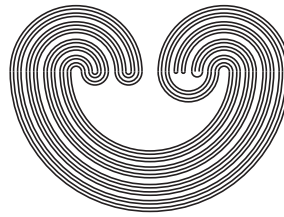


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## HOMEOMORPHISM GROUPS ON $C(X)$

by

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## HOMEOMORPHISM GROUPS ON $C(X)$

R. A. MCCOY

**ABSTRACT.** A study is made of two large subgroups, algebraic and bimonotone, of the group of homeomorphisms on  $C(X)$ , where  $C(X)$  is the space of continuous real-valued functions on  $X$  with the compact-open topology. The bimonotone subgroup is generated by the union of the subgroup of horizontal homeomorphisms on  $C(X)$  and the subgroup of vertical homeomorphisms on  $C(X)$ . These subgroups are shown to intersect only at the identity. When  $X$  is compact, the horizontal and vertical subgroups are shown to be closed subgroups of the group of homeomorphisms on  $C(X)$  with the uniform topology, and hence closed in this group with the fine topology; which is known to be a topological group. A number of examples are given, and questions are raised throughout.

### 1. INTRODUCTION

For the space  $C(X)$  of continuous real-valued functions on topological space  $X$ , where  $C(X)$  has the compact-open topology, the group  $\mathcal{H}(C(X))$  of homeomorphisms on  $C(X)$  has some interesting subgroups that are defined naturally using the functions on  $C(X)$ . We will look at two such subgroups that are essentially disjoint, intersecting only at the identity, and are large enough to contain most other familiar subgroups.

One of these subgroups of  $\mathcal{H}(C(X))$  is generated algebraically, so we call it the *algebraic* subgroup of  $\mathcal{H}(C(X))$  and denote it by  $\mathcal{A}(C(X))$ . The second of these subgroups of  $\mathcal{H}(C(X))$  is generated analytically using two kinds of homeomorphisms—horizontal and vertical. This latter subgroup was studied in [10] (see also [12] and [8]), in which it is called the *monotone* subgroup of  $\mathcal{H}(C(X))$ . We will use the more appropriate name of *bimonotone* subgroup of  $\mathcal{H}(C(X))$  and denote it by  $\mathcal{B}(C(X))$ .

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*Key words and phrases.* Homeomorphism group on  $C(X)$ , Algebraic homeomorphism on  $C(X)$ , Bimonotone homeomorphism on  $C(X)$ , Horizontal homeomorphism on  $C(X)$ , Vertical homeomorphism on  $C(X)$ , uniform topology, fine topology, topological group.

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When we consider the group  $\mathcal{H}(C(X))$  as a topological space, in order to check whether a subgroup is closed, we use either the uniform topology, denoted by  $\mathcal{H}_u(C(X))$ , or the fine topology, denoted by  $\mathcal{H}_f(C(X))$ . In both cases, we need  $C(X)$  to be a metric space. Now  $C(X)$  is metrizable whenever  $X$  is hemicompact (see [1] or [13]). However, to facilitate our arguments, we take  $X$  to be compact and use the supremum metric on  $C(X)$ , which is compatible with the compact-open topology for compact  $X$ . The space  $\mathcal{H}_u(C(X))$  is easier to work in than  $\mathcal{H}_f(C(X))$  because  $\mathcal{H}_u(C(X))$  is itself a metric space generated by the supremum metric on  $\mathcal{H}(C(X))$ . On the other hand, even though  $\mathcal{H}_f(C(X))$  is not metrizable, it is a topological group whenever  $C(X)$  is metrizable (see [2], [3] or [9]), and is therefore a natural space to study. We show that, for a compact space  $X$ , the subgroup of  $\mathcal{H}(C(X))$  consisting of the horizontal homeomorphisms on  $C(X)$  and the subgroup of  $\mathcal{H}(C(X))$  consisting of the vertical homeomorphisms on  $C(X)$  are both closed in  $\mathcal{H}_u(C(X))$ , and are thus closed subgroups of the topological group  $\mathcal{H}_f(C(X))$  since the topology on  $\mathcal{H}_f(C(X))$  is finer than that on  $\mathcal{H}_u(C(X))$ .

For most of our examples, we use for  $X$  the closed unit interval  $\mathbb{I} = [0, 1]$ , so in this case  $\mathcal{H}_u(C(\mathbb{I}))$  and  $\mathcal{H}_f(C(\mathbb{I}))$  are defined with the former being a metric space and the latter being a topological group. In section 4, we consider an example of a member of  $\mathcal{H}(C(\mathbb{I}))$  that is the pointwise limit of a sequence of members of  $\mathcal{A}(C(\mathbb{I}))$ , but does not appear to be in any of the subgroups of  $\mathcal{H}(C(\mathbb{I}))$  that we are considering, nor even in their closures. This suggests additional ways of constructing homeomorphisms on  $C(X)$ ; although, whether such homeomorphisms are “naturally defined” is a matter of opinion.

A number of questions are raised throughout as we look at the results and examples. Also there are four proofs that are rather long and detailed, so these are relegated to the last two sections for those who are interested in going through the details.

When a space  $X$  is not specified as compact, we at least assume that  $X$  is a Tychonoff space so that  $C(X)$  has sufficiently many elements for needed constructions. We use  $\mathbb{N}$  for the set of natural numbers,  $\mathbb{R}$  for the space of real numbers, and  $\mathbf{e}$  for the identity homeomorphism on  $C(X)$ . One can use [5] as a reference for topological properties and function spaces. Also, properties of the compact-open topology can be found in [1] or [13], properties of the uniform topology can be found in [13] or [15], and properties of the fine topology can be found in [14], [6] or [4]. In addition, topological properties of homeomorphism groups with the fine topology can be found in [11].

## 2. THE BIMONOTONE GROUP

We begin with the bimonotone subgroup of  $\mathcal{H}(C(X))$  because the horizontal and vertical homeomorphisms that are in it are familiar and have been studied in [12], [8] and [10]. Perhaps the horizontal homeomorphisms are the most familiar, so we start with them.

Each  $\mu$  in the group  $\mathcal{H}(X)$  of homeomorphisms on  $X$  induces a  $\hat{\mu}$  in  $\mathcal{H}(C(X))$  defined by

$$\hat{\mu}(f) = f\mu^{-1}$$

for all  $f \in C(X)$ . We call a member  $u$  of  $\mathcal{H}(C(X))$  a *horizontal* homeomorphism on  $C(X)$  provided that  $u = \hat{\mu}$  for some  $\mu \in \mathcal{H}(X)$ . The term “horizontal” is used because only the first coordinate is changed under  $\hat{\mu}$  when identifying each function in  $C(X)$  with its graph as a subset of  $X \times \mathbb{R}$ . The set of horizontal homeomorphisms in  $\mathcal{H}(C(X))$  is a subgroup of  $\mathcal{H}(C(X))$ , which we denote by  $\mathcal{H}or(C(X))$ . Note that the homeomorphisms in  $\mathcal{H}or(C(X))$  are *linear* in the sense that they preserve the vector space structure of  $C(X)$ .

**Example 2.1.** For the closed unit interval  $\mathbb{I}$ , define  $\mu \in \mathcal{H}(\mathbb{I})$  by

$$\mu(x) = x^2$$

for all  $x \in \mathbb{I}$ . Then if  $u = \hat{\mu}$ , we have

$$u(f)(x) = f(\sqrt{x})$$

for all  $f \in C(\mathbb{I})$  and  $x \in \mathbb{I}$ . That is, each  $f(x)$  in  $C(\mathbb{I})$  is mapped to  $f(\sqrt{x})$  in  $C(\mathbb{I})$  under this horizontal homeomorphism  $u$  on  $C(\mathbb{I})$ .

The proof of the following theorem is given in section 5.

**Theorem 2.2.** *For every compact space  $X$ ,  $\mathcal{H}or(C(X))$  is closed in the metric space  $\mathcal{H}_u(C(X))$ .*

**Corollary 2.3.** *For every compact space  $X$ ,  $\mathcal{H}or(C(X))$  is a closed subgroup of the topological group  $\mathcal{H}_f(C(X))$ .*

To define the vertical homeomorphisms on  $C(X)$ , we need to first define the group  $\mathcal{F}\mathcal{H}(X)$  of fiber homeomorphisms on  $X \times \mathbb{R}$ . A *fiber homeomorphism* on  $X \times \mathbb{R}$  is a homeomorphism  $\nu$  on  $X \times \mathbb{R}$  such that  $\nu(\{x\} \times \mathbb{R}) = \{x\} \times \mathbb{R}$  for all  $x \in X$ . Each  $\nu$  in  $\mathcal{F}\mathcal{H}(X)$  induces a  $\hat{\nu}$  in  $\mathcal{H}(C(X))$  defined by

$$\hat{\nu}(f) = \nu(f)$$

for all  $f \in C(X)$ , where  $\hat{\nu}(f)$  and the  $f$  on the right side of the equality are identified with their graphs as subsets of  $X \times \mathbb{R}$ . Another way to define  $\hat{\nu}$  is by

$$\hat{\nu}(f)(x) = \pi_2\nu(\langle x, f(x) \rangle)$$

for all  $f \in C(X)$  and  $x \in X$ , where  $\pi_2 : X \times \mathbb{R} \rightarrow \mathbb{R}$  is the projection map. We call a member  $v$  of  $\mathcal{H}(C(X))$  a *vertical* homeomorphism on  $C(X)$  provided that  $v = \hat{v}$  for some  $\nu \in \mathcal{FH}(X)$ . The term “vertical” is used because only the second coordinate is changed under  $\hat{v}$  when identifying each function in  $C(X)$  with its graph as a subset of  $X \times \mathbb{R}$ . The set of vertical homeomorphisms in  $\mathcal{H}(C(X))$  is a subgroup of  $\mathcal{H}(C(X))$ , which we denote by  $\mathcal{Ver}(C(X))$ .

**Example 2.4.** Define  $\nu \in \mathcal{FH}(\mathbb{I})$  by

$$\nu(\langle x, t \rangle) = \langle x, t + x^2 \rangle$$

for all  $\langle x, t \rangle \in \mathbb{I} \times \mathbb{R}$ . Then if  $v = \hat{\nu}$ , we have

$$v(f)(x) = f(x) + x^2$$

for all  $f \in C(\mathbb{I})$  and  $x \in \mathbb{I}$ . That is, each  $f$  in  $C(\mathbb{I})$  is mapped to  $f + g$  in  $C(\mathbb{I})$  under this vertical homeomorphism  $v$  on  $C(\mathbb{I})$ , where  $g \in C(\mathbb{I})$  is defined by  $g(x) = x^2$ . In other words,  $v$  is a translation on  $C(\mathbb{I})$  obtained by adding  $g$  to each member.

