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by

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A DIGITAL PRETOPOLOGY AND ONE OF ITS QUOTIENTS

JOSEF ŠLAPAL

Abstract. We introduce a certain pretopology on the digital plane $\mathbb{Z}^2$ and present a digital analogue of the Jordan curve theorem for it. We then discuss a topology on $\mathbb{Z}^2$ which is shown to be a quotient pretopology of the pretopology introduced. This fact is used to prove a digital Jordan curve theorem also for this topology.

1. Introduction

In the classical approach to digital topology (see e.g. [11] and [12]), graph theoretic tools are used for structuring $\mathbb{Z}^2$, namely the well-known binary relations of 4-adjacency and 8-adjacency. Unfortunately, neither 4-adjacency nor 8-adjacency itself allows an analogue of the Jordan curve theorem - cf. [8]. To overcome this, a combination of the two binary relations has to be used. Despite this inconvenience, the graph-theoretic approach is used to solve many problems of digital image processing and to create useful graphic software. In [5], a new, purely topological approach to the problem was proposed which utilizes a convenient topology on $\mathbb{Z}^2$, called Khalimsky topology (cf. [4]), for structuring the digital plane. At present, this topology represents an important concept in digital topology which has been studied and used by many authors, e.g., [2] and [6]-[9]. The possibility of structuring $\mathbb{Z}^2$ by using certain closure operators more general than the Kuratowski ones is discussed in [13] and [14].

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In [15], a new digital topology on $\mathbb{Z}^2$ has been introduced and studied. This topology was shown to have some advantages over the Khalimsky one. More precisely, it was proved in [15] that all cycles in a certain natural graph with the vertex set $\mathbb{Z}^2$ are Jordan curves with respect to the new topology. The topology from [15] was further investigated in [16] where four of its quotient pretopologies including the Khalimsky topology were studied. In the present paper, we continue the study from [15] and [16]. We introduce a new pretopology on $\mathbb{Z}^2$ and present a Jordan curve theorem for it. The theorem is then used to determine Jordan curves among the simple closed curves in a topology which is a quotient pretopology of the pretopology introduced. Since Jordan curves represent boundaries of regions in digital images, the structures proposed by the two pretopologies may especially be used for solving problems of computer image processing which are related to these boundaries.

2. Preliminaries

Recall that *pretopologies* are generalizations of the Kuratowski closure operators obtained by omitting the requirement of idempotency. Pretopological spaces were studied in great detail by E. Čech in [1] (and, therefore, they are sometimes called *Čech closure spaces*). We will work with some basic topological concepts (see e.g. [3]) naturally extended from topological spaces to pretopological ones. For example, a pretopology p on a set X is called a $T_{\frac{1}{2}}$*-pretopology* if each singleton subset of X is closed or open (so that $T_{\frac{1}{2}}$ implies T_0) and it is called *Alexandroff* if $pA = \bigcup_{x\in A} p\{x\}$ whenever $A \subseteq X$ (such pretopologies are called *quasi-discrete* in [1]).

We will need the following lemma, which follows from [14], Corollary 1.5:

Lemma 2.1. *Let (X, p) be a pretopological space, $e : X \to Y$ be a surjection and let q be the quotient pretopology of p on Y generated by e. Let e have connected fibres (i.e., the property that $e^{-1}(\{y\})$ is connected in (X, p) for every point $y \in Y$) and let $B \subseteq Y$ be a subset. Then B is connected in (Y, q) if and only if $e^{-1}(B)$ is connected in (X, p).*

Let us note that, for a topological space (X, p) and a quotient topology of p, the statement of the previous Lemma need not be true. It is true if, for example, B is closed or open in (Y, q).

By a *graph* on a set V, we always mean an undirected simple graph without loops whose vertex set is V. Recall that a *path* in a graph is a finite (nonempty) sequence $x_0, x_1, ..., x_n$ of pairwise different vertices such that x_{i-1} and x_i are adjacent (i.e., joined by an edge) whenever $i \in \{1, 2, ...n\}$. By a *cycle* in a graph we understand any finite set of at least three vertices which can be ordered into a path whose first and last members are adjacent.

