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ESSENTIALLY PSEUDORADIAL SPACES

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Abstract. The concept of essential sequence and essential space is given within the class of pseudoradial spaces. This concept tries to clarify the behaviour of convergent (long) sequences. Various examples and implications are proved among classes of pseudoradial (and radial, semiradial, almost radial) essential spaces. It is shown that essentially radial spaces are the same as pseudoradial Whyburn spaces. A list of open problems is given.

1. Introduction and first results

Since the beginning of General Topology, the idea was raised of describing the topology of spaces using converging sequences [F]. However, as we are going to deepen in this paper, sequences are not enough to describe the generality of topological spaces.

A sequence is a map $S : \mathbb{N} \to X$, where X is a set. In order to consider the concept of convergence, we will suppose that X is a topological space; furthermore, if not differently stated, we will restrict to Hausdorff spaces. We can think of $\mathbb{N}$ as represented by the ordinal ω. From this point of view it is immediate to step from the ordinal ω to an arbitrary ordinal λ. A *long sequence* is a map $S : \lambda \to X$, and will be denoted $\langle x_\alpha \rangle_{\alpha<\lambda}$. If X is a topological space it is natural to say that the (long) sequence S converges to a point $x \in X$ if given a neighborhood U of x, there is an ordinal δ such that for all ordinals $\eta > \delta$ we have $x_\eta \in U$. We recall that the cofinality of a limit ordinal $\alpha > 0$ (noted $\mathrm{cf}(\alpha)$) is the least ordinal β such that there is an increasing sequence $\langle \delta_\xi \rangle_{\xi<\beta}$ with $\lim_{\xi\to\beta} \delta_\xi = \sup\{\delta_\xi : \xi < \beta\} = \alpha$, while for a successor ordinal γ we define $\mathrm{cf}(\gamma) = 1$. A *subsequence* of a (long) sequence $S = \langle x_\alpha \rangle_{\alpha<\lambda}$ is a sequence $S' = \langle x_{\alpha_\beta} \rangle_{\beta<\eta}$, with $\langle \alpha_\beta \rangle_{\beta<\eta}$ cofinal in λ . We recall that for any

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ordinal number $\mathrm{cf}(\mathrm{cf}(\alpha)) = \mathrm{cf}(\alpha) \leq \alpha$, that $\mathrm{cf}(\alpha)$ is a cardinal number and that a cardinal number κ is said to be *regular* if $\mathrm{cf}(\kappa) = \kappa$ [J, K]. If we are interested in converging long sequences we can restrict ourselves to sequences whose lengths are ordinals which are regular cardinals. This fact is in the mathematical folklore but we present it here for the sake of completeness (see for example [V]).

Proposition 1.1. *Let $S = \langle x_\beta \rangle_{\beta<\gamma}$ a sequence converging to x and $\lambda = \mathrm{cf}\,\gamma$. Then there is a subsequence $\langle x_{\beta_\delta} \rangle_{\delta<\lambda}$ which converges to x.*

Long sequences are not sufficient in general to reach every point in the closure of a subset A of a topological space X. Work on transfinite sequences may have been discouraged by an example of G. Birkhoff [Bi] showing that if $X = 2^{\mathfrak{c}}$ and A is a subset of elements with a finite number of 1s, then any element x with an uncountable number of 1s belongs to the closure of A but no (long) sequence ranging in it can converge to x. Birkhoff showed that under the Continuum Hypothesis (CH) iterating twice the process of taking (long) sequential closures one obtains the topological closure. But it was shown in [Ny1] that for this space (depending on set theoretical axioms), there is some subset whose topological closure is not attained by any iteration of (long) sequential closures.

It can be interesting to find which (long) sequences converge in a given topological space X. There are spaces in which no usual sequence converges, as $\beta\omega$ and $\omega^\star$ (see e.g. [E]); however it is known that in $\beta\omega$ there are converging sequences of length ω_1, as it was observed in [BSV] and in [vM], without using the Continuum Hypothesis. Moreover it was proved in [JS] that every compact T_2 space of uncountable tightness contains a convergent sequence of length ω_1 and if CH holds every non-first countable compact T_2 space does.

