TOPOLOGY
PROCEEDINGS
Volume 39, 2012
Pages 27–39
http://topology.auburn.edu/tp/

Intrinsic Definition of Strong Shape for Compact Metric Spaces

by

Nikita Shekutkovski

Electronically published on April 14, 2011

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/
TOPOLOGY PROCEEDINGS
Volume 39 (2012)
Pages 27-39
E-Published on April 14, 2011

INTRINSIC DEFINITION OF STRONG SHAPE FOR COMPACT METRIC SPACES

NIKITA SHEKUTKOVSKI

Abstract. The best known approach to the intrinsic definition of shape of compact metric spaces is by use of near continuous functions $f : X \to Y$ and the corresponding notion of homotopy. A new definition of strong shape will be presented based on higher homotopies of this type.

1. Introduction

The best known approach to the intrinsic definition of shape of (metric) spaces is by use of near continuous functions $f : X \to Y$. The idea of ε-continuity (continuity up to $\varepsilon > 0$) leads to continuity up to some covering $\mathcal{V}$ of Y, i.e., $\mathcal{V}$-continuity and the corresponding $\mathcal{V}$-homotopy.

The first approach of this type for compact metric spaces was given in [2]. We also refer the reader to [7]. In [3], the approach is generalized to paracompact spaces.

In this paper, we present, for the first time, an intrinsic approach to strong shape theory and construct the strong shape category of compact metric spaces. The approach combines continuity up to a covering and the corresponding homotopies of second order, used in [4], in constructing the strong shape category of metric compacta. The homotopies of second order are used in actually the same way, for construction of a category of inverse systems in the paper [8].

In [4], it is shown that the approach using continuous homotopies of second order leads to standard strong shape theory in the case of compact metric spaces. It is shown that this approach is equivalent with the first definition of the strong shape category SSh given by J. Brendan Quigley [6] in 1973. The author believes that the strong shape category constructed in this paper and the category SSh are isomorphic.

2010 *Mathematics Subject Classification.* Primary 54C56, 55P55; Secondary 54C08.

Key words and phrases. Continuity up to a covering, intrinsic definition of shape, proximate sequence, strong shape.

2. Continuity Up to a Covering

Let X and Y be compact metric spaces. By a covering, we understand an open covering.

Definition 2.1. Suppose $\mathcal{V}$ is a finite covering of Y. A function $f : X \to Y$ is *$\mathcal{V}$-continuous at point* $x \in X$ if there exists a neighborhood U_x of x, and $V \in \mathcal{V}$ such that

$$f(U_x) \subseteq V.$$

A function $f : X \to Y$ is *$\mathcal{V}$-continuous* if it is $\mathcal{V}$-continuous at every point $x \in X$. In this case, the family of all U_x form a covering of X, and since X is compact, there exists a finite subcovering. By this, $f : X \to Y$ is *$\mathcal{V}$-continuous* if there exists a finite covering $\mathcal{U}$ of X, such that for any $x \in X$, there exists a neighborhood $U \in \mathcal{U}$ of x, and $V \in \mathcal{V}$ such that $f(U) \subseteq V$. We denote: there exists $\mathcal{U}$, such that $f\ (\mathcal{U}) \prec \mathcal{V}$.

If $f : X \to Y$ is $\mathcal{V}$-continuous, then $f : X \to Y$ is $\mathcal{W}$-continuous for any $\mathcal{W}$ such that $\mathcal{V} \prec \mathcal{W}$.

If $\mathcal{V}$ is a finite covering of Y and $V \in \mathcal{V}$, the open set $\mathrm{st}(V)$ (star of V) is the union of all $W \in \mathcal{V}$ such that $W \bigcap V \neq \emptyset$. We form a new covering of Y, $\mathrm{st}(\mathcal{V}) = \{\mathrm{st}(V) | V \in \mathcal{V}\}$.

Theorem 2.2. *Suppose $\mathcal{V}$ is a finite covering of Y, $X = X_1 \bigcup X_2$, X_i closed, i =1,2, and $f_i : X_i \to Y$, $\mathcal{V}$-continuous functions, i =1,2, such that $f_1(x) = f_2(x)$ for all $x \in X_1 \bigcap X_2$. We define a function $f : X \to Y$ by*

$$f(x) = f_i(x), \textit{for } x \in X_i,\ i{=}1,2.$$

Then

(1) *if $x \in \mathit{Int}X_1$ or $x \in \mathit{Int}X_2$, then $f : X \to Y$ is $\mathcal{V}$-continuous at x;*
(2) *if $x \in \mathrm{Fr}X_1$ and $x \in \mathrm{Fr}X_2$, then $f : X \to Y$ is $\mathrm{st}(\mathcal{V})$-continuous at x.*

Proof. (1) If $x \in \mathrm{Int}\ X_1$ or $x \in \mathrm{Int}\ X_2$, then there exists an open subset U of X, $x \in U$, such that $f(U) \subseteq V$ for some $V \in \mathcal{V}$; i.e., $f : X \to Y$ is $\mathcal{V}$-continuous at x.

(2) If $x \in \mathrm{Fr}X_1$ and $x \in \mathrm{Fr}X_2$, since X_i are closed, i =1,2, then $x \in X_1$ and $x \in X_2$. There exist open subsets U_i' of X_i, $x \in U_i'$, and $V_i \in \mathcal{V}$, such that $f(U_i') \subseteq V_i$, for i =1,2. Then $V_1 \bigcap V_2 \neq \emptyset$, since $f_1(U_1') \bigcap f_2(U_2') \neq \emptyset$.