The *connectedness graph* of a pretopology p on X is the graph on X in which a pair of vertices $x,\ y$ is adjacent if and only if $x \neq y$ and $\{x, y\}$ is a connected subset of (X, p). In the sequel, only connected Alexandroff pretopologies on $\mathbb{Z}^2$ will be dealt with. In the connectedness graphs of these pretopologies, the closed points will be ringed and the mixed ones (i.e., the points that are neither closed nor open) boxed. Thus, the points neither ringed nor boxed will be open (note that none of the points of $\mathbb{Z}^2$ may be both closed and open).

For every point $(x, y) \in \mathbb{Z}^2$, we denote by $A_4(x, y)$ and $A_8(x, y)$ the set of all points that are 4-adjacent to (x, y) and that of all points that are 8-adjacent to (x, y), respectively. Thus, $A_4(x, y) = \{(x + i, y + j);\ i, j \in \{-1, 0, 1\},\ ij=0,\ i+j\neq 0\}$ and $A_8(x, y)=A_4(x, y)\cup\{(x+i, y+j);\ i, j \in \{-1, 1\}\}$. For natural reasons related to possible applications of our results in digital image processing, only such pretopologies on $\mathbb{Z}^2$ will be dealt with whose connectedness graphs are subgraphs of the 8-adjacency graph. (It is well known [3] that there are exactly two topologies on $\mathbb{Z}^2$ whose connectedness graphs lie between the 4-adjacency and 8-adjacency graphs. These topologies are the Khalimsky and Marcus-Wyse ones - see below.)

By a *(digital) simple closed curve* in a pretopological space $(\mathbb{Z}^2, p)$ we mean, in accordance with [16], a nonempty, finite and connected subset $C \subseteq \mathbb{Z}^2$ such that, for each point $x \in C$, there are exactly two points of C adjacent to x in the connectedness graph of p. A simple closed curve C in $(\mathbb{Z}^2, p)$ is said to be a *(digital) Jordan curve* if it separates $(\mathbb{Z}^2, p)$ into precisely two components (i.e., if the subspace $\mathbb{Z}^2 - C$ of $(\mathbb{Z}^2, p)$ consists of precisely two components).

Definition 2.2. The *square-diagonal graph* is the graph on $\mathbb{Z}^2$ in which two points $z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in \mathbb{Z}^2$ are adjacent if and only if one of the following four conditions is fulfilled:

(1) $|y_1 - y_2| = 1$ and $x_1 = x_2 = 4k$ for some $k \in \mathbb{Z}$,
(2) $|x_1 - x_2| = 1$ and $y_1 = y_2 = 4l$ for some $l \in \mathbb{Z}$;
(3) $x_1 - x_2 = y_1 - y_2 = \pm 1$ and $x_1 - 4k = y_1$ for some $k \in \mathbb{Z}$,
(4) $x_1 - x_2 = y_2 - y_1 = \pm 1$ and $x_1 = 4l - y_1$ for some $l \in \mathbb{Z}$.

A portion of the square-diagonal graph is shown in Figure 1.

When studying digital images, it may be helpful to equip $\mathbb{Z}^2$ with a closure operator with respect to which some or even all cycles in the square-diagonal graph are Jordan curves.

Recall [5] that the Khalimsky topology on $\mathbb{Z}^2$ is the Alexandroff topology t given as follows:

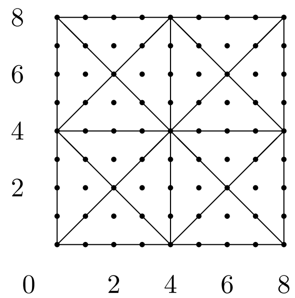

FIGURE 1. A portion of the square-diagonal graph.

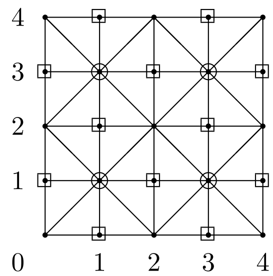

FIGURE 2. A portion of the connectedness graph of the Khalimsky topology.

For any $z = (x, y) \in \mathbb{Z}^2$,

$$t\{z\} = \begin{cases} \{z\} \cup A_8(z) \text{ if } x, y \text{ are even,} \\ \{(x+i, y); i \in \{-1, 0, 1\}\} \text{ if } x \text{ is even and } y \text{ is odd,} \\ \{(x, y+j); j \in \{-1, 0, 1\}\} \text{ if } x \text{ is odd and } y \text{ is even,} \\ \{z\} \text{ otherwise.} \end{cases}$$

The Khalimsky topology is connected and T_0; a portion of its connectedness graph is shown in Figure 2.

