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by

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PROPERTIES D AND aD ARE DIFFERENT

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Abstract. Under ($\diamondsuit^*$) we construct a locally countable, locally compact, 0-dimensional T_2 space X of size ω_1 which is aD, but not even linearly D. This consistently answers a question of Alexander V. Arhangel'skii, whether aD implies D. Furthermore, we answer two problems concerning characterization of linearly D-spaces, raised by Hongfeng Guo and Heikki Junnila.

1. Introduction

The notion of a D-space was probably first introduced by Eric K. van Douwen. See Gary Gruenhage's paper [6], which also gives a full review on what we know and do not know about D-spaces. A. V. Arhangel'skii and R. Z. Buzyakova [2] defined a weakening of property D, called aD. In [1], Arhangel'skii asked the following:

> **Problem 4.6.** Is there a Tychonoff aD-space which is not a D-space?

In section 5, we construct such a space under ($\diamondsuit^*$). Before that we consider another weakening of property D. Recently, Hongfeng Guo and Heikki Junnila [7] introduced the notion of linearly D-spaces and proved several nice results concerning the topic. In sections 3 and 4, we answer the following two questions from [7] in negative (in ZFC):

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Problem 2.5. Let X be a T_1 (linearly) D-space and let $A \subseteq X$ have uncountable regular cardinality. Does A either have a complete accumulation point or a subset of size $|A|$ which is closed and discrete in X?

Problem 2.6. Is a T_1-space X linearly D provided that, for every set $A \subseteq X$ of uncountable regular cardinality, either A has a complete accumulation point or $\overline{A}$ has a subset of size $|A|$ which is closed and discrete in X?

As we will see, these questions arise naturally. The constructions from sections 3 and 4 can be considered as preparation to the space defined in section 5. Our construction to Problem 4.6 also answers (consistently) the following from [7]:

Problem 2.12. Is every aD-space linearly D?

2. Definitions

An *open neighborhood assignment* (*ONA*) on a space (X, τ) is a map $U : X \to \tau$ such that $x \in U(x)$ for every $x \in X$. X is said to be a *D-space* if for every neighborhood assignment U, one can find a closed discrete $D \subseteq X$ such that $X = \bigcup_{d \in D} U(d) = \bigcup U[D]$ (such a set D is called a *kernel for* U). In [2], Arhangel'skii and Buzyakova introduced property aD:

Definition 2.1. A space (X, τ) is said to be aD if and only if for each closed $F \subseteq X$ and for each open cover $\mathcal{U}$ of X, there is a closed discrete $A \subseteq F$ and $\phi : A \to \mathcal{U}$ with $a \in \phi(a)$ such that $F \subseteq \cup\phi[A]$.

It is clear that D-spaces are aD. A space X is *irreducible* if and only if every open cover $\mathcal{U}$ has a *minimal open refinement* $\mathcal{U}_0$, meaning that no proper subfamily of $\mathcal{U}_0$ covers X. Later, in [1], Arhangel'skii showed the following equivalence.

Theorem 2.2 ([1, Theorem 1.8])**.** *A T_1-space X is an aD-space if and only if every closed subspace of X is irreducible.*

Another generalization of property D is due to Guo and Junnila [7]. For a space X, a cover $\mathcal{U}$ is *monotone* if and only if it is linearly ordered by inclusion.

Definition 2.3. A space (X,τ) is said to be *linearly D* if and only if for any ONA $U : X \to \tau$ for which $\{U(x) : x \in X\}$ is monotone, one can find a closed discrete set $D \subseteq X$ such that $X = \bigcup U[D]$.

We cite two results from [7]. A set $D \subseteq X$ is said to be *$\mathcal{U}$-big* for a cover $\mathcal{U}$ if and only if there is no $U \in \mathcal{U}$ such that $D \subseteq U$.

Theorem 2.4 ([7, Theorem 2.2])**.** *The following are equivalent for a T_1-space X:*

(1) *X is linearly D.*
(2) *For every non-trivial monotone open cover $\mathcal{U}$ of X, there exists a closed discrete $\mathcal{U}$-big set in X.*
(3) *For every subset $A \subseteq X$ of uncountable regular cardinality κ, there is a closed discrete subset B of X, such that for every neighborhood U of B, we have $|U \cap A| = \kappa$.*

In Problem 2.5 the authors ask whether condition (3) can be made stronger.

Theorem 2.5 ([7, Proposition 2.4])**.** *A T_1-space X is linearly D if, and only if, for every set $A \subseteq X$ of uncountable regular cardinality, either the set A has a complete accumulation point or there exists a closed discrete set D of size $|A|$ and a disjoint family $\{A_d : d \in D\}$ of subsets of A such that $d \in \overline{A_d}$ for every $d \in D$.*

In Problem 2.6 the authors ask whether the second condition of this dichotomy can be weakened.

3. On Problem 2.5 from [7]

In this section, we give a negative answer to Problem 2.5. For this, let us use the following notion for a space X. We say that X *satisfies* $(*)$ if and only if

$(*)$ for every regular, uncountable cardinal κ and $A \in [X]^\kappa$, there is a complete accumulation point of A or A has a subset of size κ which is closed discrete in X.

Problem 2.5 can be rephrased as to whether property D implies $(*)$. We will show

(1) there exists a locally countable T_2 D-space X with cardinality ω_1 which does not satisfy $(*)$;

(2) the existence of a locally countable, locally compact, T_2 D-space X with cardinality $< 2^\omega$ which does not satisfy $(*)$ is independent of ZFC;
(3) there exists a locally countable, locally compact T_3 (even 0-dimensional) D-space X with cardinality 2^ω which does not satisfy $(*)$.

First, let us observe the following.

Proposition 3.1. *Suppose that the space X is the union of a closed discrete set and a D-subspace. Then X is a D-space.*

Proof: Let $X = Y \cup Z$ such that Y is closed discrete and Z is a D-space. Let U be an ONA on X. $Z_0 = Z \setminus \bigcup\{U(y) : y \in Y\}$ is a closed subspace of the D-space Z; thus, Z_0 is a D-space too. There is a closed discrete kernel D_0 for the ONA $U|_{Z_0}$ on Z_0. Then $D = D_0 \cup Y$ is a closed discrete kernel for U. □

Proposition 3.2. *There exists a locally countable T_2 D-space X with cardinality ω_1 which does not satisfy $(*)$.*

Proof: Let $X = \omega_1 \times 2$. We define the topology on X as follows. Let $\omega_1 \times \{0\}$ be discrete. For $\alpha < \omega_1$, let $(\alpha, 1)$ have the neighborhood base

$$\big\{\{(\alpha,1)\} \cup \big((\beta,\alpha) \times \{0\}\big) : \beta < \alpha\big\}.$$

Clearly, X is a locally countable, T_2 space. Observe that $\omega_1 \times \{1\} \subseteq X$ is closed discrete and $\omega_1 \times \{0\} \subseteq X$ is discrete, hence D. Thus, X is a D-space by Proposition 3.1. Let $A = \omega_1 \times \{0\}$. Then, clearly, any infinite subset of A has an accumulation point in X. Thus, X does not satisfy $(*)$, since there is no full accumulation point of A and any infinite subset of A is not closed discrete in X. □

This answers Problem 2.5 in the negative direction by a T_2 counterexample.