There are open subsets U_i of X, $U_i' = U_i \bigcap X_i$, i=1,2. We put $U = U_1 \bigcap U_2$. Then U is a neighborhood of x in X and

$$\begin{aligned} f(U) &= f_1(U \textstyle\bigcap X_1) \bigcup f_2(U \bigcap X_2) \\ &\subseteq f_1(U_1 \textstyle\bigcap X_1) \bigcup f_2(U_2 \bigcap X_2) = f_1(U_1') \bigcup f_2(U_2') \subseteq V_1 \bigcup V_2\,. \end{aligned}$$

It follows that $f : X \to Y$ is $\mathrm{st}(\mathcal{V})$-continuous at x. □

Definition 2.3. The functions $f, g : X \to Y$ are $\mathcal{V}$-*homotopic* if there exists a function $F : I \times X \to Y$ such that

(1) $F : I \times X \to Y$ is $\mathrm{st}(\mathcal{V})$-continuous;
(2) there exists a neighborhood $N = [0, \varepsilon) \bigcup (1-\varepsilon, 1]$, $\epsilon > 0$, of $\partial I = \{0, 1\}$ such that $F|_{N \times X}$ is $\mathcal{V}$-continuous;
(3) $F(0, x) = f(x)$ and $F(1, x) = g(x)$.

We denote the relation of homotopy by $f \underset{\mathcal{V}}{\sim} g$.

Proposition 2.4. *The relation of homotopy $f \underset{\mathcal{V}}{\sim} g$ is an equivalence relation.*

Proof. If $f \underset{\mathcal{V}}{\sim} g$ by a $\mathcal{V}$-homotopy $F : I \times X \to Y$ and $g \underset{\mathcal{V}}{\sim} h$ by a $\mathcal{V}$-homotopy $G : I \times X \to Y$, we define a homotopy $H : I \times X \to Y$ by

$$H(s, x) = \begin{cases} F(2s, x), & 0 \leq s \leq \frac{1}{2} \\ G(2s-1, x), & \frac{1}{2} \leq s \leq 1 \,. \end{cases}$$

Since F is $\mathcal{V}$-continuous on $\left(\frac{1}{2} - \delta, \frac{1}{2}\right] \times X$ for some δ, $\frac{1}{2} > \delta > 0$, and G is $\mathcal{V}$-continuous on $\left[\frac{1}{2}, \frac{1}{2} + \varepsilon\right) \times X$ for some ε, $\frac{1}{2} > \varepsilon > 0$, by Theorem 2.2, it follows that H is $\mathrm{st}(\mathcal{V})$-continuous on $\left(\frac{1}{2} - \delta, \frac{1}{2} + \varepsilon\right) \times X$; i.e., $H : I \times X \to Y$ is $\mathrm{st}(\mathcal{V})$-continuous. We have checked only one of several conditions. □

Remark 2.5. In Definition 2.3, we cannot replace conditions (1) and (2) by the expected condition: $F : I \times X \to Y$ is $\mathcal{V}$-continuous. A simple example in [1] shows that in this case the usual concatenation of two homotopies which are $\mathcal{V}$-continuous is not always $\mathcal{V}$-continuous.

At the end of this section, an intrinsic definition of shape is presented which somewhat differs from the definition in [2]. However, it is shown that the definition of a shape morphism in this paper coincides with the definition in [2] and, consequently, coincides with the standard definition.

The main notion for the intrinsic definition of shape for compact metric spaces is the notion of proximate sequence from X to Y.

A sequence of finite coverings, $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \ldots$ of a compact metric space such that for any covering $\mathcal{V}$, there exists n, such that $\mathcal{V} \succ \mathcal{V}_n$ we call a *cofinal sequence of finite coverings.*

Definition 2.6. A sequence (f_n) of functions $f_n : X \to Y$ is a *proximate sequence* from X to Y if there exists a cofinal sequence of finite coverings of Y, $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \ldots$, and for all indices $m \geq n$, f_n and f_m are $\mathcal{V}_n$-homotopic. In this case, we say that (f_n) is a proximate sequence over $(\mathcal{V}_n)$.

If (f_n) and (f'_n) are proximate sequences from X to Y, then there exists a cofinal sequence of finite coverings $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \ldots$ such that (f_n) and (f'_n) are proximate sequences over $(\mathcal{V}_n)$.

Two proximate sequences (f_n) and $(f'_n) : X \to Y$ are *homotopic* if for some cofinal sequence of finite coverings $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \ldots$, (f_n) and (f'_n) are proximate sequences over $(\mathcal{V}_n)$, and for all integers, f_n and f'_n are $(\mathcal{V}_n)$-homotopic. We say that (f_n) and (f'_n) are homotopic over $(\mathcal{V}_n)$.

By Remark 2.5 and Proposition 2.4, it follows that the relation of homotopy of two proximate sequences (f_n) and (f'_n) is an equivalence relation. We denote the equivalence class by $[(f_n)]$.

In [2], the set of $\mathcal{V}$-homotopy classes of $\mathcal{V}$-continuous functions $f : X \to Y$ is denoted by $[(X, Y)]_{\mathcal{V}}$. The homotopy class of f is denoted by $[f]_{\mathcal{V}}$. If $\mathcal{V} \succ \mathcal{V}'$ and $f \underset{\mathcal{V}'}{\sim} g$, then $f \underset{\mathcal{V}}{\sim} g$. So the map $p_{\mathcal{V}\mathcal{V}'} : [X, Y]_{\mathcal{V}'} \to [X, Y]_{\mathcal{V}}$, defined by $p_{\mathcal{V}\mathcal{V}'}([f]_{\mathcal{V}'}) = [f]_{\mathcal{V}}$, is well defined, and

$$([X, Y]_{\mathcal{V}}, p_{\mathcal{V}\mathcal{V}'}, \mathcal{V} \text{ finite covering})$$

is an inverse system in the category of sets and functions.

The inverse limit of this inverse system is denoted by $\varprojlim_{\mathcal{V}} [X, Y]_{\mathcal{V}}$. In [2], a bijection is established between $\varprojlim_{\mathcal{V}} [X, Y]_{\mathcal{V}}$ and the set of all shape morphisms from X to Y.

