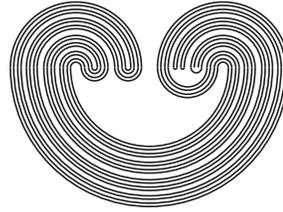


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## D-SPACES, TREES, AND AN ANSWER TO A PROBLEM OF BUZYAKOVA

by

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## D-SPACES, TREES, AND AN ANSWER TO A PROBLEM OF BUZYAKOVA

PETER J. NYIKOS

**ABSTRACT.** A ZFC example is given of a Tychonoff space in which the extent of every subspace equals its Lindelöf degree, yet the space is not a D-space, answering a question of Raushan Z. Buzyakova. The example is the tree of compact subsets of a stationary, co-stationary subset of  $\omega_1$ . A simple method is given of embedding any tree as a dense open subspace of a tree which is a D-space. Some classes of trees that are D-spaces are discussed, along with some open problems about D-spaces.

### 1. INTRODUCTION

This paper was motivated by the following question, posed by Raushan Z. Buzyakova [2, Question 3.6].

**Question 1.1.** If  $X$  is a space such that  $e(Y) = l(Y)$  for all subspaces  $Y$  of  $X$ , is  $X$  a D-space?

**Definition 1.2.** The *extent* of a space  $X$ , designated  $e(X)$ , is the supremum of the cardinalities of its closed discrete subspaces. The *Lindelöf degree* (*Lindelöf number*) of  $X$ , designated  $l(X)$ , is the least  $\kappa$  such that every open cover of  $X$  has a subcover of cardinality  $\leq \kappa$ .

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A well-known elementary fact is that  $e(X) \leq l(X)$  for all topological spaces  $X$ . But from now on, “space” will mean “Hausdorff space.” Moreover, the examples we give of trees are all Hausdorff (hence, Tychonoff).

**Definition 1.3.** A *neighborhood assignment* (*neighbornet*) on a space  $X$  is a family of sets indexed by the points of  $X$ , each one a neighborhood of the indexing point. A *D-space* (*dually discrete space*, respectively) is a space  $X$  such that for every neighbornet  $\mathcal{V} = \{V_x : x \in X\}$ , there is a closed discrete subset (discrete subset, respectively)  $D$  of  $X$  such that  $\{V_x : x \in D\}$  covers  $X$ .

In the above definition, we may confine our attention to those  $\mathcal{V}$  whose members are open — the *open neighborhood assignments*. This is because if a shrinking of a neighbornet has a [closed] discrete subspace associated with it as above, then so does the original neighbornet. Hence, one can also confine oneself to neighborhoods from a given base for the topology or a system of neighborhood bases for the points.

Despite the similarity in the definitions of the two concepts, the class of dually discrete spaces is radically larger than the class of D-spaces. But one thing the two classes share is the wide range of uncertainty as to which spaces do or do not belong. On the one hand, we do not know whether every subspace of a compact hereditarily Lindelöf space is dually discrete; on the other hand, we also do not know of a consistent example of  $\theta$ -refinable space that is not a D-space. In this respect, we have not advanced at all since Eric K. van Douwen and Washek F. Pfeffer [3] introduced the concept of D-spaces in 1979. They remarked that there was no satisfactory space that was known not to be a D-space, with “satisfactory” meaning “having a covering property at least as strong as metacompactness or supracompactness.”

Question 1.1 is the seventh of ten then-open problems about D-spaces that were repeated by Todd Eisworth in [5]. In section 2, we give a ZFC counterexample that is a tree with the interval topology.

**Definition 1.4.** A *tree* is a partially ordered set (poset) in which the set of predecessors of each element is well ordered.

If  $T$  is a tree and  $t \in T$ , then  $t^\downarrow = \{x \in T : x \leq t\}$  and  $\nabla_t(T)$  (also denoted  $\nabla_t$  or  $t^\uparrow$  if the tree is clear from context) is  $\{x \in T : t \leq x\}$ .

The *height* of  $t \in T$ , denoted  $ht(t)$ , is the order type of  $t^\downarrow$ , and if  $\alpha$  is an ordinal, then  $T(\alpha) = \{t : ht(t) = \alpha\}$ . Some authors write  $T_\alpha$  for  $T(\alpha)$ . The *height* of  $T$ , denoted  $ht(T)$ , is the least  $\alpha$  such that  $T(\alpha) = \emptyset$ .