We can see from Example 2.4 that the homeomorphisms in  $\mathcal{Ver}(C(X))$  are, in general, not linear. In fact, for the  $v$  in this example,  $v(f_0)(1) = 1 \neq 0$ , where  $f_0$  is the constant 0 function in  $C(\mathbb{I})$ , and hence  $v(f_0) \neq f_0$ . So  $v$  is not linear, as we know for translations. Example 2.4 can be generalized to show that the group of translations on  $C(X)$  is a subgroup of  $\mathcal{Ver}(C(X))$ . On the other hand, some subgroups of  $\mathcal{Ver}(C(X))$  may be linear, such as the group of dilations on  $C(X)$ ; that is, multiplications by the nonzero elements of  $\mathbb{R}$ .

Like the horizontal group, the vertical group  $\mathcal{Ver}(C(X))$  for compact  $X$  is closed in  $\mathcal{H}(C(X))$  when it has the uniform or fine topology, although its proof, given in section 5, is entirely different than that for the horizontal group.

**Theorem 2.5.** *For every compact space  $X$ ,  $\mathcal{Ver}(C(X))$  is closed in the metric space  $\mathcal{H}_u(C(X))$ .*

**Corollary 2.6.** *For every compact space  $X$ ,  $\mathcal{Ver}(C(X))$  is a closed subgroup of the topological group  $\mathcal{H}_f(C(X))$ .*

It is not surprising that the two groups  $\mathcal{Hor}(C(X))$  and  $\mathcal{Ver}(C(X))$  have no nontrivial members in common.

**Theorem 2.7.** *For every space  $X$ ,  $\mathcal{Hor}(C(X)) \cap \mathcal{Ver}(C(X)) = \{\mathbf{e}\}$ .*

*Proof.* Let  $u \in \mathcal{H}or(C(X))$  and  $v \in \mathcal{V}er(C(X))$  with  $v \neq \mathbf{e}$ . Then  $u = \hat{\mu}$  for some  $\mu \in \mathcal{H}(X)$  and  $v = \hat{\nu}$  for some  $\nu \in \mathcal{F}\mathcal{H}(X)$ . Since  $v \neq \mathbf{e}$ , there exist  $x \in X$  and  $s, t \in \mathbb{R}$  with  $s \neq t$  such that  $\nu(\langle x, s \rangle) = \langle x, t \rangle$ . Let  $f$  be the constant function in  $C(X)$  with value  $s$ . Then

$$\begin{aligned} v(f)(x) &= \pi_2 \nu(\langle x, f(x) \rangle) \\ &= \pi_2 \nu(\langle x, s \rangle) \\ &= \pi_2(\langle x, t \rangle) \\ &= t, \end{aligned}$$

while

$$u(f)(x) = fu^{-1}(x) = s,$$

showing that  $u \neq v$ .  $\square$

Now define the *bimonotone* subgroup  $\mathcal{B}(C(X))$  of  $\mathcal{H}(C(X))$  to be the subgroup of  $\mathcal{H}(C(X))$  generated by  $\mathcal{H}or(C(X)) \cup \mathcal{V}er(C(X))$ . The members of  $\mathcal{B}(C(X))$  can be characterized by the following theorem.

**Theorem 2.8.** *For every space  $X$ , the following are equivalent for a member  $h$  of the group  $\mathcal{H}(C(X))$ .*

- (1)  $h \in \mathcal{B}(C(X))$ ;
- (2)  $h$  can be uniquely factored as  $h = vu$  for some  $u \in \mathcal{H}or(C(X))$  and  $v \in \mathcal{V}er(C(X))$ ;
- (3)  $h$  can be uniquely factored as  $h = uv$  for some  $u \in \mathcal{H}or(C(X))$  and  $v \in \mathcal{V}er(C(X))$ .

*Proof.* The uniqueness follows from Theorem 2.7 because, for example, if  $v_1 u_1 = v_2 u_2$  then  $u_1 u_2^{-1} = v_1^{-1} v_2$ ; so that  $u_1 u_2^{-1} = \mathbf{e} = v_1^{-1} v_2$ , showing that  $u_1 = u_2$  and  $v_1 = v_2$ .

It remains to show that if  $u \in \mathcal{H}or(C(X))$  and  $v_1 \in \mathcal{V}er(C(X))$ , there is a  $v_2 \in \mathcal{V}er(C(X))$  such that  $v_1 u = u v_2$ . Then by successive interchanges and simplifications, it would follow that each member of  $\mathcal{B}(C(X))$  could be written as  $vu$  or  $uv$ .

So let  $u \in \mathcal{H}or(C(X))$  and  $v_1 \in \mathcal{V}er(C(X))$ ; say  $u = \hat{\mu}$  for  $\mu \in \mathcal{H}(X)$  and  $v_1 = \hat{\nu}_1$  for  $\nu_1 \in \mathcal{F}\mathcal{H}(X)$ . Define  $\nu_2 \in \mathcal{F}\mathcal{H}(X)$  by

$$\nu_2(\langle x, t \rangle) = \langle x, \pi_2 \nu_1(\langle \mu(x), t \rangle) \rangle$$

for all  $\langle x, t \rangle \in X \times \mathbb{R}$ , and let  $v_2 = \hat{\nu}_2$ .

To see that  $v_1 u = u v_2$ , let  $f \in C(X)$  and  $x \in X$ . Then

$$\begin{aligned} v_1 u(f)(x) &= \hat{\nu}_1(\hat{\mu}(f))(x) \\ &= \hat{\nu}_1(f \mu^{-1})(x) \\ &= \pi_2 \nu_1(\langle x, f \mu^{-1}(x) \rangle), \end{aligned}$$

while

$$\begin{aligned}
uv_2(f)(x) &= \hat{\mu}(\hat{\nu}_2(f))(x) \\
&= \hat{\nu}_2(f)\mu^{-1}(x) \\
&= \pi_2\nu_2(\langle\mu^{-1}(x), f\mu^{-1}(x)\rangle) \\
&= \pi_2\langle\mu^{-1}(x), \pi_2\nu_1(\langle x, f\mu^{-1}(x)\rangle)\rangle \\
&= \pi_2\nu_1(\langle x, f\mu^{-1}(x)\rangle).
\end{aligned}$$

(Note that changing  $\mu(x)$  to  $\mu^{-1}(x)$  in the definition of  $\nu_2$  gives the reverse order  $uv_1 = v_2u$ .)  $\square$

Because of Theorems 2.2 and 2.5 and the factorization in Theorem 2.8, it is reasonable to ask the following.

**Question 2.9.** Is  $\mathcal{B}(C(X))$  closed in  $\mathcal{H}_u(C(X))$  for compact  $X$ ?

Also the unique factorization in Theorem 2.8 is needed for the next theorem.

**Theorem 2.10.** *For every space  $X$ ,  $\mathcal{V}er(C(X))$  is a normal subgroup of  $\mathcal{B}(C(X))$ , and the quotient group  $\mathcal{B}(C(X))/\mathcal{V}er(C(X))$  is isomorphic to  $\mathcal{H}or(C(X))$ .*

*Proof.* Let  $v \in \mathcal{V}er(C(X))$  and  $b \in \mathcal{B}(C(X))$ . We want to see that  $b^{-1}vb \in \mathcal{V}er(C(X))$ . By Theorem 2.8,  $b = v_1u$  for some  $u \in \mathcal{H}or(C(X))$  and  $v_1 \in \mathcal{V}er(C(X))$ . So

$$b^{-1}vb = u^{-1}v_1^{-1}vv_1u.$$

Now  $u = \hat{\mu}$ ,  $v = \hat{\nu}$  and  $v_1 = \hat{\nu}_1$  for some  $\mu \in \mathcal{H}(X)$  and  $\nu, \nu_1 \in \mathcal{F}\mathcal{H}(X)$ . Let  $\nu_2 = \nu_1^{-1}\nu\nu_1$ , which is in  $\mathcal{F}\mathcal{H}(X)$ , and let

$$v_2 = \hat{\nu}_2 = \hat{\nu}_1^{-1}\hat{\nu}\hat{\nu}_1 = v_1^{-1}vv_1.$$

Then  $b^{-1}vb = u^{-1}v_2u$ .

Let  $M$  be the homeomorphism  $\mu \times \text{id} : X \times \mathbb{R} \rightarrow X \times \mathbb{R}$  defined by

$$\mu \times \text{id}(\langle x, t \rangle) = \langle \mu(x), t \rangle$$

for all  $\langle x, t \rangle \in X \times \mathbb{R}$ , and let  $M^{-1}$  be its inverse  $\mu^{-1} \times \text{id}$ . Then define  $\nu_3 = M^{-1}\nu_2M$ , which is easily seen to be in  $\mathcal{F}\mathcal{H}(X)$ . Note that for each  $\langle x, t \rangle \in X \times \mathbb{R}$ ,

$$\pi_2\nu_3(\langle x, t \rangle) = \pi_2\nu_2(\langle \mu(x), t \rangle).$$

Let  $v_3 = \hat{\nu}_3$ , which is in  $\mathcal{V}er(C(X))$ .

To show that  $u^{-1}v_2u = v_3$ , let  $f \in C(X)$  and  $x \in X$ . Then

$$\begin{aligned}
u^{-1}v_2u(f)(x) &= u^{-1}\left(v_2(u(f))\right)(x) \\
&= v_2(u(f))(\mu(x)) \\
&= \pi_2\nu_2\left(\langle\mu(x), u(f)(\mu(x))\rangle\right) \\
&= \pi_2\nu_2\left(\langle\mu(x), f\mu^{-1}(\mu(x))\rangle\right) \\
&= \pi_2\nu_2\left(\langle\mu(x), f(x)\rangle\right) \\
&= \pi_2\nu_2\left(\langle x, f(x)\rangle\right) \\
&= v_3(f)(x).
\end{aligned}$$

This shows that  $b^{-1}vb \in \mathcal{V}er(C(X))$ , and hence  $\mathcal{V}er(C(X))$  is a normal subgroup of  $\mathcal{B}(C(X))$ .