Theorem 2.7. *There is a bijection between the set* $\varprojlim_{\mathcal{V}} [X, Y]_{\mathcal{V}}$ *and the set of homotopy classes* $[(f_n)]$ *of proximate sequences* $(f_n) : X \to Y$.

Proof. Suppose $(f_n) : X \to Y$ is a proximate sequence over $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \ldots$. Then the inverse limit $\varprojlim_{\mathcal{V}} [X, Y]_{\mathcal{V}}$ is isomorphic to the inverse limit $\varprojlim_{n} [X, Y]_{\mathcal{V}_n}$ of the inverse sequence

$$([X, Y]_{\mathcal{V}_n}, p_{\mathcal{V}_n\mathcal{V}_m}, n \in N),$$

where for $n < m$, $p_{\mathcal{V}_n\mathcal{V}_m}([f]_{\mathcal{V}_m}) = [f]_{\mathcal{V}_n}$.

With the homotopy class $[(f_n)]$ of the proximate sequence $(f_n) : X \to Y$, we associate the element

$$([f_n]_{\mathcal{V}_n}) = ([f_1]_{\mathcal{V}_1}, [f_2]_{\mathcal{V}_2}, \ldots)$$

of $\varprojlim_{n} [X, Y]_{\mathcal{V}_n}$. First, this is an element of $\varprojlim_{n} [X, Y]_{\mathcal{V}_n}$ since for $n < m$, from $f_n \underset{\mathcal{V}_n}{\sim} f_m$, it follows $p_{\mathcal{V}_n\mathcal{V}_m}([f_m]_{\mathcal{V}_m}) = [f_n]_{\mathcal{V}_n}$.

Second, if (f_n) and (f'_n) are homotopic over $(\mathcal{V}n)$, then for all integers $f_n \underset{\mathcal{V}_n}{\sim} f'_n$, and it follows $([f_n]_{\mathcal{V}_n}) = ([f'_n]_{\mathcal{V}_n})$.

Third, for a given element $([g_n]_{\mathcal{V}_n}) = ([g_1]_{\mathcal{V}_1}, [g_2]_{\mathcal{V}_2}, ...)$ of $\varprojlim_n [X, Y]_{\mathcal{V}_n}$, we can find a proximate sequence $(g_n) : X \to Y$ such that $([g_n]_{\mathcal{V}_n})$ is associated with the homotopy class $[(g_n)]$.

This proves the theorem. □

Let $(f_n) : X \to Y$ be a proximate sequence over $(\mathcal{V}_n)$, and let $(g_k) : Y \to Z$ be a proximate sequence over $(\mathcal{W}_k)$. For a covering $\mathcal{W}_k$ of Z, there exists a covering $\mathcal{V}_{n_k}$ of Y such that $g(\mathcal{V}_{n_k}) \prec \mathcal{W}_k$. Then the *composition* of these two proximate sequences is the proximate sequence $(h_k) = (g_k f_{n_k}) : X \to Z$.

Compact metric spaces and homotopy classes of proximate sequences $[(f_n)]$ form the *shape category* of metric compacta. Actually, the verification that a category is obtained is given in the next section.

3. Intrinsic Definition of Strong Shape

The sequence of pairs $(f_n, f_{n,n+1})$ of functions $f_n : X \to Y$ and $f_{n,n+1} : I \times X \to Y$ is a *strong proximate sequence* from X to Y if there exists a cofinal sequence of finite coverings, $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \ldots$ of Y, such that for each natural number n, $f_n : X \to Y$ is a $(\mathcal{V}_n)$-continuous function and $f_{n,n+1} : I \times X \to Y$ is a homotopy connecting $(\mathcal{V}_n)$-continuous functions $f_n : X \to Y$ and $f_{n+1} : X \to Y$.

We say that $(f_n, f_{n,n+1})$ is a strong proximate sequence over $(\mathcal{V}_n)$.

If $(f_n, f_{n,n+1})$ and $(f'_n, f'_{n,n+1})$ are strong proximate sequences from X to Y, then there exists a cofinal sequence of finite coverings $(\mathcal{V}_n)$ such that $(f_n, f_{n,n+1})$ and $(f'_n, f'_{n,n+1})$ are strong proximate sequences over $(\mathcal{V}_n)$.

Two strong proximate sequences $(f_n, f_{n,n+1})$ and $(f'_n, f'_{n,n+1}) : X \to Y$ are *homotopic* if there exists a strong proximate sequence $(F_n, F_{n,n+1}) : I \times X \to Y$ over $(\mathcal{V}_n)$ such that

(1) $F_n : I \times X \to Y$ is a homotopy between $\mathcal{V}_n$-continuous maps f_n and f'_n;

(2) $F_{n,n+1} : I \times I \times X \to Y$ is a $\mathrm{st}^2(\mathcal{V}_n)$-continuous function, and there exists a neighborhood N of ∂I^2 such that $F_{n,n+1}|_{N \times X}$ is $\mathrm{st}(\mathcal{V}_n)$-continuous, and

$$F_{n,n+1}(t, 0, x) = f_{n,n+1}(t, x), \quad F_{n,n+1}(t, 1, x) = f'_{n,n+1}(t, x).$$

We denote the homotopy relation by $(f_n, f_{n,n+1}) \approx (f'_n, f'_{n,n+1})$.