The question whether a regular space with this property exists is natural. First, we will show that the existence of a "nice" regular counterexample with cardinality below 2^ω is independent. We will need a weakening of the *axiom* (t) which was introduced by István Juhász in [9].

Definition 3.3. The *weak (t) axiom*: there exists a *weak (t)-sequence* $\{A_\alpha : \alpha \in \lim(\omega_1)\}$, meaning that $A_\alpha \subseteq \alpha$ is an ω-sequence converging to α, and for every $X \in [\omega_1]^{\omega_1}$, there is a limit α such that $|X \cap A_\alpha| = \omega$.

The existence of such sequences is independent of ZFC. Under MA, there is no weak (t)-sequence, and adding one Cohen real to any model adds a (weak) (t)-sequence (see [9]).

Proposition 3.4. *Suppose the weak (t)-axiom. Then there exists a locally compact, locally countable, 0-dimensional T_2 D-space X which does not satisfy* $(*)$.

Proof: Suppose that $\mathcal{A} = \{A_\alpha : \alpha \in \lim(\omega_1)\}$ is a weak (t)-sequence. Let $X = \omega_1 \times 2$. Define the topology on X as follows. Let $\omega_1 \times \{0\}$ be discrete. For $\alpha \in \lim(\omega_1)$, let $(\alpha, 1)$ have the neighborhood base

$$\big\{\{(\alpha,1)\} \cup \big((A_\alpha \setminus \beta) \times \{0\}\big) : \beta < \alpha\big\}.$$

For successor $\alpha < \omega_1$, let $(\alpha, 1)$ be discrete. Clearly, X is a locally countable, locally compact, 0-dimensional T_2 space. Notice that $\omega_1 \times \{1\} \subseteq X$ is closed discrete and $\omega_1 \times \{0\} \subseteq X$ is discrete, hence D. Thus, X is a D-space by Proposition 3.1. Let $A = \omega_1 \times \{0\}$. We prove that any uncountable $B \subseteq A$ is not closed discrete in X; hence, X does not satisfy $(*)$. Let $B_0 = \{\alpha < \omega_1 : (\alpha, 0) \in B\} \in [\omega_1]^{\omega_1}$. Since $\mathcal{A}$ is a weak (t)-sequence, there is $\alpha \in \lim(\omega_1)$ such that $|A_\alpha \cap B_0| = \omega$. Clearly, B accumulates to $(\alpha, 1)$; thus, B is not closed discrete in X. $\square$

Remark 3.5. In [8], Tetsuya Ishiu uses guessing sequences to refine the standard topology on an ordinal.

Now our aim is to prove Proposition 3.8 which implies that under MA there is no such space. The following was proved by Zoltán T. Balogh (actually more, but we need only this).

Theorem 3.6 ([3, Theorem 2.2])**.** *Suppose* MA. *Then for any locally countable, locally compact space X of cardinality $< 2^\omega$, exactly one of the following is true.*

- *X is the countable union of closed discrete subspaces.*
- *X contains a perfect preimage of ω_1 with the order topology.*

From this and the following observation, we can deduce Proposition 3.8.

Proposition 3.7 ([4, Proposition 7])**.** *If the space X is the countable union of a closed D-subspace, then X is a D-space.*

Proposition 3.8. *Suppose* MA*. Then for any locally countable, locally compact space X of cardinality $< 2^\omega$, the following are equivalent.*

(0) *X is σ-closed discrete;*
(1) *X is a D-space;*
(2) *X is a linearly D-space;*
(3) *X satisfies $(*)$.*

Proof: The implications (0) $\Rightarrow$ (1) $\Rightarrow$ (2) (by Proposition 3.7) and (0) $\Rightarrow$ (3) are straightforward.

(3) $\Rightarrow$ (2) by Theorem 2.4.

We need only to show (2) $\Rightarrow$ (0). Suppose that X is linearly D; by Theorem 3.6, we need to show that X does not contain any perfect preimage of ω_1.

Claim 1. (i) If the space F is a perfect preimage of ω_1, then F is countably compact, non compact.

(ii) If X is first-countable and $F \subseteq X$ is a perfect preimage of ω_1, then F is closed in X.

Proof of Claim: (i) It is known that under perfect mappings, the preimage of a compact space is compact (see [5, Theorem 3.7.2]). Take any countably infinite $A \subseteq F$ and perfect surjection $f : F \to \omega_1$. There is some $\alpha < \omega_1$ such that $f[A] \subseteq \alpha + 1$. Thus, A is the subset of the compact set $f^{-1}[\alpha + 1]$.

(ii) is a consequence of (i).

Suppose $F \subseteq X$ is a closed subspace. Then F is linearly D; hence, if F is countably compact, F is compact too. By the claim, F cannot be a perfect preimage of ω_1. $\square$

Finally, we give a regular counterexample to the problem in ZFC without any further set-theoretic assumptions.

Theorem 3.9. *There exists a locally countable, locally compact, 0-dimensional T_2 D-space X with cardinality 2^ω such that X does not satisfy $(*)$.*

Proof: Let $\{C_\alpha : \alpha < 2^\omega\}$ denote an enumeration of the closed dense-in-itself subsets of $\mathbb{R}$. Let $\{Q_\alpha^\beta : \beta < 2^\omega\}$ denote an enumeration of all countable subsets of $\mathbb{R}$ such that $C_\alpha \subseteq \overline{Q_\alpha^\beta}$ (Euclidean closure taken). Enumerate the pairs (α, β) from $2^\omega \times 2^\omega$ in order type 2^ω: $\{p_\delta : \delta < 2^\omega\}$. We define a topology on $X = \mathbb{R} \times 2$ as follows. Let $\mathbb{R} \times \{0\}$ be discrete, and we define the topology on $\mathbb{R} \times \{1\}$ by induction. In step δ for $p_\delta = (\alpha, \beta)$, pick a point $x_\delta \in C_\alpha \setminus \{x_{\delta'} : \delta' < \delta\}$ and let $(x_\delta, 1)$ have the neighborhood base

$$\big\{\{(x_\delta, 1)\} \cup \{(x_\delta^n, 0) : n \geq m\} : m < \omega\big\}$$

where $\{x_\delta^n : n < \omega\} \subseteq Q_\alpha^\beta \setminus \{x_\delta\}$ is any sequence converging to x_δ in the Euclidean sense. Let the remaining points $(\mathbb{R} \setminus \{x_\delta : \delta < 2^\omega\}) \times \{1\}$ be discrete. Clearly, this gives us a locally countable, locally compact, 0-dimensional T_2 space. $\mathbb{R} \times \{1\}$ is closed discrete and $\mathbb{R} \times \{0\}$ is discrete, hence a D-space. Thus, X is a D-space by Proposition 3.1.

We claim that there is no uncountable subset of $A = \mathbb{R} \times \{0\} \subseteq X$ such that it is closed discrete in X with this topology; this implies that X does not satisfy $(*)$. Let $B \in [A]^{\omega_1}$ and $B_0 = \{x \in \mathbb{R} : (x, 0) \in B\}$. Then there is $\alpha < 2^\omega$ such that $C_\alpha \subseteq B_0'$ (where B_0' denotes the Euclidean accumulation points of B_0) and $\beta < 2^\omega$ such that $Q_\alpha^\beta \subseteq B_0$. By definition, in step δ, where $p_\delta = (\alpha, \beta)$, we defined the topology on X in such a way that $(x_\delta, 1)$ is in the closure of $Q_\alpha^\beta \times \{0\}$ and thus in the closure of B. Therefore, B is not closed in X. $\square$

4. On Problem 2.6 from [7]

Now our aim is to answer Problem 2.6 in the negative. For this, we will say that a space X *satisfies* $(**)$ if and only if

$(**)$ for every regular, uncountable cardinal κ and $A \in [X]^\kappa$, there is a complete accumulation point of A or there is $D \in [A]^\kappa$ which is closed discrete in X.