**Definition 1.5.** If  $T$  is a tree, the *interval topology* (sometimes referred to simply as the “tree topology”) on  $T$  is the topology whose base is the set of all intervals of the form  $(s, t] = \{x \in T : s < x \leq t\}$  together with all singletons  $\{m\}$  such that  $m$  is a minimal element of  $T$ .

Section 3 gives some positive results on when a tree is a D-space, the general question of when is a tree a D-space in its interval topology, while in section 4 we discuss some open problems, first for trees and then for spaces in general.

## 2. THE MAIN COUNTEREXAMPLE

The author’s interest in Question 1.1 was kindled by a consistent example, due to Tetsuya Ishiu [9], of a space  $X$  that is not a D-space even though  $e(Y) = l(Y)$  for all *closed* subspaces. The counterexample to Question 1.1 presented here is done just using the usual (ZFC) axioms of set theory.

The following lemma has an easy proof.

**Lemma 2.1.** *A subset  $D$  of a tree  $T$  is closed discrete if and only if every infinite ascending sequence in  $D$  is unbounded above in  $T$ .*

**Lemma 2.2.** *A tree  $T$  has a cofinal closed discrete subspace if and only if it has a cofinal subset that is the countable union of antichains.*

*Proof:* If  $D$  is a closed discrete subspace, then Lemma 2.1 implies it is a subtree of height  $\leq \omega$ , and every level in a tree is obviously an antichain. Conversely, if  $\{A_n : n \in \omega\}$  is a set of antichains whose union is cofinal in a tree  $T$ , let  $D_n \subset A_n$  be defined by induction as follows.  $D_0 = A_0$ , and if  $D_n$  has been defined, let

$$D_{n+1} = \{t \in A_{n+1} : (\forall d \in D_0, \dots, D_n) \neg(t \leq d)\}.$$

Then  $D = \bigcup_{n=0}^{\infty} D_n$  is clearly cofinal, and any ascending chain in  $D$  is associated with strictly increasing subscripts and hence is unbounded above in  $T$ .  $\square$

**Theorem 2.3.** *If a tree is a D-space, then every branch is of countable cofinality, and the tree has a cofinal subset which is a closed discrete subspace.*

*Proof:* Let  $V_t = t^\perp$  for all points  $t \in T$ . If  $D$  is as in the definition of a D-space, then  $D$  is obviously a cofinal closed discrete subspace of  $T$ .

If  $B$  is a branch of  $T$ , let  $V_t = t^\perp$  for all  $t \in B$ , while if  $t \notin B$ , then let  $V_t = t^\perp$  if  $t^\perp$  does not meet  $B$ . Otherwise, by our Hausdorff assumption,  $B$  is closed, so we can let  $V_t$  be any interval  $(s, t]$  that does not meet  $B$ . Then if  $D$  is as before,  $D \cap B$  must be countable and cofinal in  $B$  by Lemma 2.1.  $\square$

We will return to the general question of when is a tree a D-space in section 3. For now, we just note that the converse of Theorem 2.3 is far from true. Any tree can be embedded as a closed subtree of a tree with a cofinal antichain, and every closed subspace of a D-space is a D-space. But there are examples of trees in which every branch is countable, but which are not D-spaces. A Souslin tree is a consistent example. Clearly, a Souslin tree does not have a countable cofinal subset, but every union of countably many antichains in a Souslin tree is countable.

Here is a very different, ZFC example of a non-D-space which gives a negative answer to Question 1.1.

**Example 2.4.** Let  $E$  be a stationary, co-stationary subset of  $\omega_1$ . Members of the tree  $T(E)$  are the compact subsets of  $E$ , ordered by end extension  $<_T$ . That is, if  $c_1$  and  $c_2$  are compact subsets of  $E$ , then  $c_1 <_T c_2$  if and only if  $c_1 \subset c_2$  and  $\alpha < \beta$  for all  $\alpha \in c_1$  and  $\beta \in c_2 \setminus c_1$ . Since  $E$  does not contain a club, every branch of  $T(E)$  is countable.

**Definition 2.5.** Call a tree *robust* if for every  $t \in T$  and every  $\alpha$  such that  $ht(t) < \alpha < ht(T)$ , there exists  $x \in T$  such that  $t < x$  and  $ht(x) = \alpha$ . In other words, each point of  $T$  has successors at every level above its own.