Theorem 3.0.1. *The relation of homotopy of strong proximate sequences is an equivalence relation.*

Proof. We will prove the transitivity of the relation. If $(f_n, f_{n,n+1}) \approx (f'_n, f'_{n,n+1})$ by a homotopy $(F_n, F_{n,n+1}) : I \times X \to Y$ over $(\mathcal{V}_n)$ and $(f'_n, f'_{n,n+1}) \approx (f''_n, f''_{n,n+1})$ by a homotopy $(G_n, G_{n,n+1}) : I \times X \to Y$

over $(\mathcal{V}_n)$, we define a homotopy $(H_n, H_{n,n+1}) : I \times X \to Y$, defining $H_n : I \times X \to Y$ by

$$H_n(s,x) = \begin{cases} F_n(2s,x), & 0 \le s \le \frac{1}{2} \\ G_n(2s-1,x), & \frac{1}{2} \le s \le 1 \end{cases}$$

and $H_{n,n+1} : I \times I \times X \to Y$ by

$$H_{n,n+1}(t,s,x) = \begin{cases} F_{n,n+1}(t,2s,x), & 0 \le s \le \frac{1}{2} \\ G_{n,n+1}(t,2s-1,x), & \frac{1}{2} \le s \le 1\,. \end{cases}$$

Since $F_{n,n+1}$ is st$(\mathcal{V})$-continuous on $I \times \left(\frac{1}{2}-\delta, \frac{1}{2}\right] \times X$, for some δ, $\frac{1}{2} > \delta > 0$, and $G_{n,n+1}$ is st$(\mathcal{V})$-continuous on $I \times \left[\frac{1}{2}, \frac{1}{2}+\varepsilon\right) \times X$, for some ε, $\frac{1}{2} > \varepsilon > 0$, by Theorem 2.2, it follows that $H_{n,n+1}$ is st$^2(\mathcal{V})$-continuous on $I \times \left(\frac{1}{2}-\delta, \frac{1}{2}+\varepsilon\right) \times X$; i.e., $H_{n,n+1} : I \times I \times X \to Y$ is st$^2(\mathcal{V})$-continuous. □

We denote, by $[(f_n, f_{n,n+1})]$, the homotopy class of a strong proximate sequence $(f_n, f_{n,n+1})$.

Suppose $(f_n, f_{n,n+1}) : X \to Y$ is a strong proximate sequence. If $n \le n'$ are two indices, we define $f_{n,n} : I \times X \to Y$ by

$$f_{n,n}(t,x) = f_n(x),$$

and for $n < n'$, and $k = n' - n$,

$$\begin{aligned} f_{n,n'}(t,x) &= f_{n,n+1} * f_{n+1,n+2} * \cdots * f_{n'-1,n'}(t,x) \\ &= \begin{cases} f_{n,n+1}(kt,x), & 0 \le t \le \frac{1}{k} \\ f_{n+1,n+2}(kt-1,x), & \frac{1}{k} \le t \le \frac{2}{k} \\ \qquad \cdots & \\ f_{n'-1,n'}(kt-k+1,x), & \frac{k-1}{k} \le t \le 1. \end{cases} \end{aligned}$$

If $(f_n, f_{n,n+1})$ is a strong proximate sequence over $(\mathcal{V}_n)$, then the function $f_{n,n'}$ is a $\mathcal{V}_n$-homotopy connecting functions f_n and $f_{n'}$.

Definition 3.0.2. A *strong proximate subsequence* $(g_k, g_{k,k+1})$ of the strong proximate sequence $(f_n, f_{n,n+1})$ is defined by a strictly increasing sequence of integers $n_1 < n_2 < \ldots$ and by maps $g_k = f_{n_k}$ and $g_{k,k+1} : I \times X \to Y$,

$$g_{k,k+1} = f_{n_k,n_k+1} * f_{n_k+1,n_k+2} * \ldots * f_{n_{k+1}-1,n_{k+1}}.$$

In order to prove the next theorem, we need the following functions defined on 2-simplexes. In the plane with orthogonal coordinates (t,s), let e_0, e_1, and e_2 be the vertices of a simplex $\Delta=[e_0,e_1,e_2]$, and let (b_0,b_1,b_2) be barycentric coordinates in the simplex such that $e_0=(1,0,0)$, $e_1=(0,1,0)$, and $e_2=(0,0,1)$. The map $L : \Delta \to I$, defined by $L(t,s) = b_2$, is continuous and $L(e_0) = 0$, $L(e_1) = 0$, and $L(e_2) = 1$. The restriction of the function $L : \Delta \to I$ to the 1-simplex $[e_0,e_2]$ is a linear map.

The same is true for the 1-simplex $[e_1, e_2]$, while on $[e_0, e_1]$, L equals 0. We define a function $\Phi_{n,n'} : \Delta \times X \to Y$ by

$$\Phi_{n,n'}(t, s, x) = f_{n,n'}(L(t, s), x).$$

Then $\Phi_{n,n'} : \Delta \times X \to Y$ is st$(\mathcal{V}_n)$-continuous and $\Phi_{n,n'}(e_0, x) = f_n(x)$, $\Phi_{n,n'}(e_1, x) = f_n(x)$, and $\Phi_{n,n'}(e_2, x) = f_{n'}(x)$. Moreover, $\Phi_{n,n'} : \Delta \times X \to Y$ is $\mathcal{V}_n$-continuous at (e_0, x), (e_1, x), and (e_2, x) for any $x \in X$. For example, since $f_{n,n'} : I \times X \to Y$ is $\mathcal{V}_n$-continuous at $(0, x)$, there exist $[0, \varepsilon) \subseteq I$, U a neighborhood of x in X, and $V \in \mathcal{V}_n$ such that $f_{n,n'}([0, \varepsilon) \times U) \subseteq V$. Then $L^{-1}([0, \varepsilon))$ is a neighborhood of e_0, and

$$\Phi_{n,n'}(L^{-1}[0, \varepsilon) \times U) = f_{n,n'}(LL^{-1}[0, \varepsilon) \times U) \subseteq f_{n,n'}([0, \varepsilon) \times U) \subseteq V,$$

i.e., is $\mathcal{V}_n$-continuous at (e_0, x).

Also, we define a function $\Phi^{-1}_{n,n'} : \Delta \times X \to Y$ by

$$\Phi^{-1}_{n,n'}(t, s, x) = f_{n,n'}1 - (L(t, s), x).$$

Then $\Phi^{-1}_{n,n'} : \Delta \times X \to Y$ is st$(\mathcal{V}_n)$-continuous and $\Phi^{-1}_{n,n'}(e_0, x) = f_{n'}(x)$, $\Phi^{-1}_{n,n'}(e_1, x) = f_{n'}(x)$, and $\Phi^{-1}_{n,n'}(e_2, x) = f_n(x)$.