Problem 2.6 can be rephrased as whether $(**)$ implies linearly D. We prove that

(1) the existence of a locally countable, locally compact, non-linearly D space X with cardinality $< 2^\omega$ which satisfies $(**)$ is independent of ZFC;

(2) there is a locally countable, locally compact, 0-dimensional Hausdorff space X with cardinality 2^ω which satisfies $(**)$, but is not linearly D.

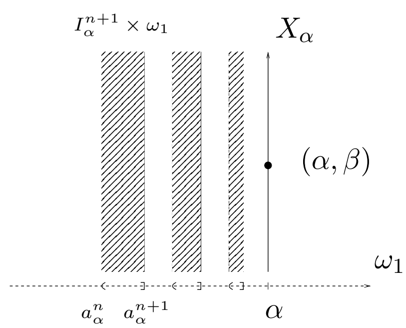
We will use the following notations in this section. $\{A_\alpha : \alpha \in \lim(\omega_1)\}$ denotes a $\clubsuit$-*sequence*: for every $\alpha < \omega_1$, $A_\alpha \subseteq \alpha$ is an ω-type sequence converging to α, and for every $X \in [\omega_1]^{\omega_1}$, there is some $\alpha < \omega_1$ such that $A_\alpha \subseteq X$. $\clubsuit$ means that there is a $\clubsuit$-sequence. For every $\alpha \in \omega_1$, enumerate increasingly A_α as $\{a^n_\alpha : n \in \omega\}$. Let $\{M^\beta : \beta \in \omega_1\} \subseteq [\omega]^\omega$ be an arbitrary almost disjoint family on ω.

Theorem 4.1. *Suppose $\clubsuit$. Then there is a 0-dimensional T_2 space X of cardinality ω_1 such that X is not linearly D, but satisfies $(**)$.*

Proof: First, we introduce some further notations for the intervals between the points in the A_α's. For each $\alpha \in \omega_1$, let $\{I^n_\alpha : n \in \omega\}$ denote the disjoint open sets in ω_1, $I^0_\alpha = (0, a^0_\alpha]$ and $I^{n+1}_\alpha = (a^n_\alpha, a^{n+1}_\alpha]$ for $n \in \omega$. We will define a topology on $X = \omega_1 \times \omega_1$. Let $\{\alpha\} \times \omega_1$ be discrete for successor α. For $\alpha \in \lim(\omega_1)$ and $\beta \in \omega_1$, a neighborhood base for the point (α, β) consists of sets

$$U((\alpha,\beta),E) = \{(\alpha,\beta)\} \cup \bigcup\{I^n_\alpha \times \omega_1 : n \in M^\beta \setminus E\},$$

where $E \in [\omega]^{<\omega}$. Observe that if $\beta, \beta' \in \omega_1$ and $\beta \neq \beta'$, then $E = M^\beta \cap M^{\beta'}$ is finite; thus, $U((\alpha,\beta),E) \cap U((\alpha,\beta'),E) = \emptyset$. This way, we defined a 0-dimensional T_2 topology. Note that the set $X_\alpha = \{\alpha\} \times \omega_1$ for $\alpha \in \omega_1$ is closed discrete. Let $\pi(A) = \{\alpha \in \omega_1 : A \cap X_\alpha \neq \emptyset\}$ for $A \subseteq X$.



Claim 1. If $|\pi(A)| = \omega_1$ for $A \subseteq X$, then there are stationary many $\alpha \in \omega_1$ such that $X_\alpha \subseteq A'$.

Proof of Claim 1: Since $\{A_\alpha : \alpha \in \lim(\omega_1)\}$ is a $\clubsuit$-sequence, the set $S = \{\alpha \in \omega_1 : A_\alpha \subseteq \pi(A)\}$ is stationary. For $\alpha \in S$, we clearly have $X_\alpha \subseteq A'$, since by definition for any U neighborhood of any point $(\alpha, \beta) \in X_\alpha$, U intersects A in infinitely many points.

This claim has the following corollaries.

Claim 2. X satisfies $(**)$.

Proof of Claim 2: Let $A \in [X]^{\omega_1}$. If there is an $\alpha \in \omega_1$ such that $|A \cap X_\alpha| = \omega_1$, we are done. Otherwise, for all $\alpha \in \omega_1$, we have $|A \cap X_\alpha| \leq \omega$ so $|\pi(A)| = \omega_1$. By Claim 1, there is an $\alpha \in \lim(\omega_1)$ such that $X_\alpha \subseteq \overline{A}$ and X_α is closed discrete.

Claim 3. X is not linearly D.

Proof of Claim 3: Suppose that $D \subseteq X$ is closed discrete. Then $\pi(D)$ is countable by Claim 1. Hence, there is no closed discrete set which is big for the open cover $\{\alpha \times \omega_1 : \alpha < \omega_1\}$. Thus, X is not linearly D by Theorem 2.4.

This completes the proof of this theorem. $\square$

Remark 4.2. If we modify the neighborhoods to be

$$\Big\{\{(\alpha, \beta)\} \cup \bigcup \{I_\alpha^n \times [0, \beta] : n \in M^\beta \setminus E\} : E \in [\omega]^{<\omega}\Big\}$$

for (α, β) where $\beta \in \omega_1$ and $\alpha \in \lim(\omega_1)$, then we obtain a topology which is locally countable, not linearly D, satisfies $(**)$, but is not even regular.

With some further set-theoretic assumptions, we can improve the above construction.

Theorem 4.3. *Suppose $\clubsuit$ and* CH *(equivalently, $\diamondsuit$). Then there is a locally countable, locally compact, 0-dimensional T_2 space X of cardinality $\omega_1 (= 2^\omega)$ such that X is not linearly D, but satisfies $(**)$.*

Proof: For our construction, we will need an enumeration of the functions $\omega \to \omega_1$ as $\{F_\delta : \delta < \omega_1\}$; here we used CH. Let h be an arbitrary bijection $h : \omega_1 \to \omega_1 \times \omega_1$. We define a topology on $X = \omega_1 \times \omega_1$. Let $X_\alpha = \{\alpha\} \times \omega_1$, $X_{<\alpha} = \alpha \times \omega_1$. Define neighborhoods for points in X_α by induction on α. Let $(X_{<\alpha}, \tau_{<\alpha})$ denote the topology defined by the induction until step α. We have the following conditions which we will preserve during each step.

(i) $(X_{<\alpha}, \tau_{<\alpha})$ is locally countable, locally compact, 0-dimensional;

(ii) $X_{<\beta}$ is open in $X_{<\alpha}$ for $\beta < \alpha$;
(iii) for every $\beta < \alpha$ and $(\beta, \gamma) \in X_\beta$, there is some neighborhood G of (β, γ) such that $G \setminus \{(\beta, \gamma)\} \subseteq X_{<\beta}$;
(iv) for every $\beta' < \beta < \alpha$, the set $(\beta', \beta] \times \omega_1$ is clopen in $(X_{<\alpha}, \tau_{<\alpha})$.