For our next theorem, we use a definition of “Baire” that refers to the logicians’ wedge topology (also known as the Alexandroff discrete topology [11]) on posets. This is the topology whose base is the set of all wedges  $\nabla_t = \{x \in T : x \geq t\}$ . This is not a  $T_2$  topology, but it is the topology logicians refer to when they use the expressions “dense,” “open,” and “Baire” in the context of trees. These have simple order-theoretic characterizations: A dense set in this topology is one that is cofinal and an open set is one that is upwards-closed. A Baire set is defined below.

**Definition 2.6.** A poset is  $\omega$ -*distributive* or *Baire* if every countable collection of cofinal, upwards-closed sets has cofinal intersection.

The following is well-known folklore. We need only the easy implication (2)  $\implies$  (1).

**Theorem 2.7.** *Let  $T$  be a robust tree of height  $\omega_1$  in which every chain is countable. The following are equivalent.*

- (1) *No subset of the form  $\nabla_t$  has a cofinal subset which is the countable union of antichains.*
- (2)  *$T$  is Baire.*
- (3) *Forcing with  $T$  cannot collapse  $\omega_1$ .*

**Lemma 2.8** ([6]).  *$T(E)$  is a robust tree of height  $\omega_1$ .*

**Theorem 2.9.**  *$T(E)$  is not a  $D$ -space.*

*Proof:* By Lemma 2.2, Theorem 2.3, Theorem 2.7, and Lemma 2.8, it is enough to show that  $T(E)$  is Baire. This is shown in [12, Lemma 9.12] where  $T(E)$  is called  $U(E)$ , except that the proof makes no mention of the essential ingredient that  $T(E)$  is robust, implicitly used in getting extensions arbitrarily far up inside a countable elementary submodel. (Compare the proof of Theorem 3.9 below.) □

**Theorem 2.10.**  *$T(E)$  has no Aronszajn subtrees.*

*Proof:* Let  $S$  be a subtree of  $T(E)$  in which every level is countable. If  $c \in S$ , then the level of  $c$  in  $S$  is no greater than its level in  $T$ , which, in turn, is no greater than  $\max(c)$ . Suppose  $S$  is uncountable; the following argument yields a contradiction.

For  $\xi \in \omega_1$ , let

$$\alpha(\xi) = \min\{\eta : \max(c) \leq \eta \text{ for all } c \text{ in the } \xi\text{th level of } S\}.$$

Let  $\alpha_0 = 0$ ,  $\alpha_{\nu+1} = \alpha(\alpha_\nu + 1)$ , and if  $\mu$  is a limit ordinal, let  $\alpha_\mu = \sup\{\alpha_\nu : \nu < \mu\}$ .

Then  $\{\alpha_\nu : \nu < \omega_1\}$  is a club, and so it meets the complement of  $E$  in a stationary set. If  $\nu$  is a limit ordinal and  $c$  is on the  $\alpha_\nu$ th level of  $S$ , then  $\alpha_\nu \in c$ ; but if  $\alpha_\nu \notin E$ , this is impossible.  $\square$

**Theorem 2.11.** *Let  $T$  be a tree. Exactly one of the following is true.*

- (1)  $T$  either has an uncountable branch or a Souslin subtree.
- (2) Every uncountable subset of  $T$  contains an antichain of the same cardinality.

*Proof:* If (1) fails, let  $S$  be an uncountable subset of  $T$ .

If  $|S| = \omega_1$ , we use the fact that any uncountable tree without an uncountable branch is either a Souslin tree or it contains an uncountable antichain.

If  $cf(|S|) > \omega_1$ , then some level must meet  $S$  in a set of cardinality  $|S|$ , and this is an antichain.

Finally, suppose  $|S|$  is singular of cofinality  $\omega$  or  $\omega_1$ . If  $cf(|S|) = \omega$ , let  $\{\kappa_n : n \in \omega\}$  be cofinal in  $|S|$ , with  $\kappa_0 > \omega_1$ , and let  $S' = \{x \in S : |\nabla_x(S)| < |S|\}$ .