By Theorem 2.8, each  $b \in \mathcal{B}(C(X))$  can be written uniquely as  $b = vu$  for some  $u \in \mathcal{H}or(C(X))$  and  $v \in \mathcal{V}er(C(X))$ . This defines a surjection

$$\phi : \mathcal{B}(C(X)) \rightarrow \mathcal{H}or(C(X))$$

with the property that for each  $b \in \mathcal{B}(C(X))$ ,  $b = v\phi(b)$  for some  $v \in \mathcal{V}er(C(X))$ . So each coset of  $\mathcal{V}er(C(X))$  using  $b$  is equal to the coset of  $\mathcal{V}er(C(X))$  using  $\phi(b)$ . Using the argument to commute elements in Theorem 2.8, we see that  $\phi(b_1b_2) = \phi(b_1)\phi(b_2)$  and  $\phi(b^{-1}) = \phi(b)^{-1}$ . In other words,  $\phi$  is a homomorphism.

Now if  $b_1$  and  $b_2$  give the same coset of  $\mathcal{V}er(C(X))$ , then  $b_2b_1^{-1} \in \mathcal{V}er(C(X))$ . But  $b_1 = v_1\phi(b_1)$  and  $b_2 = v_2\phi(b_2)$ , so that

$$v_2\phi(b_2b_1^{-1})v_1^{-1} = v_2\phi(b_2)\phi(b_1)^{-1}v_1^{-1} \in \mathcal{V}er(C(X)).$$

By Theorem 2.7, this can only happen when  $\phi(b_2b_1^{-1}) = \mathbf{e}$ ; that is,  $\phi(b_1) = \phi(b_2)$ . So there is a one-to-one correspondence between the cosets of  $\mathcal{V}er(C(X))$  and the members of  $\mathcal{H}or(C(X))$ , which is an isomorphism since  $\phi$  is a homomorphism.  $\square$

One might wonder whether the roles of  $\mathcal{V}er(C(X))$  and  $\mathcal{H}or(C(X))$  can be reversed in Theorem 2.10. The following example shows that is not the case; and in particular,  $\mathcal{H}or(C(X))$  is in general not a normal subgroup of  $\mathcal{B}(C(X))$ .

**Example 2.11.** Let  $u = \hat{\mu}$  where  $\mu \in \mathcal{H}(\mathbb{I})$  is defined by

$$\mu(x) = \sqrt{1-x}$$

for all  $x \in \mathbb{I}$ . Let  $v = \hat{\nu}$  where  $\nu \in \mathcal{F}\mathcal{H}(X)$  is defined by

$$\nu(\langle x, t \rangle) = \langle x, t+x \rangle$$

for all  $\langle x, t \rangle \in \mathbb{I} \times \mathbb{R}$ . Finally, let  $f \in C(\mathbb{I})$  be defined by  $f(x) = x$  for all  $x \in \mathbb{I}$ .



Now  $u \in \mathcal{H}or(C(\mathbb{I}))$  and  $v \in \mathcal{V}er(C(\mathbb{I}))$ , and we need to show that

$$v^{-1}uv(f) \neq u_1(f)$$

for all  $u_1 \in \mathcal{H}or(C(\mathbb{I}))$ . This will show that  $v^{-1}uv \notin \mathcal{H}or(C(\mathbb{I}))$ , so that  $\mathcal{H}or(C(\mathbb{I}))$  is not a normal subgroup of  $\mathcal{B}(C(\mathbb{I}))$ .

First note that for each  $u_1 \in \mathcal{H}or(C(\mathbb{I}))$ , we have  $u_1 = \hat{\mu}_1$  for some  $\mu_1 \in \mathcal{H}(\mathbb{I})$ , so that for every  $x \in \mathbb{I}$ ,

$$u_1(f)(x) = f\mu_1^{-1}(x) = \mu_1^{-1}(x).$$

In other words,  $u_1(f)$  is actually in  $\mathcal{H}(\mathbb{I})$ . Thus we need only show that  $v^{-1}uv \notin \mathcal{H}(\mathbb{I})$ .

For each  $x \in \mathbb{I}$ , we have

$$\begin{aligned} v^{-1}uv(f)(x) &= v^{-1}\left(u(v(f))\right)(x) \\ &= \pi_2\nu\left(\langle x, u(v(f))(x) \rangle\right) \\ &= x + u(v(f))(x) \\ &= x + v(f)(\mu^{-1}(x)) \\ &= x + v(f)(1 - x^2) \\ &= x + \pi_2\nu\left(\langle 1 - x^2, f(1 - x^2) \rangle\right) \\ &= x + 1 - x^2 + f(1 - x^2) \\ &= 2 + 2 - 2x^2. \end{aligned}$$

One can calculate that the maximum of  $2 + x - 2x^2$  occurs at  $x = \frac{1}{4}$ , which is in  $\mathbb{I}$ . So this function is not one-to-one, and hence not in  $\mathcal{H}(\mathbb{I})$ . This finishes the example, showing that  $\mathcal{H}or(C(\mathbb{I}))$  is not a normal subgroup of  $\mathcal{B}(C(\mathbb{I}))$ .

In order for us to get a sense of what the ‘‘bimonotone’’ name of  $\mathcal{B}(C(X))$  refers to, let us define what a bimonotone homeomorphism on  $C(X)$  is. First, a member  $h$  of  $\mathcal{H}(C(X))$  is *increasing* (*decreasing*, respectively) provided that for every  $f_1, f_2 \in C(X)$ , we have  $f_1 \leq f_2$  if and only if  $h(f_1) \leq h(f_2)$  ( $h(f_1) \geq h(f_2)$ , respectively). Then  $h$  is *monotone* provided that it is either increasing or decreasing. Also the definition of  $h$  being *strictly monotone* is the same except for the use of  $<$  and  $>$ .

Now a member  $h$  of  $\mathcal{H}(C(X))$  is called *bimonotone* (see [7]) provided that for every  $f_1, f_2 \in C(X)$  with  $f_1 \leq f_2$  (i.e.,  $f_1(x) \leq f_2(x)$  for all  $x \in X$ ) and for every  $f \in C(X)$ , it is true that  $f_1 \leq f \leq f_2$  if and only if

$$\min\{h(f_1), h(f_2)\} \leq h(f) \leq \max\{h(f_1), h(f_2)\}.$$

There is a closely related concept of strictly bimonotone homeomorphism on  $C(X)$ . A member  $h$  of  $\mathcal{H}(C(X))$  is called *strictly bimonotone* (see [10]) provided that there is an open partition  $\{X_i, X_d\}$  of  $X$  such that for every  $f_1, f_2 \in C(X)$ ,

$$f_1|_{X_i} < f_2|_{X_i} \quad \text{if and only if} \quad h(f_1)|_{X_i} < h(f_2)|_{X_i}$$

and

$$f_1|_{X_d} < f_2|_{X_d} \quad \text{if and only if} \quad h(f_1)|_{X_d} > h(f_2)|_{X_d}.$$

Notice that if  $X$  is connected, then  $h$  is strictly bimonotone if and only if it is strictly monotone. One can see directly from definitions that both horizontal and vertical homeomorphisms on  $C(X)$  are strictly bimonotone.

For compact spaces, there is a nice way of characterizing the elements of  $\mathcal{B}(C(X))$  in terms of the bimonotone or strictly bimonotone homeomorphisms on  $C(X)$ . This is given by the following theorem that can be found in [10], and that ultimately comes from the Factorization Theorem in [8].

**Theorem 2.12.** *For every compact space  $X$ , the following are equivalent for a member  $h$  of the group  $\mathcal{H}(C(X))$ .*

- (1)  $h \in \mathcal{B}(C(X))$ ;
- (2)  $h$  is bimonotone;
- (3)  $h$  is strictly bimonotone.

**Corollary 2.13.** *For every compact connected space  $X$ , the following are equivalent for a member  $h$  of the group  $\mathcal{H}(C(X))$ .*

- (1)  $h \in \mathcal{B}(C(X))$ ;
- (2)  $h$  is monotone;
- (3)  $h$  is strictly monotone.

### 3. THE ALGEBRAIC GROUP

We now look at an entirely different kind of subgroup of  $\mathcal{H}(C(X))$ , the algebraic group  $\mathcal{A}(C(X))$ . Instead of analytic techniques, we need to use linear algebraic techniques to establish this subgroup.

For each  $n \in \mathbb{N}$ , let  $C^n(X)$  be the product of  $n$  copies of  $C(X)$ , and let  $X^n$  be the product of  $n$  copies of  $X$ . For each  $\alpha = \langle \alpha_1, \dots, \alpha_n \rangle$  in  $C^n(X)$  and  $p = \langle p_1, \dots, p_n \rangle$  in  $X^n$ , let  $\alpha(p)$  be the  $n \times n$  matrix with  $i$ th row equal to

$$\alpha_i(p_1), \dots, \alpha_i(p_n)$$

for all  $i = 1, \dots, n$ , and let  $|\alpha(p)|$  be the determinant of  $\alpha(p)$ . Also for each  $f \in C(X)$  and  $i = 1, \dots, n$ , let  $\alpha(p, f, i)$  be the matrix  $\alpha(p)$  with its  $i$ th row replaced by

$$f(p_1), \dots, f(p_n),$$

and let  $|\alpha(p, f, i)|$  be its determinant. Finally, let  $Q(X)$  be the set of quadruples  $\langle n, \alpha, \beta, p \rangle$  such that  $n \in \mathbb{N}$ ,  $\alpha, \beta \in C^n(X)$  and  $p \in X^n$  with  $|\alpha(p)| \neq 0 \neq |\beta(p)|$ . Note that this condition implies that the coordinates  $p_1, \dots, p_n$  of  $p$  are all distinct.

For each  $\sigma = \langle n, \alpha, \beta, p \rangle$  in  $Q(X)$ , define  $\hat{\sigma} : C(X) \rightarrow C(X)$  by

$$\hat{\sigma}(f) = f + \sum_{i=1}^n \frac{|\alpha(p, f, i)|}{|\alpha(p)|} (\beta_i - \alpha_i)$$

for all  $f \in C(X)$ . We say that  $\hat{\sigma}$  has size  $n$ . One can easily check that  $\hat{\sigma}(\alpha_i) = \beta_i$  for all  $i = 1, \dots, n$ , where  $\alpha = \langle \alpha_1, \dots, \alpha_n \rangle$  and  $\beta = \langle \beta_1, \dots, \beta_n \rangle$ .

The following theorem shows that  $\hat{\sigma} \in \mathcal{H}(C(X))$ , and its proof is given in section 6.