This way, we will get a topology τ on X by taking $\cup\{\tau_{<\alpha} : \alpha < \omega_1\}$ as a base.

For successor $\alpha < \omega_1$, just let (α, β) be a discrete point for any $\beta < \omega_1$. Clearly, this way, the inductional hypothesis will hold. Suppose now that α is a limit; we define neighborhoods of (α, β) (for any $\beta < \omega_1$) as follows. Suppose that $h(\beta) = (\delta, \gamma)$. For each $n \in \omega$, take a countable, compact, clopen $G_n \subseteq (a_\alpha^{n-1}, a_\alpha^n] \times \omega_1$ such that $(a_\alpha^n, F_\delta(n)) \in G_n$; this can be done by (i) and (iv). Then define the neighborhoods of (α, β) by the base

$$\big\{\{(\alpha, \beta)\} \cup \bigcup\{G_n : n \in M^\beta \setminus E\} : E \in [\omega]^{<\omega}\big\}.$$

It is clear that the inductional assumptions will hold for the resulting topology. It follows from the construction that (X, τ) is locally countable, locally compact, and 0-dimensional. Thus, we constructed a space (X, τ) which refines the topology from the previous theorem; hence, X is T_2. Let $\pi(A) = \{\alpha \in \omega_1 : A \cap X_\alpha \neq \emptyset\}$ for $A \subseteq X$.

Claim 1. If $|\pi(A)| = \omega_1$ for $A \subseteq X$, then there are stationary many $\alpha \in \omega_1$ such that $|X_\alpha \cap A'| = \omega_1$.

Proof of Claim 1: Since $\{A_\alpha : \alpha \in \lim(\omega_1)\}$ is a $\clubsuit$-sequence, the set $S = \{\alpha \in \omega_1 : A_\alpha \subseteq \pi(A)\}$ is stationary. Fix an $\alpha \in S$. Define $F : \omega \to \omega_1$ such that $(a_\alpha^n, F(n)) \in A$ where $A_\alpha = \{a_\alpha^n : n \in \omega\}$. So there is some δ for which we have $F_\delta = F$. We claim that $\{(\alpha, \beta) \in X_\alpha : \exists \gamma < \omega_1 : h(\beta) = (\delta, \gamma)\} \subseteq A'$. Clearly, for such an (α, β), we used $F_\delta = F$ in the induction to define the neighborhoods, from which we see that the set $\{(a_\alpha^n, F(n)) : n \in \omega\} \subseteq A$ accumulates to (α, β).

Claim 2. X satisfies $(**)$.

Proof of Claim 2: Let $A \in [X]^{\omega_1}$. If there is an $\alpha \in \omega_1$ such that $|A \cap X_\alpha| = \omega_1$, we are done. Otherwise, since $|A \cap X_\alpha| \leq \omega$, we have $|\pi(A)| = \omega_1$. By Claim 1, $\overline{A}$ intersects (stationary) many closed discrete sets X_α in ω_1 many points.

Claim 3. X is not linearly D.

Proof of Claim 3: Suppose that $D \subseteq X$ is closed discrete. Then $\pi(D)$ is countable by Claim 1. Hence, there is no closed discrete set which is big for the open cover $\{\alpha \times \omega_1 : \alpha < \omega_1\}$. Thus, X is not linearly D by Theorem 2.4.

This completes the proof of this theorem. $\square$

We can further extend the equivalences of Proposition 3.8 using Theorem 3.6.

Proposition 4.4. *Suppose* MA. *Suppose that the space X is locally countable, locally compact of cardinality less than 2^ω. Then the following are equivalent.*

(1) *X is (linearly) D;*
(4) *X satisfies $(**)$.*

Proof: (1) $\Rightarrow$ (4) trivially.

Suppose (4); then a closed uncountable subspace of X cannot be countably compact. Then by Claim 1 of the proof of Proposition 3.8, there is no perfect preimage of ω_1 in X. By Theorem 3.6, this implies that X is σ-closed-discrete, hence (linearly) D. $\square$

Under ZFC, without any further set-theoretic assumptions, we can give a counterexample.

Theorem 4.5. *There is a locally countable, locally compact, 0-dimensional T_2 space X such that X is not linearly D, but satisfies $(**)$.*

Proof: We will use the following notations: let $\{C_\alpha : \alpha < 2^\omega\}$ be an enumeration of uncountable closed dense in itself subsets of $\mathbb{R}$ and enumerate $\{Q \in [\mathbb{R} \setminus \mathbb{Q}]^\omega : C_\alpha \subseteq \overline{Q}\}$ as $\{Q_\alpha^\beta : \beta < 2^\omega\}$. Enumerate the triples $(C_\alpha, Q_\alpha^\beta, \gamma)$ for $\alpha, \beta, \gamma < 2^\omega$ in order type 2^ω: $\{t_\delta : \delta < 2^\omega\}$. We need an enumeration of all functions $F : \omega \to 2^\omega$, $\{F_\varphi : \varphi < 2^\omega\}$. Fix an $h : 2^\omega \to 2^\omega \times 2^\omega$ bijection. Furthermore, let $\{M^\epsilon : \epsilon < 2^\omega\} \subseteq [\omega]^\omega$ be an almost disjoint family on ω.

We define a topology on $X = 2^\omega \times 2^\omega$ by induction. Let $X_\delta = \{\delta\} \times 2^\omega$ for $\delta < 2^\omega$. Let $(X_{<\delta}, \tau_{<\delta})$ denote the topology defined by the induction until step $\delta < 2^\omega$ where $X_{<\delta} = \bigcup\{X_{\delta'} : \delta' < \delta\}$. In step δ, we pick a point x_δ from the real line which will help us define the neighborhoods of points in X_δ. We have the following conditions which we preserve during the induction.

(i) $(X_{<\delta}, \tau_{<\delta})$ is locally countable, locally compact, 0-dimensional;
(ii) $X_{<\delta'}$ is open in $X_{<\delta}$ for $\delta' < \delta$;
(iii) for every $\delta' < \delta$ and $(\delta', \epsilon) \in X_{\delta'}$, there is some neighborhood G of (δ', ϵ) such that $G \setminus \{(\delta', \epsilon)\} \subseteq X_{<\delta'}$;
(iv) *property (E)*: suppose $\delta' < \delta$ and $x_{\delta'} \in B$ where B is Euclidean open. If $(\delta', \epsilon) \in X_{\delta'}$, then there is some compact, countable, and clopen neighborhood G of (δ', ϵ) such that $G \subseteq \bigcup\{X_{\delta''} : x_{\delta''} \in B\}$.

This way, we will get a topology τ on X if we take $\cup\{\tau_{<\delta} : \delta < 2^\omega\}$ as a base.

Suppose we are in step $\delta \in 2^\omega$, where $t_\delta = (C_\alpha, Q_\alpha^\beta, \gamma)$. We then

- pick a point $x_\delta \in C_\alpha \setminus (\{x_{\delta'} : \delta' < \delta\} \cup \mathbb{Q})$;
- if the set $Q_\alpha^\beta \cap \{x_{\delta'} : \delta' < \delta\}$ does not accumulate to x_δ, just let each point of X_δ be discrete;
- if the set $Q_\alpha^\beta \cap \{x_{\delta'} : \delta' < \delta\}$ accumulates to x_δ, choose a sequence $\{x_{\delta'_n} : n \in \omega\} \subseteq Q_\alpha^\beta \cap \{x_{\delta'} : \delta' < \delta\}$ converging to x_δ;
- take disjoint open intervals B_n with rational endpoints containing $x_{\delta'_n}$.

