbb{P}}$ "$\check{H} \subseteq \bigcap_{n<\omega} \dot{G}_n$."

Fix $x \in X$ and $p \in \mathbb{P}$ and assume $p \Vdash$ "$\check{x} \in \bigcap_{n<\omega} \dot{G}_n$." For each $n < \omega$, since we have $p \Vdash_{\mathbb{P}}$ "$\check{x} \in \dot{G}_n$ and $\dot{G}_n = \bigcup \dot{\mathcal{T}}_n$," we can take a $\mathbb{P}$-name $\dot{T}_n$ such that $p \Vdash_{\mathbb{P}}$ "$\check{x} \in \dot{T}_n$ and $\dot{T}_n \in \dot{\mathcal{T}}_n$." Note that we have $p \Vdash_{\mathbb{P}}$ "$\dot{T}_n \in \check{\tau}$ and $\dot{T}_n \subseteq \dot{G}_n$."

By the properness of $\mathbb{P}$, we can choose $q \leq p$ and a countable set $\mathcal{C} \subseteq \tau$ so that $q \Vdash_{\mathbb{P}}$ "$\{\dot{T}_n : n < \omega\} \subseteq \check{\mathcal{C}}$." Note that $q \Vdash_{\mathbb{P}}$ "$\forall n < \omega\, (\check{x} \in \dot{T}_n)$." Let $H = \bigcap\{T \in \mathcal{C} : x \in T\}$. Then $x \in H$ and, since (X, τ) is a P-space, $H \in \tau$ holds. Now we have

$$q \Vdash_{\mathbb{P}} \text{"}\check{H} = \bigcap \{T \in \check{\mathcal{C}} : \check{x} \in T\} \subseteq \bigcap_{n<\omega} \dot{T}_n \subseteq \bigcap_{n<\omega} \dot{G}_n,\text{"}$$

which concludes the proof. □

4. The main result

In this section, we give a sufficient condition for a topological space (X, τ) and a forcing notion $\mathbb{P}$ to keep $(\check{X}, \tau^{\mathbb{P}})$ having a certain game-theoretic property in the forcing extension.

Definition 4.1. For a topological space $X = (X, \tau)$, define a cardinal $p(X)$ by letting

$$p(X) = \aleph_0 + \min\left(\{|\mathcal{G}| : \mathcal{G} \subseteq \tau \text{ and } \bigcap \mathcal{G} \notin \tau\} \cup \{|\tau|^+\}\right).$$

Note that X is a P-space if and only if $p(X) \geq \aleph_1$.

Theorem 4.2. *Let (X, τ) be a topological space, $\mathbb{P}$ a forcing notion, α an ordinal, and $\kappa = p(X)$. If*

(1) One *does not have a winning strategy in $G_1^{<\alpha}(\mathcal{O}, \mathcal{O})$ on (X, τ), and*
(2) Two *has a winning strategy in $CG^{<\alpha}(<\kappa)$ on $\mathbb{P}$,*

then

$\Vdash_{\mathbb{P}}$ "One *does not have a winning strategy in $G_1^{<\alpha}(\mathcal{O}, \mathcal{O})$ on $(\check{X}, \tau^{\mathbb{P}})$."*

Proof: Fix an enumeration of τ, say $\tau = \{T_\xi : \xi < \theta\}$ for some cardinal θ.

Suppose that $\dot{\sigma}$ is a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \text{“}\dot{\sigma} \text{ is a strategy for One in } G_1^{<\alpha}(\mathcal{O}, \mathcal{O}) \text{ on } (\check{X}, \tau^{\mathbb{P}})\text{.”}$$

Without loss of generality, we may assume that it is forced that the strategy $\dot{\sigma}$ suggests only open covers which consist of elements of τ, since τ is a base of $\tau^{\mathbb{P}}$ in a generic extension, and taking refinements will not help Two win easier. Under this assumption, a sequence of initial moves for Two, played in a generic extension, against the strategy $\dot{\sigma}$ will be described in a form $\langle \check{T}_{\dot{\xi}_\beta} : \beta < \delta \rangle$, where $\delta < \alpha$ and each $\dot{\xi}_\beta$ is a $\mathbb{P}$-name for an ordinal.