Finally, if Δ is the half square, $\Delta = \{(t, s)|t \geq s\} \subseteq I \times I$, and $n \leq n' \leq n''$ are three indices and $n < n''$, we put $v = (\frac{n'-n}{n''-n}, \frac{n'-n}{n''-n})$ and we define a function $f_{nn'n''} : \Delta \times X \to Y$ by

$$f_{nn'n''}(t, s, x) = \begin{cases} \Phi^{-1}_{nn'}(t, s, x), & (t, s) \in [(1, 0), v, (0, 0)] \\ \Phi_{n'n''}(t, s, x), & (t, s) \in [v, (1, 0), (1, 1)]. \end{cases}$$

Theorem 3.0.3. *If $(f_n, f_{n,n+1})$ is a strong proximate sequence over $(\mathcal{V}_n)$ and $(g_k, g_{k,k+1})$ is a subsequence, then $(g_k, g_{k,k+1})$ is also a strong proximate sequence over $(\mathcal{V}_n)$. The strong proximate sequences $(f_n, f_{n,n+1})$ and $(g_k, g_{k,k+1})$ are homotopic.*

Proof. We have to define a homotopy $(H_k, H_{k,k+1})$ between strong proximate sequences $(f_k, f_{k,k+1})$ and $(g_k, g_{k,k+1})$. First, we define a homotopy $H_k : I \times X \to Y$ by $H_k(t, x) = f_{k,n_k}(t, x)$. Now, we have to define $H_{k,k+1} : I \times I \times X \to Y$.

Case 1: If $n_k = k$ and $n_{k+1} = k + 1$, then $(f_k, f_{k,k+1}) = (g_k, g_{k,k+1})$.

Case 2: $n_k = k$ and $n_{k+1} > k + 1$. We put $p = n_{k+1} - k$,

$$T = \{(t, s)|0 \leq t \leq 1 + (\frac{1}{p} - 1) \cdot s\} \subseteq I \times I,$$

and define $H_{k,k+1} : I \times I \times X \to Y$ by

$$H_{k,k+1}(t, s, x) = \begin{cases} f_{k,k+1}(\frac{p}{(1-s)p+s} \cdot t, x), & (t, s, x) \in T \times X \\ \Phi_{k+1,n_{k+1}}(t, s, x), & (t, s) \in [(1, \frac{1}{p}), (1, 0), (1, 1)]. \end{cases}$$

The function is well defined along the line $t = 1 + (\frac{1}{p} - 1) \cdot s$. The two points of this line which belong to $\partial(I \times I)$ are $(1, 0)$ and $(\frac{1}{p}, 1)$. The function $\Phi_{k+1,n_{k+1}}$ is $\mathcal{V}_k$-continuous at $(1, 0, x)$ and $(\frac{1}{p}, 1, x)$. We have to prove that the function defined by the expression $f_{k,k+1}(\frac{p}{(1-s)p+s} \cdot t, x)$ is also $\mathcal{V}_k$-continuous at these points. Then, by Theorem 2.2, $H_{k,k+1} : I \times I \times X \to Y$ will be $\mathrm{st}(\mathcal{V}_k)$-continuous at these points, and condition (2) for a homotopy of strong proximate sequences will be satisfied. If we put $h(t, s) = \frac{p}{(1-s)p+s} \cdot t$, $h : T \to I$, then $h(1, 0) = 1$ and $h(\frac{1}{p}, 1) = 1$ and

$$f_{k,k+1}(h(t, s), x) = f_{k,k+1}(\frac{p}{(1 - s)p + s} \cdot t, x).$$

Since $f_{k,k+1} : I \times X \to Y$ is $\mathcal{V}_k$-continuous at $(1, 0, x)$, there exist $(1 - \varepsilon, 1] \subseteq I$, a neighborhood U of x in X, and $V \in \mathcal{V}_k$ such that $f_{k,k+1}((1 - \varepsilon, 1] \times U) \subseteq V$. Then $h^{-1}((1 - \varepsilon, 1])$ is a neighborhood of $(1, 0)$, and

$$f_{k,k+1}(h(h^{-1}(1 - \varepsilon, 1]) \times U) \subseteq f_{k,k+1}((1 - \varepsilon, 1] \times U) \subseteq V;$$

i.e., the function defined by the expression $f_{k,k+1}(\frac{p}{(1-s)p+s} \cdot t, x)$ is $\mathcal{V}_k$-continuous at $(1, 0, x)$.

Case 3: $n_k > k$ and $n_{k+1} > k+1$. In order to define $H_{k,k+1} : I \times I \times X \to Y$, we decompose $I \times I = \Delta_1 \bigcup P \bigcup \Delta_2$, where Δ_1 and Δ_2 are the simplexes, $\Delta_1 = [(1, 0), (0, \frac{1}{n_k-k}), (0, 0)]$ and $\Delta_2 = [(0, 1), (1, 1 - \frac{1}{n_k-k}), (1, 1)]$ and Π is the parallelogram defined by the vertices $(0, \frac{1}{n_k-k})$, $(0, 1)$, $(1, 0)$, $(1, 1 - \frac{1}{n_k-k})$. We put

$$H_{k,k+1}(t, s, x) = \begin{cases} \Phi^{-1}_{k,k+1}(t, s, x), & (t, s, x) \in \Delta_1 \times X \\ f_{k,n_k}(\frac{t}{n_k-k} + s, x), & (t, s, x) \in \Pi \times X \\ \Phi_{k+1,n_{k+1}}(t, s, x), & (t, s, x) \in \Delta_2 \times X. \end{cases} \qquad \square$$