Now we are ready to define a neighborhood of a point (δ, ϵ). Suppose $h(\epsilon) = (\varphi, \rho)$.

- Consider the points $(\delta'_n, F_\varphi(n))$ in $X_{\delta'_n}$;
- by property (E), we can take compact, countable, and clopen neighborhoods G_n of $(\delta'_n, F_\varphi(n))$ such that $G_n \subseteq \bigcup\{X_{\delta''} : x_{\delta''} \in B_n\}$. Observe that $\bigcup\{G_n : n \in \omega\}$ is closed in $(X_{<\delta}, \tau_{<\delta})$.

Let

$$U((\delta, \epsilon), E) = \{(\delta, \epsilon)\} \cup \bigcup\{G_n : n \in M^\epsilon \setminus E\}$$

for $E \in [\omega]^{<\omega}$, and let

$$\left\{U((\delta, \epsilon), E) : E \in [\omega]^{<\omega}\right\}$$

be a neighborhood base for (δ, ϵ). Note that if $\epsilon \neq \epsilon' < 2^\omega$ and $E = M^\epsilon \cap M^{\epsilon'}$, then $U((\delta, \epsilon), E) \cap U((\delta, \epsilon'), E) = \emptyset$; this yields that the resulting topology will be T_2. We need to check that the inductional assumptions still hold. Clearly, $U((\delta, \epsilon), E)$ is countable and compact; we need to check that it is clopen. Since $U((\delta, \epsilon), E) \cap X_{<\delta} = \bigcup\{G_n : n \in \omega\}$ is closed in $X_{<\delta}$, we need only to check that

$(\delta,\epsilon') \notin \overline{U((\delta,\epsilon),E)}$ for $\epsilon \neq \epsilon' < 2^\omega$. Let $F = M^\epsilon \cap M^{\epsilon'} \in [\omega]^{<\omega}$; then $U((\delta,\epsilon),E) \cap U((\delta,\epsilon'),F) = \emptyset$. Conditions (ii) and (iii) will clearly hold. We need to check (iv) property (E). For points in $X_{<\delta}$, this will still hold. Consider a Euclidean open B such that $x_\delta \in B$ and a new point: $(\delta,\epsilon) \in X_\delta$. Using the notations of the definition of a basic neighborhood for (δ,ϵ), there is some $m \in \omega$ such that $\bigcup\{B_n : n \geq m\} \subseteq B$. So for the neighborhood

$$G = \{(\delta,\epsilon)\} \cup \bigcup\{G_n : n \in M^\epsilon, n \geq m\},$$

we have that $G \subseteq \bigcup\{X_{\delta''} : x_{\delta''} \in B\}$, since $B_n \subseteq B$ for $n \geq m$, and hence $G_n \subseteq \bigcup\{X_{\delta''} : x_{\delta''} \in B_n\} \subseteq \bigcup\{X_{\delta''} : x_{\delta''} \in B\}$.

It is clear that (X,τ) is a locally countable, locally compact, and 0-dimensional space. It is straightforward by condition (iii) that each X_δ is closed discrete. Let $\pi(A) = \{\delta < 2^\omega : A \cap X_\delta \neq \emptyset\}$ and $\pi_0(A) = \{x_\delta : \delta \in \pi(A)\} \subseteq \mathbb{R}$ for $A \subseteq X$.

Claim 1. If $|\pi(A)| > \omega$ for $A \subseteq X$, then there are 2^ω many $\delta < 2^\omega$ such that $|X_\delta \cap A'| = 2^\omega$.

Proof of Claim 1: There is $\alpha < 2^\omega$ such that $C_\alpha \subseteq \overline{\pi_0(A)}$ (Euclidean closure taken) and $\beta < 2^\omega$ such that $Q_\alpha^\beta \subseteq \pi_0(A)$. Let

$$D = \big\{\delta < 2^\omega : \exists\gamma < 2^\omega\big(t_\delta = (C_\alpha, Q_\alpha^\beta, \gamma)\big) \\ \text{and } \forall\delta' < 2^\omega(x_{\delta'} \in Q_\alpha^\beta \Rightarrow \delta' < \delta)\big\}.$$

Take a $\delta \in D$. Clearly, we did not define X_δ to be discrete since all points in Q_α^β are of the form $x_{\delta'}$ where $\delta' < \delta$. So at step δ in the induction, we chose some convergent sequence $\{x_{\delta'_n} : n \in \omega\}$ from Q_α^β where $\delta'_n < \delta$. Let $F : \omega \to 2^\omega$ such that $(\delta'_n, F(n)) \in A$. There is some $\varphi \in 2^\omega$ such that $F = F_\varphi$. We claim that $\{(\delta,\epsilon) \in X_\delta : \exists\rho < 2^\omega : h(\epsilon) = (\varphi,\rho)\} \subseteq A'$. For such a point (δ,ϵ), we used $F_\varphi = F$ for the definition of basic neighborhoods; thus, the set $\{(\delta'_n, F(n)) : n \in \omega\} \subseteq A$ accumulates to (δ,ϵ).

Claim 2. X satisfies $(**)$.

Proof of Claim 2: Let $A \in [X]^\kappa$ such that κ is an uncountable, regular cardinal. If there is a $\delta < 2^\omega$ such that $|A \cap X_\delta| = \kappa$, we are done. Otherwise, $|\pi(A)| > \omega$ since $|A \cap X_\delta| < \kappa$ for all $\delta < 2^\omega$. By Claim 1, $\overline{A}$ intersects (continuum) many closed discrete sets X_δ, in 2^ω many points.

Claim 3. X is not linearly D.

Proof of Claim 3: Suppose that $D \subseteq X$ is closed discrete. Then $\pi(D)$ is countable by Claim 1. Hence, there is no closed discrete set which is big for the open cover $\{X_{<\delta} : \delta < 2^\omega\}$. Thus, X is not linearly D by Theorem 2.4.

This completes the proof of this theorem. $\square$

Remark 4.6. Peter J. Nyikos [10] gave an example of a space T which is not a D-space; however, $e(F) = L(F)$ for every closed $F \subseteq T$. From [10, Theorem 1.11], one can see that T is linearly D (use the characterization of linear D property by Theorem 2.4). Applying Claim 1, just above, to our last construction, we get the following corollary.

Corollary 4.7. *There exists a Hausdorff space X of cardinality 2^ω such that X is locally countable, locally compact, 0-dimensional, not linearly D; however, $e(F) = L(F)$ for every closed subset $F \subseteq X$.*

5. Consistently on property D and aD

Our main goal in this section is to construct a space which is not linearly D, but every closed subset of it is irreducible, hence aD by Theorem 2.2.

We will use the following set-theoretical assumption.

($\diamondsuit^*$) There is a $\diamondsuit^*$-*sequence*, meaning that there exists an $\{\mathcal{A}_\alpha : \alpha \in \lim(\omega_1)\}$ such that $\mathcal{A}_\alpha \subseteq [\alpha]^\omega$ is countable, and for every $X \subseteq \omega_1$, there is a club $C \subseteq \omega_1$ such that $X \cap \alpha \in \mathcal{A}_\alpha$ for all $\alpha \in C$.