We will prove the following statement: For any $p \in \mathbb{P}$, there are $q \leq p$, a sequence $\langle \dot{\xi}_\beta : \beta < \alpha \rangle$ of $\mathbb{P}$-names for ordinals, and $\gamma < \alpha$ such that

$$q \Vdash_{\mathbb{P}} \text{“}\forall \delta < \gamma \left(\check{T}_{\dot{\xi}_\delta} \in \dot{\sigma}(\langle \check{T}_{\dot{\xi}_\beta} : \beta < \delta \rangle)\right) \text{ and } \bigcup \{\check{T}_{\dot{\xi}_\delta} : \delta < \gamma\} = \check{X}\text{.”}$$

This means that $\langle \check{T}_{\dot{\xi}_\beta} : \beta < \alpha \rangle$ describes winning moves for Two against the given strategy $\dot{\sigma}$ for One in a generic extension.

Fix $p \in \mathbb{P}$. By the assumption, Two has a winning strategy ρ in the game $CG^{<\alpha}(<\kappa)$ on $\mathbb{P}$ below p.

We will define a strategy Σ for One in the game $G_1^{<\alpha}(\mathcal{O}, \mathcal{O})$ on (X, τ), which cannot be a winning strategy by the assumption.

We construct Σ by induction on $\delta < \alpha$. As an additional induction hypothesis, we assume that, with each sequence $\langle H_\beta : \beta < \delta\rangle$ describing TWO's possible initial moves against Σ before the inning δ, a sequence $\langle \dot\xi_\beta : \beta < \delta\rangle$ of $\mathbb{P}$-names for ordinals is associated. We will define ONE's move $\Sigma(\langle H_\beta : \beta < \delta\rangle)$ in the inning δ and associate a $\mathbb{P}$-name $\dot\xi_\delta$ with TWO's response H_δ.

Let $\dot{\mathcal{U}}^\delta$ be a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}$ "$\dot{\mathcal{U}}^\delta = \dot\sigma(\langle \check{T}_{\dot\xi_\beta} : \beta < \delta\rangle)$." Since we have $\Vdash_{\mathbb{P}}$ "$\dot{\mathcal{U}}^\delta \subseteq \check\tau$ and $\bigcup \dot{\mathcal{U}}^\delta = \check{X}$," for each $x \in X$, we can take a $\mathbb{P}$-name $\dot\eta^\delta_x$ for an ordinal so that

$$\Vdash_{\mathbb{P}} \text{“}\check{x} \in \check{T}_{\dot\eta^\delta_x} \text{ and } \check{T}_{\dot\eta^\delta_x} \in \dot{\mathcal{U}}^\delta.\text{”}$$

For each $x \in X$, let $F_x = F\left(\langle \dot\xi_\beta : \beta < \delta\rangle, x\right) = \rho\left(\langle \dot\xi_\beta : \beta < \delta\rangle^\frown\langle \dot\eta^\delta_x\rangle\right)$, and $G_x = G\left(\langle \dot\xi_\beta : \beta < \delta\rangle, x\right) = \bigcap\{T_\xi : \xi \in F_x \text{ and } x \in T_\xi\}$. Note that since $|F_x| < \kappa = p(X)$, and, by the definition of $p(X)$, G_x is an open set containing x. Now let

$$\Sigma(\langle H_\beta : \beta < \delta\rangle) = \{G_x : x \in X\}.$$

Suppose that TWO picks H_δ from the cover $\{G_x : x \in X\}$ as a move in the inning δ. We pick $x_\delta \in X$ such that $H_\delta = G_{x_\delta}$ and let $\dot\xi_\delta = \dot\eta^\delta_{x_\delta}$. This completes the induction step at δ.

Since Σ is not a winning strategy, we can find a sequence $\langle H_\beta : \beta < \alpha\rangle$, which describes TWO's winning moves against Σ, and the associated sequence $\langle \dot\xi_\beta : \beta < \alpha\rangle$ of $\mathbb{P}$-names of ordinals. For each $\delta < \alpha$, let $F_\delta = \rho(\langle \dot\xi_\beta : \beta \le \delta\rangle)$. Find $\gamma < \alpha$ such that $\{H_\beta : \beta < \gamma\}$ covers X. Since ρ is a winning strategy for TWO in the game $CG^{<\alpha}(<\kappa)$ on $\mathbb{P}$ below p, we can find $q \le p$ such that $q \Vdash_{\mathbb{P}}$ "$\forall \delta < \gamma\,(\dot\xi_\delta \in \check{F}_\delta)$."