**Case 1:**  $|S'| = |S|$ . In this case, if  $|\nabla_x(S)| < \kappa_n$  for some  $n$  and all  $x \in S'$ , then the minimal members of  $S'$  are an antichain of size  $|S|$ . Otherwise, pick the least  $\theta$  such that the  $\theta$ th level  $S'(\theta)$  of  $S'$  is infinite, and pick distinct  $t_n \in S'(\theta)$  such that  $|\nabla_{t_n}(S)| > \kappa_n$ . Since  $\kappa_n > \omega_1$ , there must be an antichain  $L_n$  in  $\nabla_{t_n}(S)$  of cardinality  $> \kappa_n$ ; then  $\bigcup_{n=0}^{\infty} L_n$  is as desired.

**Case 2:**  $|S'| < |S|$ . Let  $U = S \setminus S'$ , let  $\theta$  be the least ordinal such that  $U(\theta)$  is infinite, and pick distinct  $t_n$  in  $U(\theta)$  and antichains  $L_n$  as above.

If  $cf(|S|) = \omega_1$ , just replace countable sets with sets of size  $\omega_1$  in the above constructions.  $\square$

**Corollary 2.12.** *If  $X$  is any subspace of  $T(E)$ , then  $e(X) = l(X)$ . Moreover, extent is always attained (except for  $\omega$  under some definitions of extent).*

$T(E)$  does not seem to satisfy any of the nice topological properties that van Douwen was interested in and which are featured in most of the other problems in [5]. Even the following question is open (and a negative answer seems likely).

**Problem 2.13.** *Is  $T(E)$  countably metacompact?*

One nice property that  $T(E)$  does have is realcompactness.

**Definition 2.14.** A *zero-set* of a space  $X$  is a set of the form  $f^{-1}\{0\}$  for some continuous function  $f : X \rightarrow \mathbb{R}$ . A *Z-filter* on a space  $X$  is a proper filter  $\mathcal{U}$  of the lattice of zero-sets. That is, if  $Z_0$  and  $Z_1$  are zero-sets in  $\mathcal{U}$ , then their intersection is in  $\mathcal{U}$ , every zero-set containing one in  $\mathcal{U}$  is itself in  $\mathcal{U}$ , and the empty set is not in  $\mathcal{U}$ .

A *Z-ultrafilter* is a Z-filter  $\mathcal{U}$  that is maximal among all Z-filters. Equivalently, if  $Z$  is a zero-set not in  $\mathcal{U}$ , then it is disjoint from some member of  $\mathcal{U}$ .

Given a collection  $\mathcal{S}$  of zero-sets of a space  $X$ , the Z-filter generated by  $\mathcal{S}$  is denoted  $\langle \mathcal{S} \rangle$  and is the upwards closure of the set of all finite meets of members of  $\mathcal{S}$  in the lattice of zero-sets.

**Definition 2.15.** Let  $\mathcal{U}$  be a Z-ultrafilter.  $\mathcal{U}$  is *free* if  $\bigcap \mathcal{U} = \emptyset$ , and it is *fixed* if it is not free. A space is *realcompact* if it is Tychonoff, and every Z-ultrafilter with the countable intersection property (c.i.p.) is fixed.

For the next theorem we need the following classical facts.

(1) If a space can be partitioned into clopen sets, every Z-ultrafilter on the set is naturally associated with an ultrafilter on the index set, and this ultrafilter has the c.i.p. if and only if the original Z-ultrafilter has it.

(2) [7, §12.2] No free ultrafilter on a set of cardinality smaller than the first measurable cardinal  $\mathfrak{M}_1$  can have the c.i.p., but there is a free ultrafilter with the c.i.p. on every discrete space of cardinality  $\geq \mathfrak{M}_1$ .

**Theorem 2.16.** *Let  $T$  be a tree. The following are equivalent.*

- *Every chain in  $T$  is countable, and  $|T| < \mathfrak{M}_1$ , and  $T$  is Hausdorff.*
- *$T$  is hereditarily realcompact.*

*Proof:* If  $T$  has an uncountable chain  $C$ , then the points of  $C$  on countable levels form a copy of  $\omega_1$ , which is not realcompact, because every countably compact, realcompact space is compact [7, Problem 5H2].

If  $T$  has no uncountable branches but its cardinality is  $\geq \mathfrak{M}_1$ , then, by regularity of measurable cardinals,  $T$  has a level of cardinality  $\geq \mathfrak{M}_1$  which is a closed discrete subspace, hence not realcompact by the classical facts mentioned above.