Another well-known topology on $\mathbb{Z}^2$ is the Marcus-Wyse one (cf. [10]), i.e., the Alexandroff topology s on $\mathbb{Z}^2$ given as follows:
For any $z = (x, y) \in \mathbb{Z}^2$,

$$s\{z\} = \begin{cases} \{z\} \cup A_4(z) \text{ if } x + y \text{ is odd,} \\ \{z\} \text{ otherwise.} \end{cases}$$

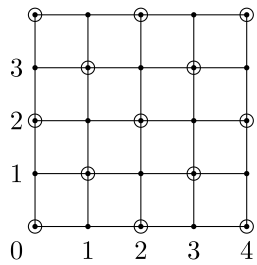

FIGURE 3. A portion of the connectedness graph of the Marcus-Wyse topology.

The Marcus-Wyse topology is connected and $T_{\frac{1}{2}}$. A portion of its connectedness graph is shown in Figure 3.

It is readily confirmed that a cycle in the square-diagonal graph is a Jordan curve in the Marcus-Wyse topological space if and only if it does not employ diagonal edges. And a cycle in the square-diagonal graph is a Jordan curve in the Khalimsky topological space if and only if it does not turn, at any of its points, at the acute angle $\frac{\pi}{4}$ - cf. [5]. It could therefore be useful to replace the Khalimsky and Marcus-Wyse topologies with some more convenient connected topologies or pretopologies on $\mathbb{Z}^2$ that allow Jordan curves to turn at the acute angle $\frac{\pi}{4}$ at some points. One such a pretopology will be introduced in the next paragraph.

3. Pretopology r and one of its quotients

We denote by r the Alexandroff pretopology on $\mathbb{Z}^2$ given as follows: For any point $z = (x, y) \in \mathbb{Z}^2$,

$$r\{z\} = \begin{cases} \{z\} \cup A_8(z) \text{ if } x = 4k,\ y = 4l,\ k, l \in \mathbb{Z}, \\ \{z\} \cup (A_8(z) - A_4(z)) \text{ if } x = 2 + 4k,\ y = 2 + 4l, \\ \hfill k, l \in \mathbb{Z}, \\ \{z\} \cup \{(x-1, y), (x+1, y)\} \text{ if } x = 2 + 4k,\ y = 1 + 2l, \\ \hfill k, l \in \mathbb{Z}, \\ \{z\} \cup \{(x, y-1), (x, y+1)\} \text{ if } x = 1 + 2k, \text{ and} \\ \hfill y = 2 + 4l),\ k, l \in \mathbb{Z}, \\ \{z\} \cup A_4(z) \text{ if either } x = 4k \text{ and } y = 2 + 4l \text{ or} \\ \hfill x = 2 + 4k \text{ and } y = 4l,\ k, l \in \mathbb{Z}, \\ \{z\} \text{ otherwise.} \end{cases}$$

Clearly, r is connected and T_0. A portion of the connectedness graph of r is shown in Figure 4.

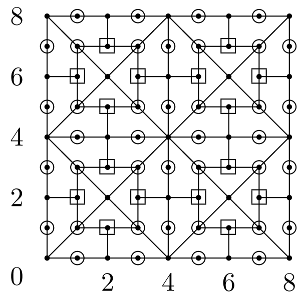

FIGURE 4. A portion of the connectedness graph of r.

The proof of the following statement is analogous to that of Theorem 11 in [15]. We decided to present it because there are some deficiencies in the proof of Theorem 11 in [15] which we have corrected here.

Theorem 3.1. *Every cycle in the square-diagonal graph is a Jordan curve in* $(\mathbb{Z}^2, r)$.