Fix $\delta < \gamma$. By the construction of the sequence $\langle \dot\xi_\beta : \beta < \alpha\rangle$, we have $H_\delta = G\left(\langle \dot\xi_\beta : \beta < \delta\rangle, x_\delta\right)$ and $\dot\xi_\delta = \dot\eta^\delta_{x_\delta}$ for a suitable $x_\delta \in X$. Note that $F_\delta = \rho(\langle \dot\xi_\beta : \beta \le \delta\rangle) = F\left(\langle \dot\xi_\beta : \beta < \delta\rangle, x_\delta\right)$ and so $H_\delta = \bigcap\{T_\xi : \xi \in F_\delta \text{ and } x_\delta \in T_\xi\}$. Also

$$\Vdash_{\mathbb{P}} \text{“}\check{x}_\delta \in \check{T}_{\dot\xi_\delta} \text{ and } \check{T}_{\dot\xi_\delta} \in \dot\sigma(\langle \check{T}_{\dot\xi_\beta} : \beta < \delta\rangle).\text{”}$$

Since $q \Vdash_{\mathbb{P}}$ "$\dot\xi_\delta \in \check{F}_\delta$" and by the definition of H_δ, we have

$$q \Vdash_{\mathbb{P}} \text{“}\check{H}_\delta \subseteq \check{T}_{\dot\xi_\delta}.\text{”}$$

Now we see $q \Vdash_{\mathbb{P}}$ "$\forall \delta < \gamma\, (\check{H}_\delta \subseteq \check{T}_{\dot{\xi}_\delta})$," and since $\{H_\delta : \delta < \gamma\}$ covers X, we have

$$q \Vdash_{\mathbb{P}} \text{“}\forall \delta < \gamma\, (\check{T}_{\dot{\xi}_\delta} \in \dot{\sigma}(\langle \check{T}_{\dot{\xi}_\beta} : \beta < \delta \rangle)) \text{ and } \bigcup \{\check{T}_{\dot{\xi}_\delta} : \delta < \gamma\} = \check{X}.\text{”}$$

This concludes the proof. □

Remark 4.3. The reader might complain that for $\delta \geq \gamma$, q may not force $\check{T}_{\dot{\xi}_\delta}$ to be a possible move for Two. But it is unimportant, since the moves after the inning γ do not affect the payoff and so Two may disregard $\check{T}_{\dot{\xi}_\delta}$'s and take any moves to follow the rule.

A similar argument to the above proof yields the following corollary. An adaptation of the proof for the corollary is left to the reader.

Corollary 4.4. *Let (X, τ) be a topological space, $\mathbb{P}$ a forcing notion, α an ordinal, and $\kappa = p(X)$. If*

(1) Two *has a winning strategy in $G_1^{<\alpha}(\mathcal{O}, \mathcal{O})$ on (X, τ), and*
(2) Two *has a winning strategy in $CG^{<\alpha}(<\kappa)$ on $\mathbb{P}$,*

then

$$\Vdash_{\mathbb{P}} \text{“Two } \textit{has a winning strategy in } G_1^{<\alpha}(\mathcal{O}, \mathcal{O}) \textit{ on } (\check{X}, \tau^{\mathbb{P}}).\text{”}$$

5. Consequences

Theorem 4.2, with Theorem 2.6, yields the following consequence.

Corollary 5.1. *Suppose that (X, τ) is an indestructibly Lindelöf space and $\mathbb{P}$ is a forcing notion such that* Two *has a winning strategy in the game $CG^{<\omega_1}(<\aleph_0)$ on $\mathbb{P}$. Then*

$$\Vdash_{\mathbb{P}} \text{“}(\check{X}, \tau^{\mathbb{P}}) \textit{ is an indestructibly Lindelöf space.”}$$

Now consider the following three conditions on a Lindelöf space (X, τ).