We remember a classification of topological spaces concerning convergent (long) sequences. X is said to be *radial* if for every subset $A \subseteq X$ and for every $x \in \overline{A}$ there is a sequence $\langle x_\alpha \rangle_{\alpha<\kappa}$, with $x_\alpha \in A$ which converges to x. A subset $C \subseteq X$ is said to be *closed for sequences of length* $\leq \kappa$ (or *κ-sequentially closed*) if it contains all limits of convergent μ-sequences (that is sequences indexed in μ) with $\mu \leq \kappa$ and $x_\alpha \in A$ for all $\alpha < \mu$; $C \subseteq X$ is *κ-closed* if for every $B \subseteq C$ with $|B| \leq \kappa$, $\overline{B} \subseteq C$ holds. The space X is *semiradial* if every κ-sequentially closed subset is κ-closed, for all κ. It is *almost radial* if for every non-closed subset $A \subseteq X$ there is a point $x \in \overline{A} \setminus A$ and a thin sequence $\langle x_\alpha \rangle_{\alpha<\kappa}$ converging to x, with range in A; we recall that a sequence $\langle x_\alpha \rangle_{\alpha<\kappa}$ converging to x, is called *thin* if $x \notin \overline{\{x_\alpha : \alpha < \beta\}}$ for all $\beta < \kappa$. The space X is *R-monolithic* if for every subset $A \subseteq X$ if $B \subseteq \overline{A}$ is closed for sequences of length not greater than $|A|$, then B is closed. It is *pseudoradial* if for every non-closed subset

$A \subseteq X$ there is a point $x \in \overline{A} \setminus A$ and a sequence $\langle x_\alpha \rangle_{\alpha<\kappa}$, with range in A, converging to x; equivalently if the topological closure of any subset can be attained iterating a sufficient number of times the radial (or long sequential closure)[1]. Radial and pseudoradial spaces are generalizations of Fréchet-Urysohn and sequential spaces respectively. R-monolithic spaces [Be] and semiradial spaces [BG] were later introduced to provide partial answers to the still unsolved problem of the productivity of the class of compact pseudoradial spaces in ZFC.

It is well known (see for example [BG, BT]) that

$$\begin{aligned}\text{Fréchet} - \text{Urysohn} \Rightarrow \text{radial} \Rightarrow \text{semiradial} \Rightarrow \text{almost radial}\\ \Rightarrow \text{pseudoradial}\end{aligned}$$

and

$$\text{Fréchet} - \text{Urysohn} \Rightarrow \text{sequential} \Rightarrow \text{R} - \text{monolithic} \Rightarrow \text{semiradial}.$$

A space X is *Whyburn* if for every subset $A \subseteq X$, for every $x \in \overline{A}$ there is a subset $B \subseteq A$ such that $\overline{B} = B \cup \{x\}$; it is *weakly Whyburn* if for any non-closed subset $A \subseteq X$ there is some $x \in \overline{A} \setminus A$ and a subset $B \subseteq A$ such that $\overline{B} = B \cup \{x\}$ (for some results and more references see [BP], [BY] and [O]).

Example 1. The space $(\omega_1 + 1) \times (\omega + 1)$ with the product topology is R-monolithic and not radial.

Proof. The space is the product of two compact R-monolithic spaces and so it is R-monolithic, as the class of compact R-monolithic spaces is countably productive [BD]. The point $\langle \omega_1, \omega \rangle$ cannot be reached from $\omega_1 \times \omega$ by any (long) sequence. □

We would like to recall to the unfamiliar reader that a pseudoradial space of countable tightness can fail to be sequential. A zero-dimensional example of this sort is given in [JW]. The compact case is much more delicate and we have that the existence of a non-sequential compact pseudoradial space of countable tightness is undecidable in ZFC: on one hand, according to Balogh's theorem [Ba] under PFA every compact space of countable tightness is sequential, on the other hand the one-point compactification of the Ostaszewski's space, constructed assuming $\diamondsuit$ in [Os], is a non-sequential compact pseudoradial space of countable tightness.

[1]Given a subset A of a pseudoradial space X we can define $\hat{A}^0 = A$, $\hat{A}^{\alpha+1} = \widehat{\hat{A}^\alpha}$ and $\hat{A}^\alpha = \cup\{\hat{A}^\beta : \beta < \alpha\}$ if α is a limit ordinal. The order of pseudoradiality of the space [AIT1] is $\text{ops}(X) = \sup\{\alpha : \hat{A}^\alpha = \overline{A}, \forall A \subseteq X\}$.