Proof. It may easily be seen that any cycle in the square-diagonal graph is a simple closed curve in $(\mathbb{Z}^2, r)$. Let $z = (x, y) \in \mathbb{Z}^2$ be a point such that $x = 4k + p$ and $y = 4l + q$ for some $k, l, p, q \in \mathbb{Z}$ with $pq = \pm 2$. Then we define the *fundamental triangle T(z)* to be the nine-point subset of $\mathbb{Z}^2$ given as follows:

$$T(z) = \begin{cases} \{(s,t) \in \mathbb{Z}^2;\ y-1 \le t \le y+1-|s-x|\} \text{ if } x = 4k+2 \\ \qquad \text{and } y = 4l+1 \text{ for some } k, l \in \mathbb{Z}, \\ \{(s,t) \in \mathbb{Z}^2;\ y-1+|s-x| \le t \le y+1\} \text{ if } x = 4k+2 \\ \qquad \text{and } y = 4l-1 \text{ for some } k, l \in \mathbb{Z}, \\ \{(s,t) \in \mathbb{Z}^2;\ x-1 \le s \le x+1-|t-y|\} \text{ if } x = 4k+1 \\ \qquad \text{and } y = 4l+2 \text{ for some } k, l \in \mathbb{Z}, \\ \{(s,t) \in \mathbb{Z}^2;\ x-1+|t-y| \le s \le x+1\} \text{ if } x = 4k-1 \\ \qquad \text{and } y = 4l+2 \text{ for some } k, l \in \mathbb{Z}. \end{cases}$$

Graphically, the fundamental triangle $T(z)$ consists of the point z and the eight points lying on the triangle surrounding z - the four types of fundamental triangles are represented in Figure 5.

Given a fundamental triangle, we speak about its sides - it is clear from the above picture what sets are understood to be the sides (note that each side consists of five or three points and that two different fundamental triangles may have at most one common side).

Now, one can easily see that:

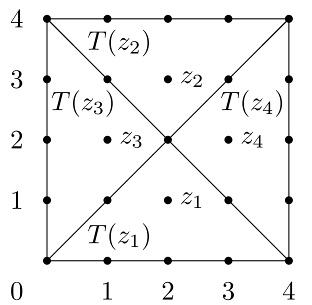

FIGURE 5. The four types of fundamental triangles.

(1) Every fundamental triangle is connected (so that the union of two fundamental triangles having a common side is connected) in $(\mathbb{Z}^2, r)$.
(2) If we subtract from a fundamental triangle some of its sides, then the resulting set is still connected in $(\mathbb{Z}^2, r)$.
(3) If S_1, S_2 are fundamental triangles having a common side D, then the set $(S_1 \cup S_2) - M$ is connected in $(\mathbb{Z}^2, r)$ whenever M is the union of some sides of S_1 or S_2 different from D.
(4) Every connected subset of $(\mathbb{Z}^2, r)$ having at most two points is a subset of a fundamental triangle.

We will now show that the following is also true:

(5) For every cycle C in the square-diagonal graph, there are sequences $\mathcal{S}_F, \mathcal{S}_I$ of fundamental triangles, $\mathcal{S}_F$ finite and $\mathcal{S}_I$ infinite, such that, whenever $\mathcal{S} \in \{\mathcal{S}_F, \mathcal{S}_I\}$, the following two conditions are satisfied:

(a) Each member of $\mathcal{S}$, excluding the first one, has a common side with at least one of its predecessors.

(b) C is the union of those sides of fundamental triangles from $\mathcal{S}$ that are not shared by two different fundamental triangles from $\mathcal{S}$.

Put $C_1 = C$ and let S_1^1 be an arbitrary fundamental triangle with $S_1^1 \cap C_1 \neq \emptyset$. For every $k \in \mathbb{Z}$, $1 \leq k$, if $S_1^1, S_2^1, ..., S_k^1$ are defined, let S_{k+1}^1 be a fundamental triangle with the following properties: $S_{k+1}^1 \cap C_1 \neq \emptyset$, S_{k+1}^1 has a side in common with S_k^1 which is not a subset of C_1 and $S_{k+1}^1 \neq S_i^1$ for all i, $1 \leq i \leq k$. Clearly, there will always be a (smallest) number $k \geq 1$ for which no such a fundamental triangle S_{k+1}^1 exists. We denote by k_1 this number so that we have defined a sequence $(S_1^1, S_2^1, ..., S_{k_1}^1)$ of fundamental triangles. Let C_2 be the union of those sides of fundamental triangles from $(S_1^1, S_2^1, ..., S_{k_1}^1)$ that are disjoint from C_1 and are not shared by two different fundamental triangles from $(S_1^1, S_2^1, ..., S_{k_1}^1)$.