(1) (X, τ) is not destroyed by $\mathrm{Fn}(\omega_1, 2, \omega_1)$.
(2) (X, τ) is indestructibly Lindelöf; that is, (X, τ) is not destroyed by any $<\omega_1$-closed forcing notion.
(3) (X, τ) is not destroyed by any forcing notion $\mathbb{P}$ on which Two has a winning strategy in $CG^{<\omega_1}(<\aleph_0)$.

Clearly (3) ⇒ (2) ⇒ (1) holds. Tall pointed out (see [19, Theorem 3]) a result due to Shelah, which claims that (1) ⇒ (2) holds. Corollary 5.1 tells us that (2) ⇒ (3) holds, and hence these three conditions are all equivalent. In fact, we can also give a direct proof of (1) ⇒ (3) by putting the argument of the proof of Theorem 4.2 into Shelah's proof.

Using Theorem 4.2 and Theorem 2.5, we see the following.

Corollary 5.2. *Suppose that* (X, τ) *has the Rothberger property and* $\mathbb{P}$ *is a forcing notion such that* Two *has a winning strategy in the game* $CG^\omega(<\aleph_0)$ *on* $\mathbb{P}$*. Then*

$$\Vdash_{\mathbb{P}} \text{“}(\check{X}, \tau^{\mathbb{P}}) \textit{ has the Rothberger property.”}$$

Let $\mathbb{B}(\kappa)$ denote the measure algebra on 2^κ. Scheepers and Tall proved that for any infinite cardinal κ, if (X, τ) has the Rothberger property, then $\Vdash_{\mathbb{B}(\kappa)}$ "$(\check{X}, \tau^{\mathbb{B}(\kappa)})$ has the Rothberger property" [18, Theorem 15]. It is known that Two has a winning strategy in the game $CG^\omega(<\aleph_0)$ on $\mathbb{B}(\kappa)$ [10], and hence Corollary 5.2 gives an alternate proof of their result.

We also remark that Corollary 5.2 extends another result due to Scheepers and Tall, which claims that for a $<\omega_1$-closed forcing notion $\mathbb{P}$ if (X, τ) has the Rothberger property then

$$\Vdash_{\mathbb{P}} \text{“}(\check{X}, \tau^{\mathbb{P}}) \text{ has the Rothberger property”}$$

[18, Theorem 21].

As we mentioned in section 2, a Lindelöf P-space has the Rothberger property. Using this fact with Theorem 2.1, Theorem 2.5, Proposition 3.1, and Theorem 4.2, we can deduce the following result.

Corollary 5.3. *Suppose that* (X, τ) *is a Lindelöf P-space and* $\mathbb{P}$ *is a forcing notion such that* Two *has a winning strategy in the game* $CG^\omega(\aleph_0)$ *on* $\mathbb{P}$*. Then*

$$\Vdash_{\mathbb{P}} \text{“}(\check{X}, \tau^{\mathbb{P}}) \textit{ is a Lindelöf P-space.”}$$

Now we can summarize these consequences of Theorem 4.2 as in Table 1. The symbol ↑ denotes "has a winning strategy in."

The table is read as follows: A property of a topological space shown in a left-hand column is preserved under forcing extension by a forcing notion with the property shown in the corresponding right-hand column.

Table 1. Summary of consequences of the main result

Topological spaces	Forcing notions	
(1) Lindelöf		
	(1) $\mathrm{Fn}(\omega_1, 2, \omega_1)$	
	(2) $<\omega_1$-closed	
	(3) Two $\uparrow CG^{<\omega_1}(1)$	
(2) indestructibly Lindelöf	(4) Two $\uparrow CG^{<\omega_1}(<\aleph_0)$	(5.1)
(3) Rothberger	(5) Two $\uparrow CG^{\omega}(<\aleph_0)$	(5.2)
(4) Lindelöf P-space	(6) Two $\uparrow CG^{\omega}(\aleph_0)$	(5.3)
	(7) proper	

Before closing this section, we state a consequence of Corollary 4.4, which gives a sufficient condition for a forcing notion to preserve the topological property "Two has a winning strategy in $G_1^{<\omega_1}(\mathcal{O}, \mathcal{O})$ on X."