So suppose  $|T| < \mathfrak{M}_1$  and every chain in  $T$  is countable. The only way a tree can fail to be Hausdorff is for two or more points on a limit level to have the same set of predecessors. This is also the only way a basic open set can fail to be clopen; hence, every Hausdorff tree is zero-dimensional and Tychonoff. Since every first countable realcompact space is hereditarily realcompact [7, Corollary 8.15], we need only show that every  $\mathcal{Z}$ -ultrafilter on  $T$  with the c.i.p. is fixed. Let  $\mathcal{U}$  be such an ultrafilter and let  $C = \{t \in T : \nabla_t \in \mathcal{U}\}$ . Then  $C$  is clearly a downwards closed chain. Because  $\mathcal{U}$  has the c.i.p. and every chain in  $T$  is countable,  $C$  has a greatest element  $t_0$ . Then  $\{t_0\}$  is in  $\mathcal{U}$  because its complement in  $\nabla_{t_0}$  can be partitioned into  $\leq \mathfrak{c}$  many clopen sets of the form  $\nabla_s$ ,  $s > t_0$ . Since  $\mathfrak{c}$  is smaller than the first measurable cardinal,  $\mathcal{U} = \langle \{t_0\} \rangle$  and is thus fixed.  $\square$

**Corollary 2.17.**  $T(E)$  is realcompact.

*Proof:* It remains only to show that  $T(E)$  is Hausdorff. Note that every non-isolated point is a compact subset  $c$  of  $\omega_1$  with a greatest element  $\alpha$  which is a limit ordinal in the closure of the ordinals in  $c$  that precede it. The predecessors of  $c$  in  $T(E)$  are initial segments of  $c$  with greatest elements. No other compact subset of  $\omega_1$  can have the same initial segments, so  $T(E)$  is Hausdorff by the criterion in the third paragraph in the proof of Theorem 2.16.  $\square$

### 3. SOME CLASSES OF TREES THAT ARE D-SPACES

Where trees with the interval topology are concerned, dual discreteness no longer causes a problem.

**Theorem 3.1** ([4]). *Every tree is dually discrete in the interval topology.*

However, we will mention some problems involving dual discreteness after presenting some classes of trees that are D-spaces.

**Definition 3.2.** Let  $L$  be a totally ordered set. A tree  $T$  is  $L$ -special if there is a  $<$ -preserving function from  $T$  to  $L$ . A tree is special if it is a countable union of antichains.

As is well known, a tree is special if, and only if, it is  $\mathbb{Q}$ -special.

**Theorem 3.3.** *Every special tree is hereditarily a D-space.*

*Proof:* A routine induction shows that if  $X$  is the countable union of closed D-subspaces, then  $X$  is a D-space. The rest is immediate from the fact that an antichain in a tree is closed discrete, and hence a D-space.  $\square$

A similar proof shows that every locally compact, subparacompact space is a D-space. Simply use the fact that every compact space is a D-space and the definition of “subparacompact” that every open cover has a  $\sigma$ -discrete closed refinement.

When this paper was originally submitted, the following questions were still open problems.

**Question 3.4.** Is every  $\mathbb{R}$ -special tree (a) a D-space? (b) hereditarily a D-space?

**Question 3.5.** If a tree is hereditarily a D-space, is it (a)  $\mathbb{R}$ -special? (b) special?

Both parts of both questions were answered by the following result of the author’s Ph.D. student, Heather Cheatum, who showed the following theorem.

**Theorem 3.6.** *Let  $\mathbb{H}$  be the lexicographically ordered Hilbert cube. Then every  $\mathbb{H}$ -special tree is hereditarily a D-space.*

Even with  $\mathbb{H}$  replaced by  $[0, 1]_{lex}^2$ , the lexicographically ordered unit square, the modified Theorem 3.6 implies that both parts of Question 3.5 have negative answers, since what is called  $\sigma\mathbb{R}$  in [12] is  $[0, 1]_{lex}^2$ -special, but not  $\mathbb{R}$ -special.

The following is a class of trees that are D-spaces, but not necessarily hereditarily D.