If $C_2 \neq \emptyset$, we construct a sequence $(S^2_1, S^2_2, ..., S^2_{k_2})$ of fundamental triangles in an analogous way to $(S^1_1, S^1_2, ..., S^1_{k_1})$ by taking C_2 instead of C_1 (and obtaining k_2 analogously to k_1). Repeating this construction, we get sequences $(S^3_1, S^3_2, ..., S^3_{k_3})$, $(S^4_1, S^4_2, ..., S^1_{k_4})$, etc. We put $\mathcal{S} = (S^1_1, S^1_2, ..., S^1_{k_1}, S^2_1, S^2_2, ..., S^2_{k_2}, S^3_1, S^3_2, ..., S^3_{k_3}, ...)$ if $C_i \neq \emptyset$ for all $i \geq 1$ and $\mathcal{S} = (S^1_1, S^1_2, ..., S^1_{k_1}, S^2_1, S^2_2, ..., S^2_{k_2}, ..., S^l_1, S^l_2, ..., S^l_{k_l})$ if $C_i \neq \emptyset$ for all i with $1 \leq i \leq l$ and $C_i = \emptyset$ for $i = l + 1$.

Further, let $S'_1 = T(z)$ be a fundamental triangle such that $z \notin S$ whenever S is a member of $\mathcal{S}$. Having defined S'_1, let $\mathcal{S}' = (S'_1, S'_2, ...)$ be a sequence of fundamental triangles defined analogously to $\mathcal{S}$ (by taking S'_1 in the role of S^1_1). Then one of the sequences $\mathcal{S}$, $\mathcal{S}'$ is finite and the other is infinite. (Indeed, $\mathcal{S}$ is finite or infinite, respectively, if and only if its first member equals such a fundamental triangle $T(z)$ for which $z = (k, l) \in \mathbb{Z}^2$ has the property that (1) k is even, l is odd and the cardinality of the set $\{(x, l) \in \mathcal{Z}^2;\ x > k\} \cap C$ is odd or even, respectively or (2) k is odd, l is even and the cardinality of the set $\{(k, y) \in \mathcal{Z}^2;\ y > l\} \cap C$ is odd or even, respectively. The same is true for $\mathcal{S}'$.) If we put $\{\mathcal{S}_F, \mathcal{S}_I\} = \{\mathcal{S}, \mathcal{S}'\}$ where $\mathcal{S}_F$ is finite and $\mathcal{S}_I$ is infinite, then the conditions (a) and (b) are clearly satisfied.

Given a cycle C in the square-diagonal graph, let S_F and S_I denote the union of all members of $\mathcal{S}_F$ and $\mathcal{S}_I$, respectively. Then $S_F \cup S_I = \mathbb{Z}^2$ and $S_F \cap S_I = C$. Let $\mathcal{S}^*_F$ and $\mathcal{S}^*_I$ be the sequences obtained from $\mathcal{S}_F$ and $\mathcal{S}_I$ by subtracting C from each member of $\mathcal{S}_F$ and $\mathcal{S}_I$, respectively. Let S^*_F and S^*_I denote the union of all members of $\mathcal{S}^*_F$ and $\mathcal{S}^*_I$, respectively. Then S^*_F and S^*_I are connected by (1), (2) and (3) and it is clear that $S^*_F = S_F - C$ and $S^*_I = S_I - C$. So, S^*_F and S^*_I are the two components of $\mathbb{Z}^2 - C$ by (4) ($S_F - C$ is the so-called *inside* component and $S_I - C$ is the so-called *outside* component). This proves the statement. □

Let v be the Alexandroff topology on $\mathbb{Z}^2$ given as follows:
For any $z = (x, y) \in \mathbb{Z}^2$,

$$v\{z\} = \begin{cases} \{(x+i, y);\ i \in \{-1, 0, 1\}\} \text{ if } x \text{ is odd and } y \text{ is even,} \\ \{(x, y+j);\ j \in \{-1, 0, 1\}\} \text{ if } x \text{ is even and } y \text{ is odd,} \\ \{z\} \cup (A_8(z) - A_4(z)) \text{ if } x, y \text{ are odd,} \\ \{z\} \text{ if } x, y \text{ are even.} \end{cases}$$

Evidently, v is connected and T_0. A portion of its connectedness graph is shown in Figure 6.