Proposition 3.0.4. *If $f : X \to Y$ is $\mathcal{W}$-continuous, then $id \times f : K \times X \to K \times Y$ is $K \times \mathcal{W}$-continuous, where K is compact and $K \times \mathcal{W} = \{K \times W | W \in \mathcal{W}\}$.*

Theorem 3.0.5. *Let $G : I \times Y \to Z$ be a $\mathrm{st}(\mathcal{W})$-continuous function and let there exist a neighborhood N of ∂I, such that $G|_{N \times Y}$ is $\mathcal{W}$-continuous. Then there exists a finite covering $\mathcal{V}$ of Y, such that for each $\mathcal{V}$-continuous function $f : X \to Y$, $G(id \times f) : I \times X \to Z$ is $\mathrm{st}(\mathcal{W})$-continuous, and $G(id \times f)|_{N' \times X}$ is $\mathcal{W}$-continuous.*

Proof. Choose a point $y \in Y$. For any $s \in I$, there exist an interval J^y_s (a neighborhood of s in I) and a neighborhood V^s_y of y such that

$$G(J^y_s \times V^s_y) \subset W^s_{*y}$$

for some $W^s_{*y} \in \mathrm{st}(\mathcal{W})$, $0 < s < 1$, and

$$G(J^y_0 \times V^0_y) \subset W^0_y, \quad G(J^y_1 \times V^1_y) \subset W^1_y$$

for some $W^0_y, W^1_y \in \mathcal{W}$.

Then $\{J_s^y | s \in I\}$ is an open covering of the interval $I = [0,1]$. There is a finite subcovering $J_{s_1}^y, J_{s_2}^y, \ldots, J_{s_n}^y$ which is minimal. We can choose the indices in such a way that

$$J_{s_1}^y = [0, b_1), J_{s_2}^y = (a_2, b_2), \ldots, J_{s_n}^y = (a_n, 1]$$

and $a_{i+1} < b_i$. If we put

$$V_y = V_y^{s_1} \cap V_y^{s_2} \cap \cdots \cap V_y^{s_n},$$

then $G((a_i, b_i) \times V_y) \subset W_{*y}^{s_i}$ and $G([0, b_1) \times V_y) \subset W_y^{s_1}$, $G((a_n, 1] \times V_y) \subset W_y^{s_n}$.

There is a finite subcovering $\mathcal{V} = \{V_z | \ z \in Z\}$ of the covering $\{V_y | \ y \in Y\}$. Then

$$\bigcup_{V_z \in \mathcal{V}} \{J_{s_i}^z \times V_z | i = 1, ..., n\}$$

is a finite covering of $I \times Y$ such that $G(J_{s_i}^z \times V_z)$ is contained in some member of $\mathrm{st}(\mathcal{W})$, $0 < s_i < 1$, and $G(J_0^z \times V_z)$ and $G(J_1^z \times V_z)$ are contained in some member of $\mathcal{W}$.

Now, if $f : X \to Y$ is a $\mathcal{V}$-continuous function, then the function $H : I \times X \to Z$, defined by

$$H(t, x) = G(t, f(x)),$$

is $\mathrm{st}(\mathcal{W})$-continuous, and $G(id \times f)|_{N' \times X}$ is $\mathcal{W}$-continuous. □

3.1. **Composition of Strong Proximate Sequences.** The composition of strong proximate sequences $(f_n, f_{n,n+1}) : X \to Y$ and $(g_k, g_{k,k+1}) : Y \to Z$ is a strong proximate sequence $(h_k, h_{k,k+1}) : X \to Z$ defined in the following way:

If $(g_k, g_{k,k+1}) : Y \to Z$ is a strong proximate sequence over $\mathcal{W}_k$, then $g_{k,k+1} : I \times Y \to Z$ is a homotopy connecting $\mathcal{W}_k$-continuous functions g_k and g_{k+1}.

(1) As in Theorem 3.0.5, by putting $G = g_{k,k+1}$, we can construct a cofinal sequence of finite coverings $\mathcal{V}_k$ of Y, such that for each $\mathcal{V}_k$-continuous function $f : X \to Y$, $G(id \times f) : I \times X \to Z$ is $\mathrm{st}(\mathcal{W}_k)$-continuous, and $G(id \times f)|_{N \times X}$ is $\mathcal{W}_k$-continuous.

We will choose a sequence of integers $n_1 < n_2 < \ldots$. There exists an integer n_1 such that for $n \geq n_1$, the function $f_{n_1,n}$ is a homotopy connecting $\mathcal{W}_1$-continuous functions f_{n_1} and f_n.

There exists an integer n_2, such that for $n \geq n_2$, the function $f_{n_2,n}$ is a homotopy connecting $\mathcal{W}_2$-continuous functions f_{n_2} and f_n. We proceed in this away and construct the sequence of integers $n_1 < n_2 < \ldots$.

From (1), it follows that the function $g_{k,k+1}(1 \times f_{n_{k+1}}) : I \times X \in Z$ is a homotopy connecting $\mathcal{W}_k$-continuous functions $g_k f_{n_{k+1}}$ and $g_{k+1} f_{n_{k+1}}$.

From (1) and Proposition 3.0.4, it follows that the function $g_k f_{n_k,n_{k+1}} : I \times X \in Z$ is a homotopy connecting $\mathcal{W}_k$-continuous functions $g_k f_{n_k}$ and $g_k f_{n_{k+1}}$.

By the two previous statements and Theorem 2.2, it follows that the sequence of pairs of functions $(h_k, h_{k,k+1})$, defined as follows, is a strong proximate sequence.

We put

$$h_k = g_k f_{n_k}$$

and

$$h_{k,k+1}(t,x) = \begin{cases} g_k f_{n_k,n_{k+1}}(2t,x), & 0 \leq t \leq \frac{1}{2} \\ g_{k,k+1}(2t-1, f_{n_{k+1}}(x)), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Here, $f_{n_k,n_{k+1}} = f_{n_k,n_k+1} * ... * f_{n_{k+1}-1,n_{k+1}}$.