Corollary 5.4. *Suppose* Two *has a winning strategy in* $G_1^{<\omega_1}(\mathcal{O}, \mathcal{O})$ *on a topological space* (X, τ) *and* $\mathbb{P}$ *is a forcing notion such that* Two *has a winning strategy in the game* $CG^{<\omega_1}(<\aleph_0)$ *on* $\mathbb{P}$*. Then*

$$\Vdash_{\mathbb{P}} \text{“Two has a winning strategy in } G_1^{<\omega_1}(\mathcal{O}, \mathcal{O}) \text{ on } (\check{X}, \tau^{\mathbb{P}})\text{.”}$$

6. Discussion

We will show that, under ZFC, the assumption "Two has a winning strategy in $CG^{\omega}(<\aleph_0)$ on $\mathbb{P}$" in Corollary 5.2 cannot be weakened to "Two has a winning strategy in $CG^{\omega}(\aleph_0)$ on $\mathbb{P}$."

We use the following famous result due to Richard Laver [15] (also found in [1, Theorem 8.3.2]). Let $\mathbb{M}$ denote the Mathias forcing notion.

Theorem 6.1. *Suppose that X is an uncountable set of real numbers. Then*

$$\Vdash_{\mathbb{M}} \text{“}\check{X} \text{ does not have strong measure zero.”}$$

It is easily checked that if a forcing notion $\mathbb{P}$ satisfies Axiom A, then Two has a winning strategy in $CG^{\omega}(\aleph_0)$ on $\mathbb{P}$. On the other hand, if Two has a winning strategy in $CG^{\omega}(<\aleph_0)$ on $\mathbb{P}$, then $\mathbb{P}$ is ω^{ω}-bounding by Theorem 2.2. The Mathias forcing $\mathbb{M}$ satisfies Axiom A but is not ω^{ω}-bounding, and so Two has a winning strategy in $CG^{\omega}(\aleph_0)$ on $\mathbb{M}$ but none in $CG^{\omega}(<\aleph_0)$ on $\mathbb{M}$.

Now assume CH and let L be an uncountable Lusin set of real numbers. It is known that L has the Rothberger property [17]. However, by Theorem 6.1, we have

$$\Vdash_{\mathbb{M}} \text{“}\check{L} \text{ does not have strong measure zero,”}$$

and a set of real numbers with the Rothberger property has strong measure zero, which implies

$$\Vdash_{\mathbb{M}} \text{“}\check{L} \text{ does not have the Rothberger property.”}$$

We do not know if the assumption “Two has a winning strategy in $CG^{<\omega_1}(<\aleph_0)$ on $\mathbb{P}$” in Corollary 5.1 can be weakened to “Two has a winning strategy in $CG^{\omega}(<\aleph_0)$ on $\mathbb{P}$.”

Question 6.2. Can we find an indestructibly Lindelöf space (X, τ) and a forcing notion $\mathbb{P}$ which satisfy the following?

(1) Two has a winning strategy in the game $CG^{\omega}(<\aleph_0)$ on $\mathbb{P}$.
(2) $\Vdash_{\mathbb{P}}$ “$(\check{X}, \tau^{\mathbb{P}})$ is not a Lindelöf space.”

Remark 6.3. As we mentioned in section 5, Two has a winning strategy in $CG^{\omega}(<\aleph_0)$ on the measure algebra $\mathbb{B}(\kappa)$ for any κ (moreover, for any fixed $\alpha < \omega_1$, Two has a winning strategy in $CG^{<\alpha}(<\aleph_0)$ on $\mathbb{B}(\kappa)$). On the other hand, it is easy to find a winning strategy for One in $CG^{<\omega_1}(<\aleph_0)$ on $\mathbb{B}(\kappa)$ (just note that any strictly decreasing sequence of real numbers has at most countable order type). Unfortunately, for any Lindelöf space (X, τ), we have $\Vdash_{\mathbb{B}(\kappa)}$ “$(\check{X}, \tau^{\mathbb{B}(\kappa)})$ is Lindelöf” (see [6]).

We do not know if the assumption “Two has a winning strategy in $CG^{\omega}(\aleph_0)$ on $\mathbb{P}$” in Corollary 5.3 can be weakened to “$\mathbb{P}$ is proper.”

Let $\mathbb{CF}$ denote the poset which adjoins a closed unbounded subset of ω_1 with finite conditions, which is due to James E. Baumgartner [2]. It is known that $\mathbb{CF}$ is proper but One has a winning strategy in $CG^{\omega}(\aleph_0)$ on $\mathbb{CF}$. So it is natural to ask the following question.