**Definition 3.7.** A tree is *branch complete* if every branch (i.e., every maximal chain) has a greatest element.

The *branch completion* of a tree  $T$  is the tree  $\tilde{T}$  obtained by adjoining a point  $t_B$  at the end of each branch  $B$  of  $T$ . That is,  $\tilde{T} = T \cup \{t_B : B \text{ is a branch of } T\}$  and if  $t_1, t_2 \in \tilde{T}$ , then  $t_1 \leq_{\tilde{T}} t_2$  if and only if either  $t_i \in T$  for  $i = 1, 2$  and  $t_1 \leq_T t_2$  or  $t_1 \in T$  and  $t_2 = t_B$  for some branch  $B$  such that  $t_1 \in B$ .

Clearly, the branch completion of every tree is branch complete. As a partial converse, if  $T$  is a branch complete Hausdorff tree in which all maximal points are non-isolated, then  $T$  is naturally isomorphic to the branch completion of its non-maximal members.

For simplicity, we write  $\tilde{T}(E)$  for  $\widetilde{T(E)}$  below. It is a tree in which every chain is countable, and  $T(E)$  is a dense, downwards closed (hence, open) subtree which is not a D-space. However,  $\tilde{T}(E)$  is a D-space.

**Theorem 3.8.** *Every branch-complete, Hausdorff tree is a D-space.*

*Proof:* Let  $T$  be branch complete. Then every closed subtree of  $T$  is also branch complete. Let  $\{V_t : t \in T\}$  be an open neighbornet and let  $D_0$  be the set of all maximal points of  $T$ . If  $D_\beta$  has been defined for all  $\beta < \alpha$ , let  $D_\alpha$  be the set of all maximal points of

$$T \setminus \bigcup_{\beta < \alpha} (\bigcup \{V_t : t \in D_\beta\}).$$

This induction ends when  $\bigcup_{\beta < \alpha} (\bigcup \{V_t : t \in D_\beta\})$  covers  $T$ , and we will be done once we show that  $D = \bigcup_{\beta < \alpha} D_\beta$  is closed discrete. In fact, the following proof shows that every chain in  $D$  is finite. Clearly, every  $D_\beta$  is an antichain. If  $d_0 < d_1 < \dots < d_n < \dots$  in  $D$ , and  $d_i \in D_{k(i)}$ , then the  $k(i)$  form a decreasing sequence which terminates after finitely many steps.  $\square$

In the case  $T = \tilde{T}(E)$ , we seem to have very little left of  $T$  after the first step.

**Theorem 3.9.** *Let  $\mathcal{U} = \langle U_t : t \in \tilde{T}(E) \rangle$  be an open neighbornet and let  $D_0$  be the antichain  $\tilde{T}(E) \setminus T(E)$ . Let  $U = \bigcup \{U_d : d \in D_0\}$  and let  $S = \tilde{T}(E) \setminus U = T(E) \setminus U$ . Then  $S$  is special.*

*Proof:* Otherwise, let  $S^* = \{s \in S : \nabla_s(S) \text{ is special}\}$ . Then  $S^*$  is special, because the minimal elements form an antichain. Let  $S' = S \setminus S^*$ . We claim that  $S' = \emptyset$ .

Suppose  $S' \neq \emptyset$ . Then  $ht(S') = \omega_1$  and  $S'$  is robust. Let  $N_0$  be a countable elementary submodel of a sufficiently large fragment of the universe containing  $E$ ,  $T(E)$ , and  $\mathcal{U}$ . Then  $\tilde{T}(E)$ ,  $D_0$ , and  $U$  are also elements of  $N_0$ . Let  $\{N_\alpha : \alpha < \omega_1\}$  be a continuous  $\in$ -chain of countable elementary submodels. Let  $\delta \notin E$  be such that  $N_\delta \cap \omega_1 = \delta$ , and let  $\alpha_n \nearrow \delta$ . Note that  $\max(t) < \delta$  for all  $t \in N_\delta$  and that  $\alpha_n \in N_\delta$  for each  $n$ .

Let  $x_0 \in S' \cap N_\delta$ . By elementarity, there exists  $x_1 \in S' \cap N_\delta$  such that  $x_1 \geq x_0$  and the height of  $x_1$  in  $S'$  is at least  $\alpha_1$  (but  $< \delta$ ). In general, with  $x_n$  defined, let  $x_{n+1} \geq x_n$ ,  $x_{n+1} \in S' \cap N_\delta$ , and  $\alpha_{n+1} \leq ht_{S'}(x_{n+1}) (< \delta)$ . Then the set of all  $x_n$  is not bounded above in  $T(E)$  because the sequence of maximum members of the  $x_n$  converges to  $\delta \notin E$ , and hence it determines a branch  $B$  of  $T(E)$ . But then  $t_B$  is in the closure of  $\{x_n : n \in \omega\}$ , and  $t_B \in D_0$ , contradicting “ $x_n \in S$  for all  $n$ .”  $\square$

From either Theorem 3.8 or Theorem 3.3 and Theorem 3.9, we obtain the following.