**Theorem 3.1.** *If  $\sigma = \langle n, \alpha, \beta, p \rangle$  in  $Q(X)$ , then the map  $\hat{\sigma}$  is a linear homeomorphism on  $C(X)$  with inverse  $\hat{\tau}$  where  $\tau = \langle n, \beta, \alpha, p \rangle$ .*

**Corollary 3.2.** *For any space  $X$  and  $n \in \mathbb{N}$ , let  $f_i, g_i \in C(X)$  for  $i = 1, \dots, n$ . If there exist  $x_i \in X$  for  $i = 1, \dots, n$  such that*

$$\begin{vmatrix} f_1(x_1) & \dots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \dots & f_n(x_n) \end{vmatrix} \neq 0 \neq \begin{vmatrix} g_1(x_1) & \dots & g_1(x_n) \\ \vdots & \ddots & \vdots \\ g_n(x_1) & \dots & g_n(x_n) \end{vmatrix},$$

*then there exists a linear homeomorphism  $h$  on  $C(X)$  such that  $h(f_i) = g_i$  for  $i = 1, \dots, n$ .*

We call a member  $a$  of  $\mathcal{H}(C(X))$  an *algebraic homeomorphism* on  $C(X)$  provided that  $a = \hat{\sigma}$  for some  $\sigma \in Q(X)$ . We denote the set of algebraic homeomorphisms in  $\mathcal{H}(C(X))$  by  $\mathcal{A}(C(X))$ . The fact that  $\mathcal{A}(C(X))$  is a subgroup of  $\mathcal{H}(C(X))$  follows from the next theorem, whose proof is given in section 6.

**Theorem 3.3.** *If  $a_1$  and  $a_2$  are members of  $\mathcal{A}(C(X))$  having sizes  $n_1$  and  $n_2$ , respectively, then  $a_2 a_1 = a$  for some  $a$  in  $\mathcal{A}(C(X))$  with size  $n$  satisfying  $\max\{n_1, n_2\} \leq n \leq n_1 + n_2$ .*

**Example 3.4.** For an example of an algebraic homeomorphism on  $C(\mathbb{I})$  having size 1, take  $n = 1$ , define

$$\alpha_1(x) = 2 \quad \text{and} \quad \beta_1(x) = x^3$$

for all  $x \in \mathbb{I}$ , and let  $p_1 = 1$ . Then with  $\alpha = \langle \alpha_1 \rangle$ ,  $\beta = \langle \beta_1 \rangle$  and  $p = \langle p_1 \rangle$ , take  $\sigma = \langle n, \alpha, \beta, p \rangle$ . We see that  $\sigma \in Q(\mathbb{I})$ , so let  $a = \hat{\sigma}$ . Then

$$a(f)(x) = f(x) + \frac{1}{2}f(1)(x^3 - 2)$$

for all  $f \in C(\mathbb{I})$  and  $x \in \mathbb{I}$ . That is, each  $f(x)$  in  $C(\mathbb{I})$  is mapped to  $f(x) + \frac{1}{2}f(1)(x^3 - 2)$  in  $C(\mathbb{I})$  under this algebraic homeomorphism  $a$  on  $C(\mathbb{I})$ .

**Example 3.5.** For an example of an algebraic homeomorphism on  $C(\mathbb{I})$  having size 2, take  $n = 2$ , define

$$\alpha_1(x) = 3 \quad , \quad \alpha_2(x) = x \quad , \quad \beta_1(x) = x^2 \quad , \quad \beta_2(x) = 1$$

for all  $x \in \mathbb{I}$ , and let  $p_1 = 0$  and  $p_2 = 1$ . Then with  $\alpha = \langle \alpha_1, \alpha_2 \rangle$ ,  $\beta = \langle \beta_1, \beta_2 \rangle$  and  $p = \langle p_1, p_2 \rangle$ , take  $\sigma = \langle n, \alpha, \beta, p \rangle$ . One can check that  $\sigma \in Q(\mathbb{I})$ , so take  $a = \hat{\sigma}$ . Then

$$\begin{aligned} a(f)(x) &= f(x) + \frac{\begin{vmatrix} f(0) & f(1) \\ 0 & 1 \end{vmatrix}}{\begin{vmatrix} 3 & 3 \\ 0 & 1 \end{vmatrix}}(x^2 - 3) + \frac{\begin{vmatrix} 3 & 3 \\ f(0) & f(1) \end{vmatrix}}{\begin{vmatrix} 3 & 3 \\ 0 & 1 \end{vmatrix}}(1 - x) \\ &= f(x) + \frac{1}{3}f(0)(x^2 + 2x - 5) + f(1)(1 - x) \end{aligned}$$

for all  $f \in C(\mathbb{I})$  and  $x \in \mathbb{I}$ . That is, each  $f(x)$  in  $C(\mathbb{I})$  is mapped to  $f(x) + \frac{1}{3}f(0)(x^2 + 2x - 5) + f(1)(1 - x)$  in  $C(\mathbb{I})$  under this algebraic homeomorphism  $a$  on  $C(\mathbb{I})$ .

As an additional example, one might try writing the composition of the  $a$  in Example 3.4 followed by the  $a$  in Example 3.5 so that it is in the form  $\hat{\sigma}$  for  $\sigma = \langle 2, \alpha, \beta^*, p \rangle \in Q(\mathbb{I})$  using the same  $p$  and  $\alpha$  as in Example 3.5—then  $\beta^*$  must be calculated, as in the proof of Theorem 3.3 (answer:  $\beta_1^*(x) = 3x^3 + 5x - 17$ ,  $\beta_2^*(x) = \frac{1}{6}(3x^3 - 2x^2 - 7x - 13)$ ).

Considering Theorems 2.2 and 2.5, the next question is natural to ask.

**Question 3.6.** Is  $\mathcal{A}(C(X))$  closed in  $\mathcal{H}_u(C(X))$  for compact  $X$ ?

As it is the case for the subgroups  $\mathcal{H}or(C(X))$  and  $\mathcal{V}er(C(X))$  of  $\mathcal{H}(C(X))$ , so it is that the subgroups  $\mathcal{A}(C(X))$  and  $\mathcal{B}(C(X))$  have no nontrivial members in common, at least for a large class of spaces  $X$ .

**Theorem 3.7.** For every space  $X$  with no isolated point,  $\mathcal{A}(C(X)) \cap \mathcal{B}(C(X)) = \{\mathbf{e}\}$ .

*Proof.* Let  $a \in \mathcal{A}(C(X))$ ,  $u \in \mathcal{H}or(C(X))$  and  $v \in \mathcal{V}er(C(X))$  with  $a \neq \mathbf{e}$ . Because of Theorem 2.8, it suffices to show that  $a \neq vu$ . Now  $a = \hat{\sigma}$  for some  $\sigma = \langle n, \alpha, \beta, p \rangle \in Q(X)$ ; say  $p = \langle p_1, \dots, p_n \rangle$ . Also  $u = \hat{\mu}$  for some  $\mu \in \mathcal{H}(X)$  and  $v = \hat{\nu}$  for some  $\nu \in \mathcal{F}\mathcal{H}(X)$ .

First suppose that  $u \neq \mathbf{e}$ . Since  $X$  has no isolated point, there exists an  $x \in X$  such that  $\mu^{-1}(x) \in X \setminus \{x, p_1, \dots, p_n\}$ . Then there exists an  $f \in C(X)$  with

$$f(x) = f(p_1) = \dots = f(p_n) = 0$$

and with  $f(\mu^{-1}(x))$  such that

$$\pi_2\nu(\langle x, f(\mu^{-1}(x)) \rangle) = 1.$$

Therefore,

$$\begin{aligned} vu(f)(x) &= v(f\mu^{-1})(x) \\ &= \pi_2\nu(\langle x, f\mu^{-1}(x) \rangle) \\ &= 1, \end{aligned}$$

while

$$\begin{aligned} a(f)(x) &= \hat{\sigma}(f)(x) \\ &= f(x) + \sum_{i=1}^n \frac{|\alpha(p, f, i)|}{|\alpha(p)|} (\beta_i - \alpha_i)(x) \\ &= 0, \end{aligned}$$

showing that  $a \neq vu$ .

If  $u = \mathbf{e}$ , then we may assume that  $v \neq \mathbf{e}$ . In this case, since  $X$  has no isolated point, there exist  $x \in X \setminus \{p_1, \dots, p_n\}$  and  $s, t \in \mathbb{R}$  with  $s \neq t$  and  $\nu(\langle x, s \rangle) = \langle x, t \rangle$ . Let  $g \in C(X)$  be such that

$$g(p_1) = \dots = g(p_n) = 0 \quad \text{and} \quad g(x) = s.$$

Then

$$\begin{aligned} v(g)(x) &= \hat{\nu}(g)(x) \\ &= \pi_2\nu(\langle x, g(x) \rangle) \\ &= \pi_2\nu(\langle x, s \rangle) \\ &= t, \end{aligned}$$

while

$$\begin{aligned} a(g)(x) &= \hat{\sigma}(g)(x) \\ &= g(x) + \sum_{i=1}^n \frac{|\alpha(p, g, i)|}{|\alpha(p)|} (\beta_i - \alpha_i)(x) \\ &= s, \end{aligned}$$

showing that  $a \neq vu$ . □

Theorem 3.7 together with Corollary 2.13 have the following consequence.

**Corollary 3.8.** *If  $X$  is a compact connected space, then no member of  $\mathcal{A}(C(X))$ , except  $\mathbf{e}$ , is monotone.*

**Example 3.9.** Corollary 3.8 is illustrated with the algebraic homeomorphism  $a$  in Example 3.4, by taking

$$f_1(x) = x + 1$$

and

$$f_2(x) = -f_1(x).$$

Then  $f_1 > f_2$  on  $\mathbb{I}$ , whereas the graphs of

$$a(f_1)(x) = x^3 + x - 1$$

and

$$a(f_2)(x) = a(-f_1)(x) = -a(f_1)(x)$$

on  $\mathbb{I}$  cross at their root  $x = .682$ .

The next two examples illustrate the necessity of the hypothesis that  $X$  have no isolated point in Theorem 3.7.