Before proving the theorem, we need the following easy claim about maximal almost disjoint families (MAD, in short).

Claim 5.1. *If $\{N_i : i \in \omega\} \subseteq [\omega]^\omega$, then there is a MAD family $\mathcal{M} \subseteq [\omega]^\omega$ of size 2^ω such that for all $M \in \mathcal{M}$ and $i \in \omega$: $|M \cap N_i| = \omega$.*

Proof: We will construct the MAD family $\mathcal{M}$ on $\mathbb{Q}$. We can suppose that each N_i is dense in $\mathbb{Q}$. Let $\mathbb{R} = \{x_\alpha : \alpha < 2^\omega\}$ and for all $\alpha < 2^\omega$, let $S_\alpha \subseteq \mathbb{Q}$ such that S_α is a convergent sequence with limit point x_α, and $|S_\alpha \cap N_i| = \omega$ for all $i \in \omega$. Then $\mathcal{S} = \{S_\alpha : \alpha < 2^\omega\}$ is almost disjoint. Let $\mathcal{T} = \{T_\alpha : \alpha < \lambda\} \subseteq [\mathbb{Q}]^\omega$ such that $\mathcal{S} \cup \mathcal{T}$ is MAD. Then $\mathcal{M} = \{S_\alpha \cup T_\alpha : \alpha < \lambda\} \cup \{S_\alpha : \lambda \leq \alpha < 2^\omega\}$ is a MAD family with the desired property. $\square$

Theorem 5.2. *Suppose* ($\diamondsuit^*$). *There is a locally countable, locally compact, 0-dimensional* T_2 *space* X *of size* ω_1 *such that* X *is not linearly* D*; however, every closed subset* $F \subseteq X$ *is irreducible, and equivalently,* X *is an* aD*-space.*

Proof: We will define a topology on $X = \omega_1 \times \omega_1$. Let $X_\alpha = \{\alpha\} \times \omega_1$ and $X_{<\alpha} = \alpha \times \omega_1$ for $\alpha < \omega_1$. $\square$

Definition 5.3. The set $A \in [X]^\omega$ *runs up to* $\alpha < \omega_1$ if and only if $A = \{(\alpha_n, \beta_n) : n \in \omega)\} \subseteq X_{<\alpha}$ such that $\alpha_0 \leq ... \leq \alpha_n \leq ...$ and $\sup\{\alpha_n : n \in \omega\} = \alpha$.

Note that if $A \subseteq X$ runs up to some $\alpha < \omega_1$, then $A \cap X_\beta$ is finite for all $\beta < \omega_1$.

We need the following consequence of ($\diamondsuit^*$). Let $\pi(A) = \{\alpha \in \omega_1 : A \cap X_\alpha \neq \emptyset\}$ for $A \subseteq X$.

Claim 5.4. ($\diamondsuit^*$) *There exists a sequence* $\{A_\alpha : \alpha \in \lim(\omega_1)\} \subseteq [X]^\omega$ *with* $A_\alpha = \bigcup\{A^n_\alpha : n \in \omega\}$ *for all* $\alpha \in \lim(\omega_1)$ *such that*

(1) $|A^n_\alpha| = \omega$ *for all* $n \in \omega$*;*
(2) A_α *runs up to* α*;*
(3) *for all* $Y \subseteq X$*, if* $|\pi(Y)| = \omega_1$*, then there exists club* $C \subseteq \omega_1$ *such that for all* $\alpha \in C$ *there exists* $n \in \omega(A^n_\alpha \subseteq Y)$*.*

Proof: Let $\{\mathcal{A}_\alpha : \alpha \in \lim(\omega_1)\}$ denote a $\diamondsuit^*$-sequence. Let $i : \omega_1 \times \omega_1 \to \omega_1$ denote a bijection which maps $\big((\alpha+1) \times (\alpha+1)\big) \setminus (\alpha \times \alpha)$ to $\omega \cdot (\alpha + 1) \setminus \omega \cdot \alpha$. Let

$$\widetilde{\mathcal{A}}_\alpha = \{i^{-1}(A) : A \in \mathcal{A}_{\omega\cdot\alpha}, \sup\big(\pi(i^{-1}(A))\big) = \alpha\}$$

and let $A_\alpha = \bigcup\{A^n_\alpha : n \in \omega\}$ such that

(1) $|A^n_\alpha| = \omega$ for all $n \in \omega$;
(2) A_α runs up to α;
(3)$'$ for all $B \in \widetilde{\mathcal{A}}_\alpha$, there is $n \in \omega$ such that $A^n_\alpha \subseteq B$,

for all $\alpha \in \lim(\omega_1)$. We claim that the sequence $\{A_\alpha : \alpha \in \lim(\omega_1)\}$ has the desired properties. Let $Y \subseteq X$ such that $|\pi(Y)| = \omega_1$. There is some club $C_0 \subseteq \omega_1$ such that $Y \cap X_{<\alpha} \subseteq \alpha \times \alpha$ for $\alpha \in C_0$. There is some club $C_1 \subseteq \omega_1$ such that $\alpha \cap i[Y] \in \mathcal{A}_\alpha$ for $\alpha \in C_1$. Let $C_2 = \{\alpha < \omega_1 : \omega \cdot \alpha \in C_1\}$; clearly, C_2 is a club. Let $C = C_0 \cap C_2 \cap \pi(Y)'$. Fix some $\alpha \in C$. Then $\omega \cdot \alpha \cap i[Y] = A$ for some $A \in \mathcal{A}_{\omega\cdot\alpha}$; thus, $i[Y \cap X_{<\alpha}] = A$ since $\omega \cdot \alpha = i[\alpha \times \alpha]$ and $Y \cap X_{<\alpha} \subseteq \alpha \times \alpha$. Hence, $i^{-1}(A) = Y \cap X_{<\alpha}$ and $i^{-1}(A) \in \widetilde{\mathcal{A}}_\alpha$

because $\alpha \in \pi(Y)'$. Thus, there is $n \in \omega$ such that $A_\alpha^n \subseteq Y$ by $(3)'$. $\square$

Let $\{A_\alpha : \alpha \in \lim(\omega_1)\} \subseteq [X]^\omega$ denote a sequence with $A_\alpha = \bigcup\{A_\alpha^n : n \in \omega\}$ for $\alpha \in \lim(\omega_1)$ from Claim 5.4. We want to define the topology on X such that

- X_α is closed discrete for all $\alpha < \omega_1$;
- $X_{<\alpha}$ is open for all $\alpha \in \omega_1$,
- if $A \in [X]^\omega$ runs up to α, then A has an accumulation point in X_α;
- $X_\alpha \subseteq \overline{A_\alpha^n}$ for all $\alpha \in \lim(\omega_1)$ and $n \in \omega$.

Let $\mathcal{M}_\alpha \subseteq [A_\alpha]^\omega$ denote a MAD family on A_α for $\alpha \in \lim(\omega_1)$ such that $|M \cap A_\alpha^n| = \omega$ for all $M \in \mathcal{M}_\alpha$ and $n \in \omega$; such an $\mathcal{M}_\alpha$ exists by Claim 5.1. Enumerate $\mathcal{M}_\alpha = \{M_\alpha^\beta : \beta < \omega_1\}$.