Note that, by Proposition 3.1, a P-space is still a P-space in a forcing extension by a proper poset.

Question 6.4. Is there a Lindelöf P-space (X, τ) such that

$$\Vdash_{\mathbb{CF}} \text{“}(\check{X}, \tau^{\mathbb{CF}}) \text{ is not Lindelöf”}\,?$$

Acknowledgment. The author thanks Marion Scheepers for his helpful comments and discussion during this work.

References

[1] Tomek Bartoszyński and Haim Judah, *Set Theory: On the Structure of the Real Line*. Wellesley, MA: A. K. Peters, Ltd., 1995.

[2] James E. Baumgartner, *Applications of the proper forcing axiom*, in Handbook of Set-Theoretic Topology. Ed. Kenneth Kunen and Jerry E. Vaughan. Amsterdam: North-Holland, 1984. 913–959

[3] Peg Daniels and Gary Gruenhage, *The point-open type of subsets of the reals*, Topology Appl. **37** (1990), no. 1, 53–64.

[4] Matthew Foreman, *Games played on Boolean algebras*, J. Symbolic Logic **48** (1983), no. 3, 714–723.

[5] J. Gerlits and Zs. Nagy, *Some properties of $C(X)$. I*, Topology Appl. **14** (1982), no. 2, 151–161.

[6] Renata Grunberg, Lúcia R. Junqueira, and Franklin D. Tall, *Forcing and normality*, Topology Appl. **84** (1998), no. 1-3, 145–174.

[7] Tetsuya Ishiu, *Games of transfinite length on Boolean algebra*. Unpublished manuscript, 1997.

[8] ———, *α-properness and Axiom A*, Fund. Math. **186** (2005), no. 1, 25–37.

[9] Tetsuya Ishiu and Yasuo Yoshinobu, *Directive trees and games on posets*, Proc. Amer. Math. Soc. **130** (2002), no. 5, 1477–1485.

[10] Thomas J. Jech, *More game-theoretic properties of Boolean algebras*, Ann. Pure Appl. Logic **26** (1984), no. 1, 11–29.

[11] ———, *Multiple Forcing*. Cambridge Tracts in Mathematics, 88. Cambridge: Cambridge University Press, 1986.

[12] Thomas Jech and Saharon Shelah, *On countably closed complete Boolean algebras*, J. Symbolic Logic **61** (1996), no. 4, 1380–1386.

[13] Masaru Kada, *More on Cichoń's diagram and infinite games*, J. Symbolic Logic **65** (2000), no. 4, 1713–1724.

[14] Kenneth Kunen, *Set Theory: An Introduction to Independence Proofs*. Studies in Logic and the Foundations of Mathematics, 102. Amsterdam-New York: North-Holland Publishing Co., 1980.

[15] Richard Laver, *On the consistency of Borel's conjecture*, Acta Math. **137** (1976), no. 3-4, 151–169.

[16] Janusz Pawlikowski, *Undetermined sets of point-open games*, Fund. Math. **144** (1994), no. 3, 279–285.

[17] Fritz Rothberger, *Eine verschärfung der eigenshaft C*, Fund. Math **30** (1938), 50–55.

[18] Marion Scheepers and Franklin D. Tall, *Lindelöf indestructivity, topological games and selection principles*. Preprint.

[19] Franklin D. Tall, *On the cardinality of Lindelöf spaces with points G_δ*, Topology Appl. **63** (1995), no. 1, 21–38.

[20] Boban Veličković, *Jensen's $\Box$ principles and the Novák number of partially ordered sets*, J. Symbolic Logic **51** (1986), no. 1, 47–58.

[21] ______, *Playful Boolean algebras*, Trans. Amer. Math. Soc. **296** (1986), no. 2, 727–740.

[22] Jindřich Zapletal, *More on the cut and choose game*, Ann. Pure Appl. Logic **76** (1995), no. 3, 291–301.

Graduate School of Science; Osaka Prefecture University; 1–1 Gakuen-cho, Naka-ku, Sakai; Osaka 599–8531 JAPAN

E-mail address: `kada@mi.s.osakafu-u.ac.jp`