A question which seems not to have been considered so far is whether some proper (and non-trivial) weakening of PFA can ensure that every compact pseudoradial space of countable tightness is sequential.

The purpose of distinguishing within the class of all pseudoradial spaces of countable tightness the sequential ones has led to the introduction of the notion of almost radiality [AIT]. This is based on the following:

Proposition 1.2. *A converging thin sequence in a space X of countable tightness is always countable.*

Proof. Let $\langle x_\alpha \rangle_{\alpha<\kappa}$ be a thin sequence converging to $x \in X$. As the space has countable tightness, there exists a countable set $A \subseteq \kappa$ such that $x \in \overline{\{x_\alpha : \alpha \in A\}}$. The thinness of the sequence implies that the set A is cofinal in κ and therefore, as we are assuming that κ is a regular cardinal, we must have $\kappa = \omega$. □

Hence, we have

Corollary 1.3. *A pseudoradial space of countable tightness is sequential if and only if it is almost radial.*

The existence of a linearly ordered local base at a point x is a condition strong enough to guarantee that every sequence converging to x is thin.

This is a consequence of the following:

Proposition 1.4. *Let X be a Hausdorff (or even T_1) topological space and suppose that a point $x \in X$ has the property that $\{x\}$ is the intersection of a family $\{U_\alpha : \alpha < \eta\}$ of open sets ordered by reverse inclusion (i.e. $\alpha < \beta \Rightarrow U_\beta \subseteq U_\alpha$). If κ is a regular cardinal and $\langle x_\beta \rangle_{\beta<\kappa}$ is an injective sequence converging to x, then $\kappa = \mathrm{cf}(\eta)$ and the sequence is thin.*

Proof. By passing to a cofinal subfamily if necessary, we may assume that η is a regular cardinal. For every ordinal $\alpha < \eta$ there is an ordinal $\beta_\alpha < \kappa$ such that $\{x_\beta : \beta > \beta_\alpha\} \subseteq U_\alpha$. By the regularity of κ, if $\eta < \kappa$, we have that $\sup\{\beta_\alpha : \alpha < \eta\} = \overline{\beta} < \kappa$. So $\{x\} = \bigcap\{U_\alpha : \alpha < \eta\} \supseteq \{x_\beta : \beta > \overline{\beta}\}$ which implies $x_\beta = x$ for all $\beta > \overline{\beta}$, contradicting injectivity, then $\kappa \leq \eta$. If, by contradiction, $\kappa < \eta$, take for every $\beta < \kappa$ a neighborhood U_{α_β} of x such that $x_\beta \notin U_{\alpha_\beta}$. Let $\overline{\alpha} = \sup\{\alpha_\beta : \beta < \kappa\}$. Then $\overline{\alpha} < \eta$ and $x_\beta \notin U_{\overline{\alpha}}$ for all $\beta < \kappa$, contradicting that $x_\beta \to x$. If $\kappa = \eta$ and a sequence $\langle x_\alpha \rangle_{\alpha<\kappa}$ converges to x, if $\lambda < \kappa$, repeating the above discussion, for every $\beta < \lambda$ there is a neighborhood U_{α_β} of x such that $x_\beta \notin U_{\alpha_\beta}$. We can find $\overline{\alpha} = \sup\{\alpha_\beta : \beta < \lambda\}$. Then $\overline{\alpha} < \eta$ and $x_\beta \notin U_{\overline{\alpha}}$ for all $\beta < \lambda$ and so $x \notin \overline{\{x_\beta : \beta < \lambda\}}$ for any $\lambda < \kappa$. □

Proposition 1.5. *There is a space which is R-monolithic, weakly Whyburn and neither radial nor sequential nor Whyburn. It has order of pseudoradiality 2.*