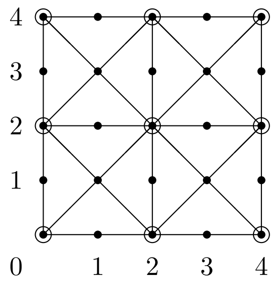

FIGURE 6. A portion of the connectedness graph of v.

Proposition 3.2. *v is the quotient pretopology of r generated by the surjection $h : \mathbb{Z}^2 \to \mathbb{Z}^2$ given as follows:*

$$h(x,y) = \begin{cases} (2k,2l) \ \textit{if} \ (x,y) \in \{(4k,4l)\} \cup A_8(4k,4l), \ k,l \in \mathbb{Z}, \\ (2k,2l+1) \ \textit{if} \ (x,y) \in \{(4k+i,4l+2); \\ \qquad\qquad\qquad i \in \{-1,0,1\}\}, \ k,l \in \mathbb{Z}, \\ (2k+1,2l) \ \textit{if} \ (x,y) \in \{(4k+2,4l+j); \\ \qquad\qquad\qquad j \in \{-1,0,1\}\}, \ k,l \in \mathbb{Z}, \\ (2k+1,2l+1) \ \textit{if} \ (x,y) = (4k+2,4l+2), \ k,l \in \mathbb{Z}. \end{cases}$$

The decomposition of the pretopological space $(\mathbb{Z}^2, r)$ given by h is demonstrated in Figure 7 by the dashed lines. Every class of the decomposition is mapped by h to its center point expressed in the bold coordinates.

Remark 3.3. The Khalimsky and Marcus-Wyse topologies are also quotient pretopologies of the topology r. The corresponding quotient maps are the same as those discussed in [16] (Theorems 10 and 11).

Analogously to r, also the topology studied in [16] has the property that v is one of its quotient topologies. But the corresponding quotient map does not have connected fibres. The advantage of the closure operator r over the topology from [16] is that h has connected fibres. Thus, the assumptions of Lemma 2.1 are satisfied if $p = r$, $q = v$ and e is the surjection h.

Using Lemma 2.1, Theorem 3.1 and Proposition 3.2, we may identify Jordan curves among the simple closed curves in $(\mathbb{Z}^2, v)$:

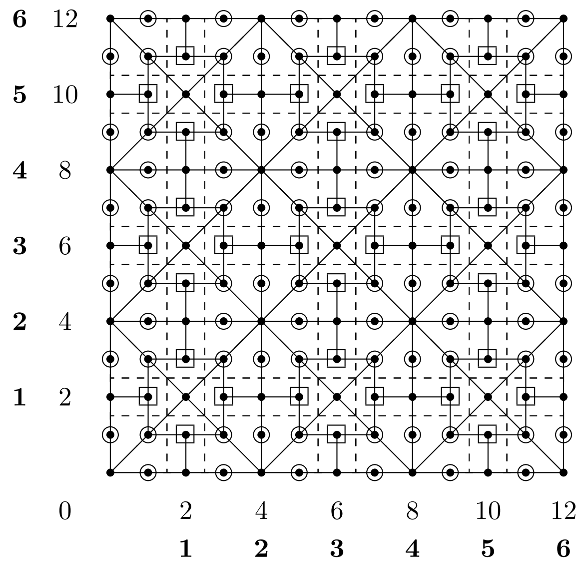

FIGURE 7. Decomposition of $(\mathbb{Z}^2, r)$ given by the surjection h.

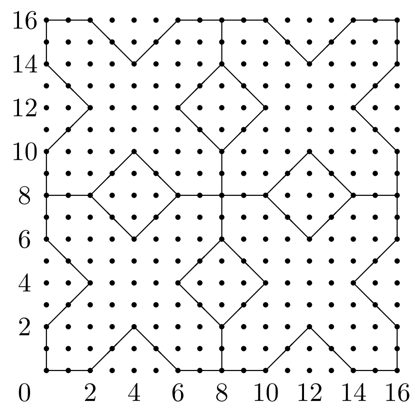

FIGURE 8. A subgraph of the connectedness graph of v.