The function is well defined since

$$g_k f_{n_k,n_{k+1}}(2 \cdot \tfrac{1}{2}, x) = g_k f_{n_{k+1}}(x) = g_{k,k+1}(2 \cdot \tfrac{1}{2} - 1, f_{n_{k+1}}(x))$$

and

$$h_{k,k+1}(0,x) = g_k f_{n_k,n_{k+1}}(0,x) = g_k f_{n_k}(0,x) = h_k(x)$$
$$h_{k,k+1}(1,x) = g_{k,k+1}(1, f_{n_{k+1}}(x)) = g_{k+1} f_{n_{k+1}}(x) = h_{k+1}(x).$$

We have to prove that the composition does not depend on the choice of subsequences.

Suppose that there is another choice of a subsequence $n'_1 < n'_2 < \dots$. The corresponding strong proximate sequence $(h'_k, h'_{k,k+1}) : X \to Z$ is defined by

$$h'_k = g_k f_{n'_k}$$

and

$$h'_{k,k+1}(t,x) = \begin{cases} g_k f_{n'_k,n'_{k+1}}(2t,x), & 0 \leq t \leq \frac{1}{2} \\ g_{k,k+1}(2t-1, f_{n'_{k+1}}(x)), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

We have to prove that $(h_k, h_{k,k+1}) : X \to Z$ and $(h'_k, h'_{k,k+1}) : X \to Z$ are homotopic.

We can construct a sequence $n''_1 < n''_2 < \dots$ such that $n_i, n'_i < n''_i$ for every i, and we define a strong proximate sequence $(h''_k, h''_{k,k+1}) : X \to Z$, putting

$$h''_k = g_k f_{n''_k}$$

and

$$h''_{k,k+1}(t,x) = \begin{cases} g_k f_{n''_k,n''_{k+1}}(2t,x), & 0 \leq t \leq \frac{1}{2} \\ g_{k,k+1}(2t-1, f_{n''_{k+1}}(x)), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

To prove that $(h_k, h_{k,k+1}) : X \to Z$ and $(h'_k, h'_{k,k+1}) : X \to Z$ are homotopic, it is enough to prove that $(h_k, h_{k,k+1}) : X \to Z$ and $(h''_k, h''_{k,k+1}) : X \to Z$ are homotopic.

We define a homotopy $(H_k, H_{k,k+1}) : I \times X \to Z$ by

$$H_k(t,x) = g_k f_{n_k n''_k}(t,x).$$

This homotopy connects $h_k = g_k f_{n_k}$ and $h''_k = g_k f_{n''_k}$.

We define $H_{k,k+1} : I \times I \times X \to Z$ by

$$H_{k,k+1}(t,s,x) = \begin{cases} g_k f_{n_k n''_k n''_{k+1}}(s,2t,x), & 0 \le t \le \frac{s}{2} \\ g_k f_{n_k n_{k+1} n''_{k+1}}(2t,s,x), & \frac{s}{2} \le t \le \frac{1}{2} \\ g_{k,k+1}(2t-1, f_{n_{k+1} n''_{k+1}}(s,x)), & \frac{1}{2} \le t \le 1\,. \end{cases}$$

Then $(H_k, H_{k,k+1}) : X \times I \to Z$ is a homotopy connecting strong proximate sequences $(h_k, h_{k,k+1}) : X \to Z$ and $(h''_k, h''_{k,k+1}) : X \to Z$.

3.2. **Composition of Homotopy Classes of Strong Proximate Sequences.** In subsection 3.1, we showed that the definition of the composition $(g_n, g_{n,n+1}) \circ (f_n, f_{n,n+1}) : X \to Z$ of strong proximate sequences $(f_n, f_{n,n+1}) : X \to Y$ and $(g_n, g_{n,n+1}) : Y \to Z$ is well defined.

Now we define composition of homotopy classes in the standard way by

$$[(g_n, g_{n,n+1})] \circ [(f_n, f_{n,n+1})] = [(g_n, g_{n,n+1}) \circ (f_n, f_{n,n+1})].$$

Remark 3.2.1. By taking a subsequence of the original strong proximate sequence $(f_n, f_{n,n+1})$, we can suppose that the composition $(h_n, h_{n,n+1}) : X \to Z$ of $(f_n, f_{n,n+1}) : X \to Y$ and $(g_n, g_{n,n+1}) : Y \to Z$ is given by

$$h_n = g_n f_n$$

and

$$h_{n,n+1}(t,x) = \begin{cases} g_n f_{n,n+1}(2t,x), & 0 \le t \le \frac{1}{2} \\ g_{n,n+1}(2t-1, f_{n+1}(x)), & \frac{1}{2} \le t \le 1. \end{cases}$$

We denote $(h_n, h_{n,n+1}) = ((gf)_n, (gf)_{n,n+1})$.

Theorem 3.2.2. *Suppose that $(f_n, f_{n,n+1}), (f'_n, f'_{n,n+1}) : X \to Y$ and $(g_n, g_{n,n+1}), (g'_n, g'_{n,n+1}) : Y \to Z$ are strong proximate sequences such that $(f_n, f_{n,n+1}) \approx (f'_n, f'_{n,n+1})$ and $(g_n, g_{n,n+1}) \approx (g'_n, g'_{n,n+1})$. Then $((gf)_n, (gf)_{n,n+1}) \approx ((g'f')_n, (g'f')_{n,n+1})$.*

Proof. If $(f_n, f_{n,n+1}) \approx (f'_n, f'_{n,n+1})$ and $(g_n, g_{n,n+1}) \approx (g'_n, g'_{n,n+1})$ by homotopies $(F_n, F_{n,n+1})$ and $(G_n, G_{n,n+1})$, respectively, then $((gf)_n, (gf)_{n,n+1}) \approx ((gf')_n, (gf')_{n,n+1})$ by a homotopy $(H_n, H_{n,n+1})$, defined by

$$H_n(s,x) = g_n F_n(s,x)$$

and

$$H_{n,n+1}(t,s,x) = \begin{cases} g_n F_{n,n+1}(2t,s,x), & 0 \le t \le \frac{1}{2} \\ g_{n,n+1}(2t-1, F_{n+1}(s,x)), & \frac{1}{2} \le t \le 1\,. \end{cases}$$