Proof. Let $X := (\omega_1 \times (\omega_1 + 1)) \cup \{\infty\}$. The topology is the following: all points of $\omega_1 \times \omega_1$ are isolated, points of type $\langle \eta, \omega_1 \rangle$ have a neighborhood base of type $U_\delta := \{\langle \eta, \alpha \rangle : \delta < \alpha \leq \omega_1\}$, while a base of neighborhoods of the point ∞ are sets

$$U_{\epsilon,f} := \{\langle \gamma, \delta \rangle : \epsilon < \gamma < \omega_1, f(\gamma) < \delta \leq \omega_1\} \cup \{\infty\}, \tag{1.1}$$

where $f(\gamma) :]\epsilon, \omega_1[\to \omega_1$ is an arbitrary function. The space is not radial, since if $A := \omega_1 \times \omega_1$, then $\infty \in \overline{A}$, but any sequence in A cannot converge to the point ∞; in fact given any sequence $S := \langle x_\alpha, y_\alpha \rangle_{\alpha < \omega_1}$, if for any $\beta < \omega_1$ the cardinality of $Y_\beta := \{y_\alpha : x_\alpha = \beta\}$ is less than ω_1, let $f(\beta) := \sup Y_\beta$ and then $U_{0,f} \cap \operatorname{im} S = \emptyset$; if there is some $\beta < \omega_1$ such that $|Y_\beta| = \omega_1$ there is in Y_β a subsequence of S which converges to $\langle \beta, \omega_1 \rangle \neq \infty$, a contradiction. Similarly we prove that X is not Whyburn: let $A := \omega_1 \times \omega_1$ and consider $\infty \in \overline{A}$. If $B \subseteq A$ and for every $\beta < \omega_1$, if $Z_\beta := B \cap (\beta \times \omega_1)$ we have $|Z_\beta| < \omega_1$, then, if $f(\beta) := \sup Z_\beta$, $U_{0,f} \cap B = \emptyset$. If for some $\beta < \omega_1$, $|Z_\beta| = \omega_1$, then $\langle \beta, \omega_1 \rangle \in \overline{B}$, so $\overline{B} \setminus A \neq \{\infty\}$. As X is a non-discrete P-space, it immediately follows that it is not sequential. Moreover, every subset is ω-closed and so every converging sequence has only its limit as accumularion point. Consequently, to prove that X is R-monolithic and weakly Whyburn it is enough to check that it is pseudoradial. Let A be a non-closed subset of X. If there exists some $\alpha < \omega_1$ such that $\langle \alpha, \omega_1 \rangle \in \overline{A} \setminus A$, then $\{\alpha\} \times \omega_1 \cap A$ is a sequence converging to $\langle \alpha, \omega_1 \rangle$ and we are done. The remaining case to consider is $\{\infty\} = \overline{A} \setminus A$. This may happen only if the set $A \cap (\omega_1 \times \{\omega_1\})$ has cardinality ω_1 and this in turn implies that $A \cap (\omega_1 \times \{\omega_1\})$ is actually a sequence converging to ∞. □

Remark 1.6. This example is simpler than the one given in Theorem 3.8 of [BY]. In that paper an example is given of a space which is R-monolithic, weakly Whyburn and neither Whyburn nor radial (nor sequential). The space considered in [BY] is $C_p(\kappa)$ where κ is an ω-inaccessible cardinal (i.e. for every $\lambda < \kappa$, we have $\lambda^\omega < \kappa$).

In connection with the previous remark, it is worth mentioning that $C_p(\kappa)$ is weakly Wyburn if and only if the cofinality of κ is ω-inaccessible [BP].

2. Essential sequences and spaces

Already in the past particular kind of sequences, like for instance the thin ones, were introduced to solve various topological problems. This could also be seen as a way to lead a sequence to its essential purpose of converging tool, trying to avoid useless "enlargements" in the initial segments of it before reaching the limit. We present a new attempt to clean up sequences to their essential nature.

Definition 2.1. Let κ be a regular cardinal, we say that a κ-sequence $\langle x_\alpha \rangle_{\alpha<\kappa}$ in a Hausdorff space is essential if it is injective, converging and

$$\overline{\{x_\alpha : \alpha < \kappa\}} = \{x_\alpha : \alpha < \kappa\} \cup \{x\}\,, \tag{2.1}$$

where $x := \lim x_\alpha$.