Theorem 3.4. *Let D be a simple closed curve in $(\mathbb{Z}^2, v)$. Then D is a Jordan curve in $(\mathbb{Z}^2, v)$ if and only if one of the following two conditions is satisfied:*

(a) *$D = A_4(z)$ where $z \in \mathbb{Z}^2$ is a point with one coordinate even and the other odd.*
(b) *D has more than four points and every pair of different points $z_1, z_2 \in D$ with both coordinates even satisfies $A_4(z_1) \cap A_4(z_2) \subseteq D$.*

Proof. By Proposition 3.2, v is the quotient pretopology of r generated by h. It is evident that condition (a) is sufficient for D to be a Jordan curve in $(\mathbb{Z}^2, v)$. Let condition (b) be satisfied. It may easily be seen that there is precisely one cycle C in the square-diagonal graph satisfying $h(C) = D$. Thus, C is a Jordan curve in $(\mathbb{Z}^2, r)$. C consists of the center points of the sets $h^{-1}(z)$, $z \in C$, and the points laying between the pairs of center points of the sets $h^{-1}(z_1)$ and $h^{-1}(z_2)$ where $z_1, z_2 \in C$ are adjacent points in the connectedness graph of v (clearly, for every pair of points $z_1, z_2 \in C$ adjacent in the connectedness graph of v, there is precisely one point lying between the center points of $h^{-1}(z_1)$ and $h^{-1}(z_2)$ - it is the only point adjacent to each of the two center points in the square-diagonal graph). Since D is a simple closed curve in $(\mathbb{Z}^2, v)$, we have $\operatorname{card}(\{(x+i, y+j);\ i, j \in \{-2, 2\}\} \cap C) = 2$ for every point $(x, y) \in C$ with $x = 4k+2$ and $y = 4l+2$ for some $k, l \in \mathbb{Z}$. Further, we have card $C > 8$ because D has more than four points. The fact that every pair of different points $z_1, z_2 \in h(C)$ with both coordinates even satisfies $A_4(z_1) \cap A_4(z_2) \subseteq h(C)$ implies that $(4k, 4l+2) \in C$ whenever $(4k, 4l), (4k, 4l+4) \in C$ $(k, l \in \mathbb{Z})$ and $(4k+2, 4l) \in C$ whenever $(4k, 4l), (4k+4, 4l) \in C$ $(k, l \in \mathbb{Z})$.

Let C_1, C_2 be the two components of the subspace $\mathbb{Z}^2 - C$ of $(\mathbb{Z}^2, r)$ and put $C'_i = C_i - h^{-1}(D)$ for $i = 1, 2$. Since C has more than four points, we have $C'_i \neq \emptyset$ for $i = 1, 2$. Let $(x, y) \in D$ be a point and write $h^{-1}(x, y)$ briefly instead of $h^{-1}(\{(x, y)\})$. Clearly, there exists a point $(x, y) \in D$ with x or y even.

(1) Suppose that x is even and y is odd. Then $h^{-1}(x, y) = \{(2x+k, 2y);\ k \in \{-1, 0, 1\}\}$ where $(2x, 2y) \in C$ and there is $i \in \{1, 2\}$ such that $(2x-1, 2y) \in C_i$ and $(2x+1, 2y) \in C_{3-i}$. We also have $\{(x, y-1), (x, y+1)\} \subseteq D$ and, consequently, $\{(2x-1, 2y-1), (2x-1, 2y+1)\} \subseteq C_i$ and $\{(2x+1, 2y-1), (2x+1, 2y+1)\} \subseteq C_{3-i}$. Suppose that D contains exactly one of the points $(x-2, y-1)$ and $(x-2, y+1)$, say $(x-2, y-1)$. Then $(x-2, y+1) \notin D$ implies $h^{-1}(x-2, y+1) = A_8(2x-4, 2y+2) \cup \{(2x-4, 2y+2)\} \subseteq C'_i$. It may easily be seen that every point of C'_i may be joined with $(2x-4, 2y+2)$ by a path contained in C'_i (in the connectedness graph of r). It follows that

C'_i is connected. Further, we have $(x+2, y-1) \notin D$ because, otherwise, $\{(x+1, y-1)\} = A_4(x, y-1) \cap A_4(x+2, y-1) \subseteq D$, so that D would contain three points adjacent to $(x, y-1)$ (in the connectedness graph of v), namely $(x-1, y-1)$, (x, y) and $(x+1, y-1)$, which is impossible. Thus, we have $h^{-1}(x+2, y-1) = A_8(2x+4, 2y-2) \cup \{(2x+4, 2y-2)\} \subseteq C'_{3-i}$. It may easily be seen that every point of C'_{3-i} may be joined with $(2x+4, 2y-2)$ by a path contained in C'_{3-i} (in the connectedness graph of r). It follows that C'_{3-i} is connected.