Also, $((gf')_n, (gf')_{n,n+1}) \approx ((g'f')_n, (g'f')_{n,n+1})$ by a homotopy $(K_n, K_{n,n+1})$, defined by

$$K_n(s,x) = G_n(s, f'_n(x))$$

and

$$K_{n,n+1}(t,s,x) = \begin{cases} G_n(s, f'_{n,n+1}(2t,s,x)), & 0 \le t \le \frac{1}{2} \\ G_{n,n+1}(2t-1, f'_{n+1}(s,x)), & \frac{1}{2} \le t \le 1\,. \end{cases}$$

Since $\approx$ is an equivalence relation, it follows that$((gf)_n, (gf)_{n,n+1}) \approx ((g'f')_n, (g'f')_{n,n+1})$. □

Theorem 3.2.3. *If $(f_n, f_{n,n+1}) : X \to Y$, $(g_n, g_{n,n+1}) : Y \to Z$, and $(h_n, h_{n,n+1}) : Z \to W$ are strong proximate sequences, then*

$$((h_n, h_{n,n+1}) \circ (g_n, g_{n,n+1})) \circ (f_n, f_{n,n+1}) \approx$$
$$(h_n, h_{n,n+1}) \circ ((g_n, g_{n,n+1}) \circ (f_n, f_{n,n+1}))\,.$$

Proof. These two maps are connected by a homotopy $(H_n, H_{n,n+1})$, defined by

$$H_n(s,x) = h_n g_n f_n(x)$$

and

$$H_{n,n+1}(t,s,x) = \begin{cases} h_n g_n f_{n,n+1}(\frac{4t}{1+s}, x), & 0 \le t \le \frac{s+1}{4} \\ h_n g_{n,n+1}(4t-1-s, f_{n+1}(x)), & \frac{s+1}{4} \le t \le \frac{s+1}{2} \\ h_{n,n+1}(\frac{4t-2-s}{2-s}, g_{n+1} f_{n+1}(x)), & \frac{s+1}{2} \le t \le 1. \end{cases}$$ □

The *identity map* $X \to X$ is defined by the strong proximate sequence $(id_X, p) : X \to X$, where $p : I \times X \to X$ is defined by $p(t,x) = x$.

Theorem 3.2.4. *Let $(f_n, f_{n,n+1}) : X \to Y$ be a strong proximate sequence. Then $(id_Y, p) \circ (f_n, f_{n,n+1}) \approx (f_n, f_{n,n+1})$ and $(f_n, f_{n,n+1}) \circ (id_X, p) \approx (f_n, f_{n,n+1})$.*

Proof. A homotopy $(H_n, H_{n,n+1})$ connecting strong proximate sequences $(id_Y, p) \circ (f_n, f_{n,n+1})$ and $(f_n, f_{n,n+1})$ is defined by

$$H_n(s,x) = f_n(x)$$

and

$$H_{n,n+1}(t,s,x) = \begin{cases} f_{n,n+1}(\frac{2t}{s+1}, x), & 0 \le t \le \frac{s+1}{2} \\ f_{n+1}(x), & \frac{s+1}{2} \le t \le 1. \end{cases}$$

The homotopy $(K_n, K_{n,n+1})$ connecting strong proximate sequences $(f_n, f_{n,n+1}) \circ (id_X, p)$ and $(f_n, f_{n,n+1})$ is defined by

$$K_n(s,x) = f_n(x)$$

and

$$K_{n,n+1}(t,s,x) = \begin{cases} f_n p(t,x), & 0 \leq t \leq \frac{1-s}{2} \\ f_{n,n+1}(\frac{2t-1+s}{1+s}, x), & \frac{1-s}{2} \leq t \leq 1. \end{cases} \qquad \square$$

By theorems 3.2.2, 3.2.3, and 3.2.4, it follows that compact metric spaces and homotopy classes of strong proximate sequences form a category which we call the strong shape category of compact metric spaces.

References

[1] Yuji Akaike and Katsuro Sakai, *Describing the proper n-shape category by using non-continuous functions*, Glas. Mat. Ser. III **33(53)** (1998), no. 2, 299–321.

[2] James E. Felt, *ε-continuity and shape*, Proc. Amer. Math. Soc. **46** (1974), 426–430.

[3] R. W. Kieboom, *An intrinsic characterization of the shape of paracompacta by means of non-continuous single-valued maps*, Bull. Belg. Math. Soc. Simon Stevin **1** (1994), no. 5, 701–711.

[4] Ju. T. Lisica and S. Mardešić, *Coherent prohomotopy and strong shape of metric compacta*, Glas. Mat. Ser. III **20(40)** (1985), no. 1, 159–186.

[5] Sibe Mardešić, *Strong Shape and Homology.* Springer Monographs in Mathematics. Berlin: Springer-Verlag, 2000.

[6] J. Brendan Quigley, *An exact sequence from the nth to the $(n-1)$-st fundamental group*, Fund. Math. **77** (1973), no. 3, 195–210.

[7] José M. R. Sanjurjo, *A non-continuous description of the shape category of compacta*, Quart. J. Math. Oxford Ser. (2) **40** (1989), no. 159, 351–359.

[8] Nikita Shekutkovski, *From inverse to coherent systems*, Topology Appl. **113** (2001), no. 113, 293–307.

Institute of Mathematics; Faculty of Natural Sciences and Mathematics; Sts. Cyril and Methodius University; 1000 Skopje, Republic of Macedonia

E-mail address: nikita@pmf.ukim.mk