**Corollary 3.10.**  *$\tilde{T}(E)$  is a  $D$ -space, yet has a dense, downwards closed (hence open) subtree  $T(E)$  which is not a  $D$ -space.*

The induction in the proof of Theorem 3.8 need not end at  $\omega$ . Remarkably enough, given any cardinal number  $\kappa$ , it is not hard to construct trees by induction in which every chain is finite, and for which the process does not terminate before some stage  $\alpha$  such that  $\kappa \leq \alpha$ . So even if every chain in the  $S$  of Theorem 3.9 is finite, we could have  $\mathfrak{c}$ -many steps still to go.

#### 4. OPEN PROBLEMS AND ONE MORE CONSTRUCTION

Cheatum’s Theorem 3.6 suggests the following problems.

**Problem 4.1.** *Let  $\alpha$  be an ordinal and let  $L_\alpha = [0, 1]^\alpha$  with the lexicographic order. (For example,  $\mathbb{H} = L_\omega$ .) Is every  $L_\alpha$ -special tree a  $D$ -space for all countable  $\alpha$ ?*

**Problem 4.2.** *What implications hold between the following statements for arbitrary Hausdorff trees  $T$ ?*

- (1)  $T$  is  $L_\alpha$ -special for some countable ordinal  $\alpha$ .
- (2)  $T$  is quasi-metrizable.
- (3)  $T$  is hereditarily  $D$ .

- (4) *Every subtree of  $T$  has a cofinal subset that is the countable union of antichains.*

The author has shown that  $(1) \implies (2) \implies (4)$ , and he conjectures that  $(1) \implies (3) \implies (4)$ . What makes these implications nontrivial is the fact that the relative topology on a subtree is not always the interval topology. Indeed, the subtree of successor levels is a closed discrete space in any tree, and it is an easy exercise to embed any tree as the set of isolated points in another tree. In contrast, the implication  $(1) \implies (2)$  in the following problem is a corollary of Theorem 2.3 and the fact that a closed subspace of a D-space is a D-space, so that only the reverse implication is open.

**Problem 4.3.** *Are the following equivalent for any Hausdorff tree  $T$ ?*

- (1)  *$T$  is a D-space.*
- (2) *Every closed subtree of  $T$  has a cofinal subset that is a countable union of antichains.*

In comparing this problem with Theorem 2.3, note that a branch of a Hausdorff tree is a closed subspace.

The following construction justifies a remark following Theorem 2.3.

**Example 4.4.** Given a tree  $T$ , let  $T^* = \{t^* : t \in T\}$ , where  $t^* \notin T$  satisfies  $t < t^*$  but  $t^*$  is incomparable to any point of  $\nabla_t \setminus \{t\}$  and to any other  $s^* \neq t^*$ . In other words,  $\nabla_{t^*}(T^*) = \{t^*\}$  and  $(t^*)^\downarrow = t^\downarrow \cup \{t\}$ . Then  $T^* \setminus T$  is a cofinal antichain of  $T^*$ , and  $T$  is closed in  $T^*$ .

Finally, Problem 4.2 suggests some general questions, beginning with the following.

**Problem 4.5.** [**Problem 4.6.**] *Is every [non-Archimedeanly] quasi-metrizable space (a) a D-space or (b) dually discrete?*

Problem 4.6 is also motivated by the fact [8, Theorem 10.3] that non-Archimedeanly quasi-metrizable spaces are those spaces with  $\sigma$ -interior-preserving bases, and an affirmative answer to Problem 4.6 would be a generalization of the result (see [1] and [5]) that all spaces with  $\sigma$ -point-finite bases (more generally, all spaces with point-countable bases) are D-spaces.

At the present time, we even have difficulty distinguishing between problems 4.5 and 4.6. There is essentially only one known example of a space that is quasi-metrizable but not non-Archimedeanly quasi-metrizable, the Kofner plane [10] and [8, p. 490]. It is apparently not known whether this space is a D-space.

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