Example 2.
• The sequence $\langle \alpha \rangle_{\alpha \in \mathrm{Lim}_2}$, where Lim_2 are the limit ordinals $\alpha < \omega_2$, is essential in $\omega_2 + 1$ with the usual topology.
• The sequence $\langle \alpha + 1 \rangle_{\alpha<\omega_1}$ is not essential in the space $\omega_1 + 1$ with the usual topology.

Although, as previously mentioned, $\omega^\star$ contains converging long sequences, we have the following:

Proposition 2.2. *In $\omega^\star$ there are no essential sequences.*

In fact, if $S \subseteq \omega^\star$ is an essential convergent sequence, then it cannot be that $|S| < 2^{\mathfrak{c}}$, since every infinite closed set has cardinality $2^{\mathfrak{c}}$. But no converging sequence in $\omega^\star$ can have length $2^{\mathfrak{c}}$, since the character is at most $\mathfrak{c}$. □

Proposition 2.3. *An injective and converging ω-sequence $\langle x_n \rangle_{n\in\omega}$ in a Hausdorff space is essential.*

Proof. If $\langle x_n \rangle_{n\in\omega}$ converges to a point x and $S = \{x_n : n \in \omega\}$, then the subspace $S \cup \{x\}$ is compact and hence closed in the space. But, this precisely means that $\overline{S} = S \cup \{x\}$, i.e. the sequence is essential. The reverse implication is obvious. □

Parallel to the above is the following:

Proposition 2.4. *Every converging sequence of length ω_1 in a Hausdorff P-space is essential.*

Remark 2.5. If the space is not Hausdorff and sequences are usual countable sequences the notion of Definition 2.1 was introduced as part of the definition of an SC-spaces in [AW] and considered, for example, in [Be1, BC]. We recall that a topological space is said to be SC if whenever a sequence $\langle x_n \rangle_{n\in\omega}$ converges to a point x, the set $\{x_n : n \in \omega\} \cup \{x\}$ is closed. The previous two propositions deal with essential sequences S with the extra property that S is discrete but this in general is not the case. For example, in the space $\omega_1 + 1$ no discrete sequence converging to ω_1 can be essential.

Remark 2.6. Observe that the notions of essential and thin are independent. The set of successor ordinals in ω_1+1 is a thin sequence converging to ω_1 which is not essential. On the other hand, if we consider the T_2 zero-dimensional space $Y = X \cup \{p\}$ described in [JW] (Corollary 1), then, by well-ordering X in type $\mathfrak{c}$, we have that X is an essential sequence converging to p which is not thin. This example is quite strong because no tail of this sequence can be thin either.

Definition 2.7. A topological space is said to be *essentially radial (resp. semiradial, almost radial, pseudoradial)* if the corresponding definitions can be formulated by using essential sequences. For instance, a space X is *essentially pseudoradial* if for every non-closed subset $A \subseteq X$ there is a point $x \in \overline{A} \setminus A$ and an essential (in X) sequence $\langle x_\alpha \rangle_{\alpha<\kappa}$ converging to x, with range in A.

It is evident that essentially radial implies radial, essentially semiradial implies semiradial, essentially almost radial implies almost radial and essentially pseudoradial implies pseudoradial. Of course, the interesting thing here is to check that the previous implications are not reversible. Below, we show that this is the case at least for radial and pseudoradial spaces.

Example 3.
(i) $X := \omega_1 + 1$, endowed with the topology where all points $\alpha < \omega_1$ are isolated and $[\beta, \omega_1]$ are fundamental neighborhoods of ω_1, is essentially radial and not sequential.
(ii) $\omega_1 + 1 = \omega_1 \cup \{\omega_1\}$ with the usual topology is radial, not essentially radial, but essentially semiradial.

Proof. (i) Is obvious.

(ii) It is well known that $\omega_1 + 1$ with the usual topology is radial. To see that it is not essentially radial, let $A := \{\beta + 1 : \beta < \omega_1\}$. We have $\omega_1 \in \overline{A} \setminus A$ and the set A is discrete. Thus, according to the previous remark, A cannot contain any essential sequence converging to ω_1.

To check essential semiradiality let $A \subseteq \omega_1 + 1$. If A is not ω-closed, then there is some $\beta \in \omega_1$ satisfying $\beta \in \overline{A} \setminus A$. Since β has a countable local base, there exists a countable, and hence essential, sequence in A converging to β. If A is not closed but ω-closed, then the only possibility is to have $\omega_1 \in \overline{A} \setminus A$ and in this case A itself with the induced order is an essential sequence converging to ω_1. □

Notice that the space $\omega_1 + 1$ has "essential order of pseudoradiality" 2 and "order of pseudoradiality" 1.