**Example 3.10.** Let  $X$  be any space containing an isolated point  $p$ . Define  $u = \mathbf{e}$  and  $v = \hat{v}$  where

$$\nu(\langle x, t \rangle) = \begin{cases} \langle x, 2t \rangle & \text{if } x = p, \\ \langle x, t \rangle & \text{otherwise} \end{cases}$$

for all  $\langle x, t \rangle \in X \times \mathbb{R}$ . Also define  $a = \hat{\sigma}$  where  $\sigma = \langle 1, \alpha, \beta, p \rangle$  and  $\alpha$  and  $\beta$  are defined for  $x \in X$  by

$$\alpha(x) = \begin{cases} 1 & \text{if } x = p, \\ 0 & \text{otherwise} \end{cases}$$

and  $\beta(x) = 2\alpha(x)$ . Then one can check that for  $f \in C(X)$  and  $x \in X$ ,

$$a(f)(x) = v(f)(x) = \begin{cases} 2f(x) & \text{if } x = p, \\ f(x) & \text{otherwise,} \end{cases}$$

showing that  $a = vu \neq \mathbf{e}$ .

**Example 3.11.** Let  $X$  be any space containing two isolated points  $p_1$  and  $p_2$ . Define  $v = \mathbf{e}$  and  $u = \hat{\mu}$  where

$$\mu(x) = \begin{cases} p_2 & \text{if } x = p_1, \\ p_1 & \text{if } x = p_2, \\ x & \text{otherwise} \end{cases}$$

for all  $x \in X$ . Also define  $a = \hat{\sigma}$  where  $\sigma = \langle 2, \alpha, \beta, p \rangle$  with  $p = \langle p_1, p_2 \rangle$  and with  $\alpha = \langle \alpha_1, \alpha_2 \rangle$  and  $\beta = \langle \beta_1, \beta_2 \rangle$  defined for  $x \in X$  by

$$\alpha_1(x) = \begin{cases} 1 & \text{if } x = p_1, \\ 0 & \text{otherwise} \end{cases}, \quad \alpha_2(x) = \begin{cases} 1 & \text{if } x = p_2, \\ 0 & \text{otherwise} \end{cases},$$

$$\beta_1(x) = \alpha_2(x), \quad \text{and} \quad \beta_2(x) = \alpha_1(x).$$

Then one can check that for  $f \in C(X)$  and  $x \in X$ ,

$$a(f)(x) = u(f)(x) = \begin{cases} f(p_2) & \text{if } x = p_1, \\ f(p_1) & \text{if } x = p_2, \\ f(x) & \text{otherwise,} \end{cases}$$

showing that  $a = vu \neq \mathbf{e}$ .

Here are two questions concerning the subgroup of  $\mathcal{H}(C(X))$ , that we call  $\mathcal{G}(C(X))$ , generated by  $\mathcal{A}(C(X)) \cup \mathcal{B}(C(X))$ .

**Question 3.12.** Can a unique factorization analog of Theorem 2.8 be established for  $\mathcal{G}(C(X))$ ?

**Question 3.13.** Is  $\mathcal{G}(C(X))$  closed in  $\mathcal{H}_u(C(X))$  for compact  $X$ ?

#### 4. A POINTWISE LIMIT EXAMPLE

This section consists of an example of a member  $\bar{a}$  of  $\mathcal{H}(C(\mathbb{I}))$  that is the pointwise limit of a sequence from  $\mathcal{A}(C(\mathbb{I}))$ . This homeomorphism  $\bar{a}$  on  $C(\mathbb{I})$  does not appear to be a member of the subgroup  $\mathcal{G}(C(\mathbb{I}))$  of  $\mathcal{H}(C(\mathbb{I}))$  generated by  $\mathcal{A}(C(\mathbb{I})) \cup \mathcal{B}(C(\mathbb{I}))$ . This suggests additional ways of constructing homeomorphisms on  $C(X)$  for general compact spaces  $X$ , and it also leads to questions about how  $\mathcal{A}(C(X))$  and  $\mathcal{B}(C(X))$  sit in  $\mathcal{H}_p(C(X))$ , where  $p$  denotes the topology of pointwise convergence.

For each  $n \in \mathbb{N}$ , let  $\alpha^n, \beta^n \in C(\mathbb{I})$  and  $p^n \in [\frac{1}{n+1}, \frac{1}{n}]$  be such that

- (1)  $\alpha^n(x) = \beta^n(x) = 0$  for all  $x \in \mathbb{I} \setminus [\frac{1}{n+1}, \frac{1}{n}]$ ; and
- (2)  $|\beta^n(x) - \alpha^n(x)| \leq \min\{|\alpha^n(p^n)|, |\beta^n(p^n)|\}$  for all  $x \in [\frac{1}{n+1}, \frac{1}{n}]$ .

Identify  $\alpha^n$ ,  $\beta^n$  and  $p^n$  with  $\langle \alpha^n \rangle$ ,  $\langle \beta^n \rangle$  and  $\langle p^n \rangle$ , and define  $\sigma^n = \langle 1, \alpha^n, \beta^n, p^n \rangle \in Q(\mathbb{I})$ . Also define

$$a^n = \hat{\sigma}^n \cdots \hat{\sigma}^1,$$

which is a member of  $\mathcal{A}(C(\mathbb{I}))$ . Then for each  $f \in C(\mathbb{I})$ ,

$$a^n(f) = f + \sum_{i=1}^n \frac{f(p^i)}{\alpha^i(p^i)} (\beta^i - \alpha^i).$$

Note that condition (1) ensures that if  $x \in [0, \frac{1}{n+1}]$ , then  $a^n(f)(x) = f(x)$ , and that if  $x \in [\frac{1}{n+1}, 1]$ , then for some  $i = 1, \dots, n$ ,  $x \in [\frac{1}{i+1}, \frac{1}{i}]$  and

$$a^n(f)(x) = f(x) + \frac{f(p^i)}{\alpha^i(p^i)} (\beta^i(x) - \alpha^i(x)).$$

We see from this that, for every  $x \in \mathbb{I}$ , the value of  $a^n(f)(x)$  does not depend on  $n$  beyond some value of  $n$ . Therefore, for each  $f \in C(\mathbb{I})$ ,

the sequence  $\langle a^n(f) \rangle_{n \in \mathbb{N}}$  in  $C(\mathbb{I})$  converges pointwise to the function  $\bar{a}(f) : \mathbb{I} \rightarrow \mathbb{R}$  defined by

$$\bar{a}(f) = f + \sum_{i=1}^{\infty} \frac{f(p^i)}{\alpha^i(p^i)} (\beta^i - \alpha^i).$$

Now for each  $n \in \mathbb{N}$  and  $f \in C(\mathbb{I})$ ,

$$\bar{a}(f) - a^n(f) = \sum_{i=n+1}^{\infty} \frac{f(p^i)}{\alpha^i(p^i)} (\beta^i - \alpha^i).$$

Condition (2) ensures that for every  $i = n+1, n+2, \dots$  and  $x \in [\frac{1}{i+1}, \frac{1}{i}]$ ,

$$\begin{aligned} |\bar{a}(f)(x) - a^n(f)(x)| &= \frac{|f(p^i)|}{|\alpha^i(p^i)|} |\beta^i(x) - \alpha^i(x)| \\ &< \frac{1}{n} |f(p^i)|. \end{aligned}$$

Since  $\lim_{i \rightarrow \infty} f(p^i) = f(0)$ , the sequence  $\langle a^n(f) \rangle_{n \in \mathbb{N}}$  converges uniformly to  $\bar{a}(f)$ , which implies that  $\bar{a}(f) \in C(\mathbb{I})$ . This gives us a function  $\bar{a} : C(\mathbb{I}) \rightarrow C(\mathbb{I})$ . It remains to show that  $\bar{a} \in \mathcal{H}(C(\mathbb{I}))$ .

For each  $n \in \mathbb{N}$  and  $f \in C(\mathbb{I})$ ,

$$(a^n)^{-1}(f) = f + \sum_{i=1}^n \frac{f(p^i)}{\beta^i(p^i)} (\alpha^i - \beta^i),$$

so that a similar argument as above shows that  $\bar{a}^{-1}(f) \in C(\mathbb{I})$  where

$$\bar{a}^{-1}(f) = f + \sum_{i=1}^{\infty} \frac{f(p^i)}{\beta^i(p^i)} (\alpha^i - \beta^i).$$

This defines the function  $\bar{a}^{-1} : C(\mathbb{I}) \rightarrow C(\mathbb{I})$ . Now let  $f \in C(\mathbb{I})$  and  $x \in \mathbb{I}$ . If  $x = 0$ , then

$$\begin{aligned} \bar{a}^{-1}\bar{a}(f)(x) &= \bar{a}^{-1}(\bar{a}(f))(x) \\ &= \bar{a}(f)(x) \\ &= f(x). \end{aligned}$$



If  $x \in [\frac{1}{i+1}, \frac{1}{i}]$  for some  $i \in \mathbb{N}$ , then

$$\begin{aligned}
\bar{a}^{-1}\bar{a}(f)(x) &= \bar{a}^{-1}(\bar{a}(f))(x) \\
&= \bar{a}(f)(x) + \frac{\bar{a}(f)(p^i)}{\beta^i(p^i)}(\alpha^i(x) - \beta^i(x)) \\
&= \left(f(x) + \frac{f(p^i)}{\alpha^i(p^i)}(\beta^i(x) - \alpha^i(x))\right) \\
&\quad + \frac{\left(f(x) + \frac{f(p^i)}{\alpha^i(p^i)}(\beta^i(p^i) - \alpha^i(p^i))\right)}{\beta^i(p^i)}(\alpha^i(x) - \beta^i(x)) \\
&= f(x) + \frac{f(p^i)}{\alpha^i(p^i)\beta^i(p^i)}\left(\beta^i(p^i)(\beta^i(x) - \alpha^i(x))\right) \\
&\quad + \alpha^i(p^i)(\alpha^i(x) - \beta^i(x)) + (\beta^i(p^i) - \alpha^i(p^i))(\alpha^i(x) - \beta^i(x)) \\
&= f(x).
\end{aligned}$$

This shows that  $\bar{a}^{-1}\bar{a}$  is the identity on  $C(\mathbb{I})$ . A similar argument shows that  $\bar{a}\bar{a}^{-1}$  is also the identity on  $C(\mathbb{I})$ , and hence  $\bar{a}$  is a bijection on  $C(\mathbb{I})$  with inverse  $\bar{a}^{-1}$ .