We define topologies $\tau_{<\alpha}$ on $X_{<\alpha}$ by induction on $\alpha < \omega_1$ such that $\tau_{<\alpha} \cap \mathcal{P}(X_{<\beta}) = \tau_{<\beta}$ for all $\beta < \alpha < \omega_1$. This way, we will get a topology τ on X if we take $\cup\{\tau_{<\alpha} : \alpha < \omega_1\}$ as a base.

Suppose $\alpha < \omega_1$ and we have defined the topology $(X_{<\alpha}, \tau_{<\alpha})$ such that

(i) $(X_{<\alpha}, \tau_{<\alpha})$ is a locally countable, locally compact, 0-dimensional T_2 space,
(ii) for all $\alpha' < \alpha$ and $x \in X_{\alpha'}$, there is some neighborhood G of x such that $G \cap X_{\alpha'} = \{x\}$,
(iii) $(\alpha_0, \alpha_1] \times \omega_1 \subseteq X_{<\alpha}$ is clopen for all $\alpha_0 < \alpha_1 < \alpha$.

If $\alpha \in \omega_1 \setminus \lim(\omega_1)$, then let X_α be discrete. Suppose $\alpha \in \lim(\omega_1)$ and let us enumerate $\{F \subseteq X_{<\alpha} \setminus A_\alpha : F \text{ runs up to } \alpha\}$ as $\{F_\alpha^\beta : \beta < \omega_1\}$.

Definition 5.5. A subspace $A \subseteq T$ of a topological space T is *completely discrete* if and only if there is a discrete family of open sets $\{G_a : a \in A\}$ such that $a \in G_a$ for all $a \in A$.

The following claim will be useful later.

Claim 5.6. *Suppose that $A = \{(\alpha_n, \beta_n) : n \in \omega\} \subseteq X$ runs up to α. Then A is completely discrete in $X_{<\alpha}$, hence closed discrete.*

Proof: Let $G_0 = (0, \alpha_0] \times \omega_1$ and $G_{n+1} = (\alpha_n, \alpha_{n+1}] \times \omega_1$ for $n \in \omega$. G_n is open for all $n \in \omega$ by inductional hypothesis (iii). Note that $\{G_n : n \in \omega\}$ is a discrete family of open sets such that $A \cap G_n$ is finite for all $n \in \omega$. Let $\mathcal{G}_n$ denote a finite, disjoint family of clopen subsets of G_n such that for all $a \in A \cap G_n$, there is exactly one $G \in \mathcal{G}_n$ such that $a \in G$. Then the discrete family $\cup\{\mathcal{G}_n : n \in \omega\}$ shows that A is completely discrete. $\square$

In step $\alpha \in \lim(\omega_1)$, we define the neighborhoods of points in $X_\alpha = \{(\alpha, \beta) : \beta < \omega_1\}$ by induction on $\beta < \omega_1$ such that

(a) $X_{<\alpha} \cup \{(\alpha, \beta') : \beta' \leq \beta\}$ is locally countable, locally compact, and 0-dimensional T_2;
(b) there is some neighborhood U of (α, β) such that $U \cap A_\alpha \subseteq M_\alpha^\beta$;
(c) M_α^β converges to (α, β);
(d) F_α^β accumulates to (α, β') for some $\beta' \leq \beta$.

We need the following lemma to carry out the induction on $\beta < \omega_1$.

Lemma 5.7. *Suppose that $(T \cup S, \tau)$ is a locally countable, locally compact, and 0-dimensional T_2 space such that T is open and S is countable. Let $D = \{d_n : n \in \omega\} \subseteq T$ be closed discrete in $T \cup S$ and completely discrete in T. Let $r \notin T \cup S$. Then there is a topology ρ on $R = T \cup S \cup \{r\}$ such that*

- *(R, ρ) is locally countable, locally compact, and 0-dimensional T_2,*
- *$\rho|_{(T \cup S)} = \tau$,*
- *D converges to r and $r \notin \overline{S}$ in (R, ρ).*

Proof: Suppose that $d_n \in G_n$ such that $\{G_n : n \in \omega\}$ is a family of open sets which is discrete in T. For each $n \in \omega$, let $\{B_i^n : i \in \omega\}$ denote a neighborhood base of d_n such that

- $G_n \supseteq B_0^n \supseteq B_1^n \supseteq ...$, and
- B_i^n is countable, compact, and clopen for all $n, i \in \omega$.

Since $S \cap D = \emptyset$, there is some clopen neighborhood U_s of each $s \in S$ such that $U_s \cap D = \emptyset$. There is $g_s : \omega \to \omega$ such that

$$U_s \cap B_{g_s(n)}^n = \emptyset \text{ for all } n \in \omega.$$

Since S is countable, there is $g : \omega \to \omega$ such that for all $s \in S$, there is some $N \in \omega$ such that $g_s(n) \leq g(n)$ for all $n \geq N$. Define the topology ρ on R as follows. Let

$$B_N = \{r\} \cup \bigcup\{B_{g(n)}^n : n \geq N\} \text{ and } \mathcal{B} = \{B_N : N \in \omega\}.$$

Let ρ be the topology on R generated by $\tau \cup \mathcal{B}$.

Clearly, $\rho|_{(T\cup S)} = \tau$. We claim that (R,ρ) is locally countable, locally compact, and 0-dimensional. Since $\mathcal{B}$ is a neighborhood base for r, it suffices to prove that each $B \in \mathcal{B}$ is countable, compact (trivial), and clopen. Let $N \in \omega$; then B_N is clopen in T since $\bigcup\{B^n_{g(n)} : n \in \omega\}$ is a family of clopen sets which is discrete in T guaranteed by the discrete family $\{G_n : n \in \omega\}$. Let $s \in S$. There is $N \in \omega$ such that $U_s \cap B^n_{g(n)} = \emptyset$ for $n \geq N$. There is some neighborhood $V \in \tau$ of s such that $V \cap \bigcup\{B^n_{g(n)} : n < N\} = \emptyset$ since s is not in the closed set $\bigcup\{B^n_{g(n)} : n < N\}$. Thus, $(U_s \cap V) \cap B_N = \emptyset$. This proves that B_N is clopen.

We claim that (R,ρ) is T_2. Let $s \in S$; then there is $N \in \omega$ such that $U_s \cap B^n_{g(n)} = \emptyset$ for $n \geq N$, and thus $B_N \cap U_s = \emptyset$. As noted before, $B_N \cap T$ is closed and clearly $\bigcap\{B_N \cap T : N \in \omega\} = \emptyset$. This yields that any point $t \in T$ and r can be separated; thus, (R,ρ) is T_2.

Clearly, D converges to r and $S \cap B = \emptyset$ for any $B \in \mathcal{B}$; thus, $r \notin \overline{S}$. $\square$

Suppose we are in step $\beta < \omega_1$ and we defined the neighborhoods of points in $X_{<\alpha} \cup \{(\alpha,\beta') : \beta' < \beta\}$. We use Lemma 5.7 to define the neighborhoods of $r = (\alpha,\beta)$. Let $T = X_{<\alpha}$ and $S = \{(\alpha,\beta') : \beta' < \beta\} \cup (A_\alpha \setminus M^\beta_\alpha)$. Note that $F^\beta_\alpha \cup M^\beta_\alpha$ runs up to α, and thus is closed and completely discrete in T by Claim 5.6. Also, M^β_α is closed discrete in $T \cup S$ by inductional hypothesis (b) for (α,β') where $\beta' < \beta$.