(2) If x is odd and y is even, then the situation is analogous to the previous one.

(3) Suppose that both x and y are even. Then, by the assumptions of the statement, $D \cap A_4(x, y) = \emptyset$ and the set $A_8(x, y) - (A_4(x, y) \cup D)$ has precisely two points $z = (p, q)$ and $z' = (p', q')$ such that $p \neq p'$ and $q \neq q'$. It follows that there is $i \in \{1, 2\}$ such that $h^{-1}(p, q) = A_8(2p, 2q) \cup \{(2p, 2q)\} \subseteq C'_i$ and $h^{-1}(p', q') = A_8(2p', 2q') \cup \{(2p', 2q')\} \subseteq C'_{3-i}$. Using arguments similar to those used in (1), we may show that C'_i and C'_{3-i} are connected.

By (1)-(3), C'_i is connected for $i = 1, 2$. We clearly have $h(C_1) \cap h(C_2) = \emptyset$ (because, otherwise, there is a point $y \in h(C_1) \cap h(C_2)$, which means that $h^{-1}(\{z\}) \cap C_1 \neq \emptyset \neq h^{-1}(\{z\}) \cap C_2$ - this is a contradiction because $h^{-1}(\{z\})$ is connected). Therefore, $h(C'_1) \cap h(C'_2) = \emptyset$. This yields $C'_i = h^{-1}(h(C'_i))$ for $i = 1, 2$, hence $h(C'_i)$ is connected for $i = 1, 2$ by Lemma 2.1. Suppose that $\mathbb{Z}^2 - D$ is connected. Then $h^{-1}(\mathbb{Z}^2 - D) = C'_1 \cup C'_2$ is connected by Lemma 2.1. This is a contradiction because $\emptyset \neq C'_i \subseteq C_i$ for $i = 1, 2$, C_1 and C_2 are disjoint and $C_1 \cup C_2$ is not connected. Therefore, $\mathbb{Z}^2 - D = h(C'_1) \cup h(C'_2)$ is not connected and, consequently, $h(C'_1)$ and $h(C'_2)$ are components of $\mathbb{Z}^2 - D$. We have shown that also condition (b) is sufficient for D to be a Jordan curve in $(\mathbb{Z}^2, v)$.

To prove necessity, suppose that D is a Jordan curve in $(\mathbb{Z}^2, v)$ and that condition (a) is not valid. Then D has more than four points because there are no simple closed curves in $(\mathbb{Z}^2, v)$ with less than four points and the simple closed curves satisfying (a) are the only four-element Jordan curves in $(\mathbb{Z}^2, v)$. (There are other four-element simple closed curves in $(\mathbb{Z}^2, v)$, namely those obtained from $A_4(z)$, $z \in \mathbb{Z}^2$ a point with one coordinate even and the other odd, by replacing one of the points of $A_4(z)$ by z. But these simple closed curves are not Jordan curves in $(\mathbb{Z}^2, v)$ because, after deleting an arbitrary one from $(\mathbb{Z}^2, v)$, we get a connected topological space.) Suppose that there are points z_1, z_2 with both coordinates even such that $A_4(z_1) \cap A_4(z_2) \not\subseteq D$. Then the point $z \in A_4(z_1) \cap A_4(z_2)$ has one coordinate even and the other odd and z is isolated in $\mathbb{Z}^2 - D$, i.e., it is not joined with any other point by a path in $\mathbb{Z}^2 - D$.

Thus, $\{z\}$ is a component of $\mathbb{Z}^2 - D$, which is a contradiction because the Jordan curves satisfying (a) are the only Jordan curves in $(\mathbb{Z}^2, v)$ having a singleton component $\{z\}$ where $z \in \mathbb{Z}^2$ is a point with one coordinate even and the other odd. □

The following result immediately follows from Theorem 3.4.

Corollary 3.5. *Every cycle in the graph a portion of which is shown in Figure 8 is a Jordan curve in* $(\mathbb{Z}^2, v)$.

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