To show that  $\bar{a}$  is continuous, let  $f \in C(\mathbb{I})$  and let  $\langle f_n \rangle_{n \in \mathbb{N}}$  be a sequence in  $C(\mathbb{I})$  that converges to  $f$ . Because of the compactness of  $\mathbb{I}$ , the compact-open topology is the same as the uniform topology, so that  $\langle f_n \rangle_{n \in \mathbb{N}}$  converges uniformly to  $f$ . To see that  $\langle \bar{a}(f_n) \rangle_{n \in \mathbb{N}}$  converges uniformly to  $\bar{a}(f)$ , let  $\varepsilon > 0$ . Now there exists an  $m \in \mathbb{N}$  such that for all  $n \in \mathbb{N}$  with  $n \geq m$  and for all  $x \in \mathbb{I}$ ,

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2}.$$

So let  $n \in \mathbb{N}$  with  $n \geq m$  and let  $x \in \mathbb{I}$ . If  $x = 0$ , then

$$|\bar{a}(f_n)(x) - \bar{a}(f)(x)| = |f_n(x) - f(x)| < \frac{\varepsilon}{2} < \varepsilon.$$

If  $x \in \mathbb{I} \setminus \{0\}$ , then  $x \in [\frac{1}{i+1}, \frac{1}{i}]$  for some  $i \in \mathbb{N}$ . So then

$$\begin{aligned}
|\bar{a}(f_n)(x) - \bar{a}(f)(x)| &= \left| \left( f_n(x) + \frac{f_n(p^i)}{\alpha^i(p^i)}(\beta^i(x) - \alpha^i(x)) \right) \right. \\
&\quad \left. - \left( f(x) + \frac{f(p^i)}{\alpha^i(p^i)}(\beta^i(x) - \alpha^i(x)) \right) \right| \\
&\leq |f_n(x) - f(x)| + \frac{|\beta^i(x) - \alpha^i(x)|}{|\alpha^i(p^i)|} |f_n(p^i) - f(p^i)| \\
&\leq |f_n(x) - f(x)| + |f_n(p^i) - f(p^i)| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

This shows that  $\langle \bar{a}(f_n) \rangle_{n \in \mathbb{N}}$  converges to  $\bar{a}(f)$  in  $C(\mathbb{I})$ , and therefore  $\bar{a}$  is continuous. A similar argument shows that  $\bar{a}^{-1}$  is continuous, so that indeed  $\bar{a} \in \mathcal{H}(C(\mathbb{I}))$ .

We point out that although the sequence  $\langle a^n \rangle_{n \in \mathbb{N}}$  in  $\mathcal{A}(C(\mathbb{I}))$  converges pointwise to  $\bar{a}$  in  $\mathcal{H}(C(\mathbb{I}))$ , the convergence is not uniform. So we cannot say that  $\bar{a}$  is in the closure of  $\mathcal{A}(C(\mathbb{I}))$  in  $\mathcal{H}_u(C(\mathbb{I}))$ , only that  $\bar{a}$  is in the closure of  $\mathcal{A}(C(\mathbb{I}))$  in  $\mathcal{H}_p(C(\mathbb{I}))$ , where  $\mathcal{H}_p(C(\mathbb{I}))$  has the topology of pointwise convergence. One question raised by this is the following.

**Question 4.1.** For compact  $X$ , is the subgroup  $\mathcal{G}(C(X))$  of  $\mathcal{H}(C(X))$ , generated by  $\mathcal{A}(C(X)) \cap \mathcal{B}(C(X))$ , dense in  $\mathcal{H}_p(C(X))$ ?

## 5. CLOSURE PROOFS

This section contains the proofs of Theorems 2.2 and 2.5 that show the horizontal and vertical groups,  $\mathcal{H}or(C(X))$  and  $\mathcal{V}er(C(X))$ , to be closed in  $\mathcal{H}(C(X))$  with the uniform and fine topologies. So let us first define these topologies. If  $X$  is compact, the supremum metric  $\rho$  on  $C(X)$  is compatible with the compact-open topology, where  $\rho$  is defined by

$$\rho(f, g) = \sup\{|f(x) - g(x)| : x \in X\}$$

for all  $f, g \in C(X)$ . In this case, the uniform topology that  $\mathcal{H}_u(C(X))$  has is defined by the supremum metric on  $\mathcal{H}(C(X))$  and has basic open sets of the form

$$B(h, \varepsilon) = \{h' \in \mathcal{H}(C(X)) : \rho(h'(f), h(f)) < \varepsilon \text{ for all } f \in C(X)\}$$

for  $h \in \mathcal{H}(C(X))$  and  $\varepsilon$  a positive number. The basic open sets in the fine topology that  $\mathcal{H}_f(C(X))$  has, are of the form

$$B(h, \varepsilon) = \{h' \in \mathcal{H}(C(X)) : \rho(h'(f), h(f)) < \varepsilon(f) \text{ for all } f \in C(X)\}$$

for  $h \in \mathcal{H}(C(X))$  and  $\varepsilon$  a positive continuous real-valued function on  $C(X)$ . Clearly the fine topology is finer than the uniform topology, so that a subset need only be shown closed in the uniform topology.

*Proof of Theorem 2.2.* Let  $h$  be in the closure of  $\mathcal{H}or(C(X))$  in  $\mathcal{H}_u(C(X))$ . We need to show that  $h \in \mathcal{H}or(C(X))$ . For each  $n \in \mathbb{N}$ , there exists a

$$u_n \in B(h, \frac{1}{n}) \cap \mathcal{H}or(C(X)).$$

Now each  $u_n = \hat{\mu}_n$  for some  $\mu_n \in \mathcal{H}(X)$ . Since  $u_n(f) = f\mu_n^{-1}$  for all  $f \in C(X)$ , it is easy to see that  $u_n^{-1} = \hat{\mu}_n^{-1}$ .

We need to know that  $u_n^{-1}$  is in  $B(h^{-1}, \frac{1}{n})$ . If  $f \in C(X)$  and  $x \in X$ , then for  $g = h^{-1}(f)$  and  $y = \mu_n(x)$ , we have

$$\begin{aligned} |u_n^{-1}(f)(x) - h^{-1}(f)(x)| &= |f\mu_n(x) - g(x)| \\ &= |h(g)(y) - g\mu_n^{-1}(y)| \\ &= |u_n(g)(y) - h(g)(y)|. \end{aligned}$$

Since  $u_n \in B(h, \frac{1}{n})$ , for all  $g \in C(X)$ ,

$$\sup\{|u_n(g)(y) - h(g)(y)| : y \in X\} < \frac{1}{n}.$$

That means for all  $f \in C(X)$ ,

$$\sup\{|u_n^{-1}(f)(x) - h^{-1}(f)(x)| : x \in X\} < \frac{1}{n},$$

showing that  $u_n^{-1} \in B(h^{-1}, \frac{1}{n})$ .

Now we need to show that the sequence  $\langle \mu_n^{-1} \rangle_{n \in \mathbb{N}}$  converges pointwise to  $\mu^{-1}$  for some  $\mu \in \mathcal{H}(X)$ , and that  $h = \hat{\mu}$ .

So let us first show how to define  $\mu^{-1}(x)$  for some given  $x$  in  $X$ . Suppose, by way of contradiction, that  $\langle \mu_n^{-1}(x) \rangle_{n \in \mathbb{N}}$  has distinct cluster points  $x_1$  and  $x_2$  in  $X$ . Then there is an  $f \in C(X)$  such that  $f(x_1) \neq f(x_2)$ . Let  $m \in \mathbb{N}$  with

$$\frac{1}{m} < \frac{1}{4}|f(x_1) - f(x_2)|.$$

Also let  $U_1$  and  $U_2$  be neighborhoods of  $x_1$  and  $x_2$  in  $X$  such that

$$f(U_1) \subseteq (f(x_1) - \frac{1}{m}, f(x_1) + \frac{1}{m}) \quad \text{and} \quad f(U_2) \subseteq (f(x_2) - \frac{1}{m}, f(x_2) + \frac{1}{m}).$$

Because  $x_1$  and  $x_2$  are cluster points of  $\langle \mu_n^{-1}(x) \rangle_{n \in \mathbb{N}}$ , there exist  $n_1, n_2 \in \mathbb{N}$ , both greater than  $n$ , such that

$$\mu_{n_1}^{-1}(x) \in U_1 \quad \text{and} \quad \mu_{n_2}^{-1}(x) \in U_2.$$

Now

$$|f(\mu_{n_1}^{-1}(x)) - h(f)(x)| < \frac{1}{n_1} \leq \frac{1}{m} \quad \text{and} \quad |f(\mu_{n_2}^{-1}(x)) - h(f)(x)| < \frac{1}{n_2} \leq \frac{1}{m},$$

so that

$$\begin{aligned} |f(\mu_{n_1}^{-1}(x)) - f(\mu_{n_2}^{-1}(x))| &\leq |f(\mu_{n_1}^{-1}(x)) - h(f)(x)| \\ &\quad + |h(f)(x) - f(\mu_{n_2}^{-1}(x))| \\ &< \frac{1}{m} + \frac{1}{m} = \frac{2}{m}. \end{aligned}$$

But also

$$|f(\mu_{n_1}^{-1}(x)) - f(x_1)| < \frac{1}{m} \quad \text{and} \quad |f(\mu_{n_2}^{-1}(x)) - f(x_2)| < \frac{1}{m},$$

so that

$$\begin{aligned} |f(x_1) - f(x_2)| &\leq |f(x_1) - f(\mu_{n_1}^{-1}(x))| \\ &\quad + |f(\mu_{n_1}^{-1}(x)) - f(\mu_{n_2}^{-1}(x))| + |f(\mu_{n_2}^{-1}(x)) - f(x_2)| \\ &< \frac{1}{m} + \frac{2}{m} + \frac{1}{m} = \frac{4}{m} \\ &< |f(x_1) - f(x_2)|. \end{aligned}$$

This is a contradiction, so we conclude that  $\langle \mu_n^{-1}(x) \rangle_{n \in \mathbb{N}}$  has at most one cluster point. Since  $X$  is compact,  $\langle \mu_n^{-1}(x) \rangle_{n \in \mathbb{N}}$  has exactly one cluster point, call it  $\mu^{-1}(x)$ . Letting  $x$  vary, this defines our function  $\mu^{-1} : X \rightarrow X$ .