Recall that $\sigma_c(X)$ (the chain character of the space X) is the smallest cardinal number λ such that for any non-closed set $A \subseteq X$ there is a sequence $\langle x_\alpha \rangle_{\alpha<\xi}$ with $\xi \leq \lambda$ converging to a point $x \in \overline{A} \setminus A$.

By considering in the above definition essential sequences, we may get the notion of "essential chain character" $E\sigma_c(X)$, definable of course for any essentially pseudoradial space: $E\sigma_c(X) \geq t(X)$.

The Ostaszewski's space X is a hereditarily separable locally countable locally compact countably compact non-compact space of cardinality ω_1. Its one-point compactification $Y = X \cup \{p\}$ is a non-sequential essentially pseudoradial space of countable tightness with $\sigma_c(X) = E\sigma_c(X) = \omega_1$. To check the essential pseudoradiality, let A be a non-closed subset of Y. If there is some $x \in X$ such that $x \in \overline{A} \setminus A$, then we may find a usual ω-sequence in A converging to x. The remaining case to consider is $A \subseteq X$ and A closed in X. As any well-ordering of A gives an essential sequence converging to p, we are done.

In a similar manner, the space produced in [JW], Corollary 1, is a ZFC essentially pseudoradial non-sequential T_3 space of countable tightness.

Proposition 2.8. *Every essentially pseudoradial space is weakly Whyburn.*

Proof. Let $A \subseteq X$ non-closed. Then there is a point $x \in \overline{A} \setminus A$ and an essential sequence $\langle x_\alpha \rangle_{\alpha<\lambda}$ converging to x. Since $\overline{\{x_\alpha : \alpha < \lambda\}} = \{x_\alpha : \alpha < \lambda\} \cup \{x\}$, then $B = \{x_\alpha : \alpha < \lambda\} \subseteq A$ satisfies the definition of a weakly Whyburn space. □

Proposition 2.9. *The following implications hold:*

$$\text{ess. radial} \Rightarrow \text{ radial} \not\Rightarrow \text{ess. radial}$$

$$\text{ess. semiradial} \Rightarrow \text{ semiradial}$$

$$\text{ess. almost radial} \Rightarrow \text{almost radial}$$

$$\text{ess. pseudoradial} \Rightarrow \text{pseudoradial} \not\Rightarrow \text{ess. pseudoradial}$$

Proof. Example 3 (ii) shows that the reverse implication does not hold in (i).

(v) A. Dow [D] has given, under $\diamondsuit$, an example of a compact pseudoradial and non-weakly Whyburn space. Because of Proposition 2.8, this implies that it is not essentially pseudoradial. □

Proposition 2.10. *A space is pseudoradial and Whyburn if and only if it is essentially radial.*

Proof. Let $S \subseteq X$ and $x \in \overline{S} \setminus S$. Since X is Whyburn, there exists $A \subseteq S$ such that $\{x\} = \overline{A} \setminus S$. Since X is pseudoradial, there is an injective sequence $\langle x_\alpha \rangle_{\alpha<\kappa}$ such that $x_\alpha \in \overline{A} \setminus \{x\} \subseteq S$ and $x_\alpha \to x \neq x_\alpha$ for all $\alpha < \kappa$. Let $C = \{x_\alpha : \alpha < \kappa\}$, then $x \in \overline{C} \setminus C$. Again, since the space is Whyburn, there is a subset $B \subseteq C$, such that $\{x\} = \overline{B} \setminus C$. So there is some $\lambda \leq \kappa$ such that if $\{x_{\alpha_\beta} : \beta < \lambda\} = C \cap \overline{B}$, then $\langle x_{\alpha_\beta} \rangle_{\beta<\lambda}$ is an essential sequence converging to x. The reverse implication is obviously true. □

The proof of the next result easily follows from the various definitions.