- If F^β_α accumulates to $x_{\beta'}$ for some $\beta' < \beta$, then let $D = M^\beta_\alpha$.
- If F^β_α is closed discrete in $T \cup S$, then let $D = M^\beta_\alpha \cup F^\beta_\alpha$.

Note that D is closed discrete in $T \cup S$. By Claim 5.7, we can define the neighborhoods of $r = (\alpha,\beta)$ such that the resulting space satisfies conditions (a), (b), (c), and (d). After carrying out the induction on β, the resulting topology on X_α clearly satisfies conditions (i), (ii), and (iii). This completes the induction.

As a base, the family $\bigcup\{\tau_{<\alpha} : \alpha \in \lim(\omega_1)\}$ generates a topology τ on X which is locally countable, locally compact, and 0-dimensional T_2. Observe that X_α is closed discrete, and $X_{<\alpha}$ is open for all $\alpha < \omega_1$ (by inductional hypotheses (ii) and (iii)) .

Claim 5.8. *Suppose that $F \subseteq X$ runs up to some $\alpha \in \lim(\omega_1)$. Then there is some $\beta < \omega_1$ such that F accumulates to (α, β). Equivalently, if $G \subseteq X$ is open and $X_\alpha \subseteq G$, then there is some $\alpha' < \alpha$ such that $(\alpha', \alpha] \times \omega_1 \subseteq G$.*

Proof: There is some $\beta < \omega_1$ such that $F = F_\alpha^\beta$. Thus, by inductional hypothesis (d), there is some $\beta' \leq \beta$ such that F accumulates to (α, β'). □

Claim 5.9. *X is not linearly D.*

Proof: If $D \subseteq X$ is closed discrete, then $\pi(D)$ is finite by Claim 5.8. Thus, there is no big closed discrete set for the cover $\{X_{<\alpha} : \alpha < \omega_1\}$. □

Our next aim is to prove that all closed subspaces of X are irreducible.

Claim 5.10. *If $|\pi(F)| = \omega$ for a closed $F \subseteq X$, then F is a D-space, hence irreducible.*

Proof: Since $F = \cup\{F \cap X_\alpha : \alpha \in \pi(F)\}$ is a countable union of closed discrete sets, F is a D-space by Proposition 3.7. We mention that if the ONA U on F has closed discrete kernel D, then we get an irreducible cover by taking the open refinement

$$\{(U(d) \setminus D) \cup \{d\} : d \in D\}.$$

□

Claim 5.11. *If $|\pi(A)| = \omega_1$ for $A \subseteq X$, then there is a club $C \subseteq \omega_1$ such that $C \times \omega_1 \subseteq A'$. As a consequence, if $\pi(U)$ is stationary for the open $U \subseteq X$, then there is some $\alpha < \omega_1$ such that $X \setminus U \subseteq \alpha \times \omega_1$.*

Proof: There is a club $C \subseteq \omega_1$ by Claim 5.4 such that for all $\alpha \in C$, there is $n \in \omega$ such that $A_\alpha^n \subseteq A$. We will prove that $X_\alpha \subseteq A'$ for $\alpha \in C$. Take any point $(\alpha, \beta) \in X_\alpha$. $|M_\alpha^\beta \cap A_\alpha^n| = \omega$ for all $\beta < \omega_1$ by the construction of the MAD family $\mathcal{M}_\alpha$, and M_α^β converges to (α, β) by inductional hypothesis (c). Thus, A_α^n accumulates to (α, β), and hence $X_\alpha \subseteq A'$. □

Claim 5.12. *If $|\pi(F)| = \omega_1$ for a closed $F \subseteq X$, then F is irreducible.*

Proof: Take an open cover of F, say $\mathcal{U}$. We can suppose that we refined it to the form $\mathcal{U} = \{U(x) : x \in F\}$, where $U(x)$ is a neighborhood of $x \in F$. From Claim 5.11, we know that there is some club $C \subseteq \omega_1$ such that $C \times \omega_1 \subseteq F$. For $\alpha \in C$, define the open set $G_\alpha = \cup\{U(x) : x \in X_\alpha\}$. For every $\alpha \in C$, there is some $\delta(\alpha) < \alpha$ such that $(\delta(\alpha), \alpha] \times \omega_1 \subseteq G_\alpha$, by Claim 5.8. So there is some $\delta < \omega_1$ and a stationary $S \subseteq C$ such that $(\delta, \alpha] \times \omega_1 \subseteq G_\alpha$ for all $\alpha \in S$. Fix some $\delta_0 > \delta$ such that $X_{\delta_0} \subseteq F$. Let $S_0 = S \setminus (\delta_0 + 1)$. For all $\alpha \in S_0$, there is $d_\alpha \in X_\alpha \subseteq F$ such that $(\delta_0, \alpha) \in U(d_\alpha)$. Let us refine the sets $U_0(d_\alpha) = \big(U(d_\alpha) \setminus (\{\delta_0\} \times S_0)\big) \cup \{(\delta_0, \alpha)\}$ for all $\alpha \in S_0$; let $\mathcal{U}_0 = \{U_0(d_\alpha) : \alpha \in S_0\}$. Clearly, $\mathcal{U}_0$ is an open refinement of $\mathcal{U}$ which is minimal and $\{d_\alpha : \alpha \in \omega_1\} \subseteq \cup\mathcal{U}_0$. Since S_0 is stationary and $S_0 \subseteq \pi[\cup\mathcal{U}_0]$, we get that there is some $\gamma < \omega_1$ such that $F_1 = F \setminus \cup\mathcal{U}_0 \subseteq \gamma \times \omega_1$ by Claim 5.11. So by Claim 5.10, the closed set F_1 is a D-space, hence irreducible. Take a minimal open refinement of the cover $\{U(x) \setminus (\{\delta_0\} \times S_0) : x \in F_1\}$; let this be $\mathcal{U}_1$. The union $\mathcal{U}_0 \cup \mathcal{U}_1$ is an open refinement of $\mathcal{U}$ which covers F and is minimal. □

This proves that all closed subspaces of X are irreducible. Hence, X is an aD-space by Theorem 2.2.

Using again the strong result of Balogh, we can observe the following.

Proposition 5.13. *Suppose* MA. *Let X be a locally countable, locally compact space of cardinality $< 2^\omega$. Then the following are equivalent.*

(1) *X is a (linearly) D-space;*
(5) *X is an aD-space.*

Proof: (1) ⇒ (5) trivially.

Suppose that X is an aD-space. It is enough to show that X does not contain any perfect preimage of ω_1. Since property aD is hereditary to closed sets, any closed countably compact subspace is compact. By Claim 1 of the proof of Proposition 3.8, there is no perfect preimage of ω_1 in X. □

Corollary 5.14. *The existence of a locally countable, locally compact space X of size ω_1 which is aD and nonlinearly D is independent of* ZFC.

However, the following remain open.

Problem 5.15. (1) *Is it consistent with* ZFC *that there exists a locally countable, locally compact space X of cardinality $< 2^\omega$ such that X is not (linearly) D, but is aD?*

(2) *Is there a* ZFC *example of a Tychonoff space X such that X is not (linearly) D, but is aD?*

$(2)'$ *Is there a* ZFC *example of a locally countable, locally compact (0-dimensional) T_2 space X such that X is not (linearly) D, but is aD?*

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