We next show that for each  $f \in C(X)$  and  $x \in X$ ,

$$f\mu^{-1}(x) = h(f)(x).$$

So let  $f \in C(X)$ , let  $x \in X$ , and let  $\varepsilon > 0$ . By the continuity of  $f$ ,  $\mu^{-1}(x)$  has a neighborhood  $U$  in  $X$  such that

$$f(U) \subseteq \left( f\mu^{-1}(x) - \frac{\varepsilon}{2}, f\mu^{-1}(x) + \frac{\varepsilon}{2} \right).$$

Choose an  $n \in \mathbb{N}$  with  $\frac{1}{n} < \frac{\varepsilon}{2}$  and such that  $\mu_n^{-1}(x) \in U$ . Then

$$\begin{aligned} |f\mu^{-1}(x) - h(f)(x)| &\leq |f\mu^{-1}(x) - f\mu_n^{-1}(x)| + |f\mu_n^{-1}(x) - h(f)(x)| \\ &< \frac{\varepsilon}{2} + \frac{1}{n} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Since  $\varepsilon$  is arbitrary, we have  $f\mu^{-1}(x) = h(f)(x)$ .

Now by using  $h^{-1}$  and the sequence  $\langle \mu_n \rangle_{n \in \mathbb{N}}$ , with the same arguments as above, we obtain a function  $\mu : X \rightarrow X$  having the property that

$$f\mu(x) = h^{-1}(f)(x)$$

for all  $f \in C(X)$  and  $x \in X$ .

We need to show that  $\mu$  and  $\mu^{-1}$  are inverse functions of each other. Suppose, by way of contradiction, that  $\mu\mu^{-1}$  is not the identity on  $X$ . So there exists an  $x \in X$  with  $\mu\mu^{-1}(x) \neq x$ . Then there is an  $f \in C(X)$  such that

$$f(\mu\mu^{-1}(x)) \neq f(x).$$

But if  $g = h^{-1}(f)$ , we have

$$\begin{aligned} f(\mu\mu^{-1}(x)) &\neq h(g)(x) \\ &= g\mu^{-1}(x) \\ &= h^{-1}(f)(\mu^{-1}(x)) \\ &= (f\mu)(\mu^{-1}(x)) \\ &= f(\mu\mu^{-1}(x)), \end{aligned}$$

which is a contradiction. Therefore,  $\mu\mu^{-1}$  is the identity on  $X$ , and a similar argument shows that  $\mu^{-1}\mu$  is the identity on  $X$ . This tells us that  $\mu^{-1}$  is a bijection with inverse  $\mu$ .

Finally, we show that  $\mu^{-1}$  is continuous. Again, by way of contradiction, suppose that  $\mu^{-1}$  is not continuous. Then since  $X$  is compact, there exist  $x, x_0 \in X$  with  $x_0 \neq \mu^{-1}(x)$  and a net  $\langle x_\lambda \rangle_{\lambda \in \Lambda}$  in  $X$  that converges to  $x$  in  $X$  while  $\langle \mu^{-1}(x_\lambda) \rangle_{\lambda \in \Lambda}$  converges to  $x_0$  in  $X$ . Let  $f \in C(X)$  be such that  $f(x_0) \neq f(\mu^{-1}(x))$ . Then  $\langle h(f)(x_\lambda) \rangle_{\lambda \in \Lambda}$  converges to  $h(f)(x) = f\mu^{-1}(x) \neq f(x_0)$ , while  $\langle h(f)(x_\lambda) \rangle_{\lambda \in \Lambda} = \langle f\mu(x_\lambda) \rangle_{\lambda \in \Lambda}$  which converges to  $f(x_0)$ . This contradiction shows that  $\mu^{-1}$  must be continuous. A similar argument shows that  $\mu$  is continuous, and completes the proof that  $\mu \in \mathcal{H}(X)$ . Therefore,  $h = \hat{\mu} \in \mathcal{H}_u(C(X))$ .  $\square$

It is surprising that the next proof of the analogous theorem for the vertical subgroup is so much different than that above for the horizontal subgroup.

*Proof of Theorem 2.5.* Let  $h$  be in the closure of  $\mathcal{V}er(C(X))$  in  $\mathcal{H}_u(C(X))$ . We need to show that  $h \in \mathcal{V}er(CX)$ . For each  $n \in \mathbb{N}$ , there exists a

$$v_n \in B(h, \frac{1}{n}) \cap \mathcal{V}er(C(X))$$

(see section 2 for the definition of  $B(h, \frac{1}{n})$ ). Now each  $v_n = \hat{\nu}_n$  for some  $\nu_n \in \mathcal{FH}(X)$ . Let  $\langle x, t \rangle \in X \times \mathbb{R}$ . For every  $f \in C(X)$  such that  $f(x) = t$  and for every  $n \in \mathbb{N}$ ,

$$|\pi_2 \nu_n(\langle x, t \rangle) - h(f)(x)| = |v_n(f)(x) - h(f)(x)| < \frac{1}{n}.$$

Therefore, for every  $f \in C(X)$  with  $f(x) = t$ , the sequence  $\langle \pi_2 \nu_n(\langle x, t \rangle) \rangle_{n \in \mathbb{N}}$  converges to  $h(f)(x)$ . So we define function  $\nu : X \times \mathbb{R} \rightarrow X \times \mathbb{R}$  by taking for every  $\langle x, t \rangle \in X \times \mathbb{R}$ ,

$$\nu(\langle x, t \rangle) = \langle x, h(f)(x) \rangle$$

where  $f \in C(X)$  is such that  $f(x) = t$ . In particular,

$$\nu(\langle x, f(x) \rangle) = \langle x, h(f)(x) \rangle$$

for all  $f \in C(X)$ . Also note that if  $f_1(x) = f_2(x)$ , then  $h(f_1)(x) = h(f_2)(x)$ .

We need to show that  $\nu \in \mathcal{FH}(X)$ , and then it follows that  $h = \hat{\nu} \in \mathcal{V}er(C(X))$ . Now it suffices to show that  $\nu$  is a homeomorphism.

First, to show that  $\nu$  maps  $X \times \mathbb{R}$  onto itself, let  $\langle x, t \rangle \in X \times \mathbb{R}$ . Then if  $f$  is the constant function in  $C(X)$  with value  $t$  and if  $s = h^{-1}(f)(x)$ , we have

$$\nu(\langle x, s \rangle) = \langle x, t \rangle,$$

showing that  $\nu$  is an onto mapping.

To show that  $\nu$  is one-to-one, the argument is much harder than to show  $\nu$  is onto. We do this by contradiction. Suppose that there exist  $x_0 \in X$  and  $f_1, f_2 \in C(X)$  such that  $f_1(x_0) \neq f_2(x_0)$  and  $h(f_1)(x_0) = h(f_2)(x_0)$ . Since

$$h(\min\{f_1, f_2\})(x_0) = h(\max\{f_1, f_2\})(x_0),$$

we may assume that  $f_1 \leq f_2$ . We can now use the fact that the members of  $\mathcal{V}er(C(X))$  are bimonotone (see the definition before Theorem 2.12). Let  $f_1 \leq f \leq f_2$ , and let  $n \in \mathbb{N}$ . Then

$$\min\{v_n(f_1), v_n(f_2)\} \leq v_n(f) \leq \max\{v_n(f_1), v_n(f_2)\}.$$

Therefore, for each  $x \in X$ ,

$$\begin{aligned} \min\{h(f_1)(x), h(f_2)(x)\} - \frac{1}{n} &< \min\{v_n(f_1)(x), v_n(f_2)(x)\} \\ &\leq v_n(f)(x) \\ &\leq \max\{v_n(f_1)(x), v_n(f_2)(x)\} \\ &< \max\{h(f_1)(x), h(f_2)(x)\} + \frac{1}{n}. \end{aligned}$$

Since this is true for all  $x \in X$  and all  $n \in \mathbb{N}$ , we have

$$\min\{h(f_1)(x), h(f_2)(x)\} \leq h(f) \leq \max\{h(f_1)(x), h(f_2)(x)\}.$$

This is true for all  $f \in C(X)$  with  $f_1 \leq f \leq f_2$ .

Now let us use some particular  $f$ 's to obtain our contradiction. Let  $\mathcal{N}$  be a neighborhood base for  $x_0$  in  $X$ . We may assume that  $x_0$  is not an isolated point of  $X$  because if it were, then the function  $f_3$  defined by  $f_3(x) = f_1(x)$  for  $x \in X \setminus \{x_0\}$  and  $f_3(x_0) = f_2(x_0)$  is such that  $h(f_3) = h(f_1)$ ; contradicting the fact that  $h$  is one-to-one. For each  $U \in \mathcal{N}$ , let  $f_U \in C(X)$  be such that

$$\begin{aligned} f_U(x_U) &= f_2(x_U) \text{ for some } x_U \in U \setminus \{x_0\}, \\ f_U(x) &= f_1(x) \text{ for all } x \in X \setminus (U \setminus \{x_0\}), \\ &\text{and } f_1 \leq f_U \leq f_2. \end{aligned}$$

Since  $h(f_1)(x_0) = h(f_2)(x_0)$ , since each  $f_U$  agrees with  $f_1$  outside of  $U$ , and since

$$\min\{h(f_1), h(f_2)\} \leq h(f_U) \leq \max\{h(f_1), h(f_2)\},$$

it follows that the net  $\langle h(f_U) \rangle_{U \in \mathcal{N}}$  in  $C(X)$  is a Cauchy net that converges to  $h(f_1)$ . Now  $h^{-1}$  is continuous on  $C(X)$ , which implies that  $\langle f_U \rangle_{U \in \mathcal{N}}$  converges uniformly to  $f_1$ . Since  $\langle x_n \rangle_{U \in \mathcal{N}}$  converges to  $x_0$ , it follows that  $\langle f_U(x_U) \rangle_{U \in \mathcal{N}}$  converges to  $f_1(x_0)$ . On the other hand,  $\langle f_U(x_U) \rangle_{U \in \mathcal{N}} = \langle f_2(x_U) \rangle_{U \in \mathcal{N}}$ , which converges to  $f_2(x_0) \neq f_1(x_0)$ . With this contradiction, we can now conclude that whenever  $f_1(x_0) \neq f_2(x_0)$  then  $h(f_1)(x_0) \neq h(f_2)(x_0)$ . This shows that  $\nu$  is one-to-one, and since it is also onto, it has an inverse  $\nu^{-1}$ .

In order to understand how  $\nu^{-1}$  must be defined, note that because

$$\nu(\langle x, f(x) \rangle) = \langle x, h(f)(x) \rangle$$

for all  $f \in C(X)$  and  $x \in X$ , we have

$$\nu^{-1}(\langle x, h(f)(x) \rangle) = \langle x, f(x) \rangle.$$

So thinking of  $g = h(f)$ , we can say that

$$\nu^{-1}(\langle x, g(x) \rangle) = \langle x, h^{-1}(g)(x) \rangle$$

for all  $g \in C(X)$  and  $x \in X$ . In particular,  $\nu^{-1} : X \times \mathbb{R} \rightarrow X \times \mathbb{R}$  is defined by

$$\nu^{-1}(\langle x, t \rangle) = \langle x, h^{-1}(f)(x) \rangle$$

for all  $\langle x, t \rangle \in X \times \mathbb{R}$  where  $f \in C(X)$  is such that  $f(x) = t$ .