Proposition 2.11. *The following logical relations hold:*

$$\text{FU} \Rightarrow \text{ess. radial} \Rightarrow \text{ess. semiradial} \Rightarrow$$

$$\Rightarrow \text{ess. almost radial} \Rightarrow \text{ess. pseudoradial}$$

and

$$\text{ess. semiradial} \not\Rightarrow \text{ess. radial} \not\Rightarrow$$

$$\not\Rightarrow \text{sequential} \Rightarrow \text{ess. semiradial} \not\Rightarrow \text{sequential}$$

(we have written ess. for essentially and FU, as usual, for Fréchet-Urysohn).

Definition 2.12. A topological space X is said to be essential if every injective convergent sequence of regular length is an essential sequence.

At first glance, the notion presented above may seem quite peculiar, but we have a characterization.

Proposition 2.13. *A space is essential if and only if the derived set of the range of any converging sequence contains only its limit.*

Proof. Let X be essential and let S be a sequence converging to x. If $p \neq x$ is an accumulation point of S, then obviously $S \setminus \{p\}$ is a sequence still converging to x, but not an essential sequence. The converse is obvious. □

The class of these spaces is not "too small". Just look at the following:

Proposition 2.14. *All T_2 spaces of countable pseudocharacter and all T_2 P-space of pseudocharacter equal to ω_1 are essential.*

Proof. It is enough to observe that in the first case all converging sequences have length ω and in the second case length ω_1. In both cases, a converging sequence has its limit as the only accumulation point and so it is an essential sequence. □

Remark 2.15. We can observe that even in the compact case the previous Proposition 2.14 cannot be reversed. Indeed the one-point compactification of a discrete space of uncountable cofinality is an essential space of uncountable pseudocharacter, it has convergent essential sequences of all cardinalities up to that of the space itself, and is Fréchet-Urysohn. However it is easy to see that an essential compact and radial space is necessarily Fréchet-Urysohn. Also an essential compact and pseudoradial space is sequential, since any essential sequence must have the first ω terms converging to the limit point. It should be noted that the space Ψ is an essential, compact, sequential space which is not Fréchet-Urysohn.

However, a compact Fréchet-Urysohn space may fail to be essential.

Example 4. There is a Fréchet-Urysohn Hausdorff and compact space which is not essential. In particular, it has an uncountable convergent ω_1-sequence.

Proof. Let X be the one-point compactification with point p of a space Y, where the underlying set of Y is the topological direct sum of the rows $(\omega_1+1)\times\{\alpha\}$ of $(\omega_1+1)\times\omega_1$, each of which is the one-point compactification of $\omega_1\times\{\alpha\}$ with the discrete topology. To check that X is not essential, let $S=\omega\times\omega_1$ be endowed with the following order: $\langle m,\alpha\rangle<\langle n,\beta\rangle$ if $\alpha<\beta$ or $m<n$ if $\alpha=\beta$. With this order S is order-isomorphic to ω_1 and then it is a sequence of length ω_1. Since a fundamental system of neighborhoods of p is the family of sets $\{X\setminus(\omega_1+1)\times F : F\in[\omega_1]^{<\omega}\}$, then S converges to p. It is clear however that $S\cup\{p\}$ is not closed in X. □

3. Open Problems

Essential pseudoradiality is a sort of combination of pseudoradiality and weak Whyburness, but a priori the points which guarantee the validity of the former are different from those involved in the latter. The following is the main question left open here:

- Is it true that pseudoradial and weakly Whyburn implies essentially pseudoradial? (see also Proposition 2.8)

Since every semiradial space is weakly Whyburn and every compact weakly Whyburn space is pseudoradial (see [Be2]), the above main question has two interesting special cases.

- Is it true that every semiradial space is essentially pseudoradial?
- Is it true that every compact weakly Whyburn space is essentially pseudoradial?

Following the argument in [Be2], we see that the above question is equivalent to:

- Let X be a compact weakly Whyburn space and p a non-isolated point of X. Is it true that there exists an essential sequence in $X\setminus\{p\}$ which converges to p?

Other basic questions are:

- Find an essentially pseudoradial space X such that $\sigma_c(X)<E\sigma_c(X)$.

• Every topological space may be embedded in a pseudoradial space (see [Sl]), using category arguments; may it be embedded in an essential pseudoradial space?
• Does there exist in ZFC a pseudoradial space which is not essentially pseudoradial?

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