It remains to show that  $\nu$  is continuous, and since  $\nu^{-1}$  is defined similarly, a similar argument shows that  $\nu^{-1}$  is continuous. Let  $\langle x, t \rangle \in X \times \mathbb{R}$ , and let  $\langle x_\lambda, t_\lambda \rangle_{\lambda \in \Lambda}$  be a net in  $X \times \mathbb{R}$  that converges to  $\langle x, t \rangle$ . Let  $f$  be the constant function in  $C(X)$  with value  $t$ , and for each  $\lambda \in \Lambda$ , let  $f_\lambda$  be the constant function in  $C(X)$  with value  $t_\lambda$ . Now  $\langle x_\lambda \rangle_{\lambda \in \Lambda}$  converges to  $x$  in  $X$ , and  $\langle t_\lambda \rangle_{\lambda \in \Lambda}$  converges to  $t$  in  $\mathbb{R}$ . The latter convergence ensures that  $\langle f_\lambda \rangle_{\lambda \in \Lambda}$  converges to  $f$  in  $C(X)$ . Then the continuity of  $h$  implies that  $\langle h(f_\lambda) \rangle_{\lambda \in \Lambda}$  converges to  $h(f)$  in  $C(X)$ , which is uniform convergence. Therefore,  $\langle h(f_\lambda)(x_\lambda) \rangle_{\lambda \in \Lambda}$  converges to  $h(f)(x)$  in  $\mathbb{R}$ , so that

$$\langle \nu(\langle x_\lambda, t_\lambda \rangle) \rangle_{\lambda \in \Lambda} = \langle \langle x_\lambda, h(f_\lambda)(x_\lambda) \rangle \rangle_{\lambda \in \Lambda}$$

converges to  $\langle x, h(f)(x) \rangle = \nu(\langle x, t \rangle)$  in  $X \times \mathbb{R}$ . This shows that  $\nu$  is continuous, and finishes the argument that  $\nu \in \mathcal{FH}(X)$ , and hence  $h = \hat{\nu} \in \mathcal{Ver}(C(X))$ .  $\square$

## 6. ALGEBRAIC PROOFS

This last section contains the proofs of Theorems 3.1 and 3.3 in Section 3 on the algebraic group. The first of these theorems ensures that the algebraic homeomorphisms on  $C(X)$  are well-defined.

*Proof of Theorem 3.1.* To prove that  $\hat{\sigma}$  is a bijection with inverse  $\hat{\tau}$ , we need to show that  $\hat{\tau}\hat{\sigma}$  is the identity on  $C(X)$ ; by symmetry,  $\hat{\sigma}\hat{\tau}$  would also be the identity on  $C(X)$ . If  $f \in C(X)$ , then

$$\begin{aligned} \hat{\tau}\hat{\sigma}(f) &= \hat{\tau}\left(f + \sum_{i=1}^n \frac{|\alpha(p, f, i)|}{|\alpha(p)|} (\beta_i - \alpha_i)\right) \\ &= f + \sum_{i=1}^n \frac{|\alpha(p, f, i)|}{|\alpha(p)|} (\beta_i - \alpha_i) + \sum_{i=1}^n \frac{|\beta(p, \hat{\sigma}(f), i)|}{|\beta(p)|} (\alpha_i - \beta_i). \end{aligned}$$

It suffices to show that for each  $m = 1, \dots, n$ ,

$$\frac{|\beta(p, \hat{\sigma}(f), m)|}{|\beta(p)|} = \frac{|\alpha(p, f, m)|}{|\alpha(p)|},$$

or equivalently,

$$|\alpha(p)||\beta(p, \hat{\sigma}(f), m)| = |\beta(p)||\alpha(p, f, m)|.$$

We can write this as

$$|\alpha(p)||\beta(p, f + \sum_{i=1}^n \frac{|\alpha(p, f, i)|}{|\alpha(p)|}(\beta_i - \alpha_i), m)| = |\beta(p)||\alpha(p, f, m)|$$

or equivalently,

$$\begin{aligned} |\alpha(p)||\beta(p, f, m)| + \sum_{i=1}^n |\alpha(p, f, i)||\beta(p, \beta_i, m)| - \sum_{i=1}^n |\alpha(p, f, i)||\beta(p, \alpha_i, m)| \\ = |\beta(p)||\alpha(p, f, m)|. \end{aligned}$$

Since  $|\beta(p, \beta_i, m)| = 0$  for  $i \neq m$ , we have

$$\sum_{i=1}^n |\alpha(p, f, i)||\beta(p, \beta_i, m)| = |\alpha(p, f, m)||\beta(p)|.$$

So we need to show that

$$\sum_{i=1}^n |\alpha(p, f, i)||\beta(p, \alpha_i, m)| = |\alpha(p)||\beta(p, f, m)|.$$

If  $A_{ij}$  and  $B_{ij}$  are the  $ij$  cofactors of  $\alpha(p)$  and  $\beta(p)$ , respectively, then we need to show that

$$\sum_{i=1}^n \left( \sum_{j=1}^n f(p_j) A_{ij} \right) \left( \sum_{j=1}^n \alpha_i(p_j) B_{mj} \right) = |\alpha(p)| \sum_{j=1}^n f(p_j) B_{mj}.$$

We do this by showing that the terms involving  $f(p_j)B_{mk}$  on both sides are equal for all  $j, k = 1, \dots, n$ .

First suppose that  $j = k$ . Then the  $f(p_j)B_{mj}$  term on the left is

$$\begin{aligned} \sum_{i=1}^n f(p_j) A_{ij} \alpha_i(p_j) B_{mj} &= \left( \sum_{i=1}^n \alpha_i(p_j) A_{ij} \right) f(p_j) B_{mj} \\ &= |\alpha(p)| f(p_j) B_{mj}, \end{aligned}$$

which is equal to the  $f(p_j)B_{mj}$  term on the right. Finally, suppose that  $j \neq k$ . Then the  $f(p_j)B_{mk}$  term on the left is

$$\sum_{i=1}^n f(p_j) A_{ij} \alpha_i(p_k) B_{mk} = \left( \sum_{i=1}^n \alpha_i(p_k) A_{ij} \right) f(p_j) B_{mk}.$$



But  $\sum_{i=1}^n \alpha_i(p_k)A_{ij} = 0$  since  $j \neq k$ , and there is no  $f(p_j)B_{mk}$  term on the right. This completes the argument that  $\hat{\sigma}$  and  $\hat{\tau}$  are inverse functions, showing that  $\hat{\sigma}$  is a bijection.

To show that  $\hat{\sigma}$  is continuous on  $C(X)$ , let  $f \in C(X)$ , let  $K$  be a compact subset of  $X$ , and let  $\varepsilon > 0$ . We can assume that  $\{p_1, \dots, p_n\} \subseteq K$ . We have

$$\hat{\sigma}(f) = f + \frac{1}{|\alpha(p)|} \sum_{i=1}^n \sum_{j=1}^n f(p_j)A_{ij}(\beta_i - \alpha_i)$$

where  $A_{ij}$  is the cofactor of the  $ij$  element of  $\alpha(p)$ . Define  $C = \max\{|A_{ij}| : 1 \leq i, j \leq n\}$  and  $M = 1 + \max\{|\beta_i(x) - \alpha_i(x)| : 1 \leq i \leq n \text{ and } x \in K\}$ . Finally, let

$$\delta = \min \left\{ \frac{\varepsilon}{2}, \frac{\varepsilon|\alpha(p)|}{2nCM} \right\}.$$

Let  $B(f, K, \delta)$  be the neighborhood of  $f$  in  $C(X)$  defined by

$$B(f, K, \delta) = \{g \in C(X) : |g(x) - f(x)| < \delta \text{ for all } x \in K\}.$$

If  $g \in B(f, K, \delta)$ , then for each  $x \in K$ ,

$$\begin{aligned} |\hat{\sigma}(f)(x) - \hat{\sigma}(g)(x)| &= \left| f(x) + \frac{1}{|\alpha(p)|} \sum_{i=1}^n \sum_{j=1}^n f(p_j)A_{ij}(\beta_i(x) - \alpha_i(x)) \right. \\ &\quad \left. - g(x) + \frac{1}{|\alpha(p)|} \sum_{i=1}^n \sum_{j=1}^n g(p_j)A_{ij}(\beta_i(x) - \alpha_i(x)) \right| \\ &< \delta + \frac{1}{|\alpha(p)|} \sum_{i=1}^n \sum_{j=1}^n |f(p_j) - g(p_j)| |A_{ij}| |\beta_i(x) - \alpha_i(x)| \\ &< \delta + \frac{CM}{|\alpha(p)|} \sum_{j=1}^n |f(p_j) - g(p_j)| \\ &< \delta + \frac{nCM}{|\alpha(p)|} \delta \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon. \end{aligned}$$

Therefore,  $\hat{\sigma}(B(f, K, \delta)) \subseteq B(\hat{\sigma}(f), K, \varepsilon)$ , showing that  $\hat{\sigma}$  is continuous on  $C(X)$ . A similar argument shows that  $\hat{\sigma}^{-1}$  is continuous on  $C(X)$ , so that  $\hat{\sigma}$  is a homeomorphism on  $C(X)$ . The fact that  $\hat{\sigma}$  is linear follows from properties of determinants.  $\square$

We now prove the second of the theorems that ensures the algebraic homeomorphisms on  $C(X)$  forms a group.

*Proof of Theorem 3.3.* For  $m = 1, 2$ , let  $a_m = \hat{\sigma}_m$  where  $\sigma_m = \langle n_m, \alpha^m, \beta^m, p^m \rangle \in Q(X)$ . We need to find a  $\sigma = \langle n, \alpha, \beta, p \rangle$  in  $Q(X)$  such that  $\hat{\sigma} = \hat{\sigma}_2 \hat{\sigma}_1$ . Let  $\phi_0 : J_0 \rightarrow \{p_1^1, \dots, p_{n_1}^1\} \cap \{p_1^2, \dots, p_{n_2}^2\}$ ,  $\phi_1 : J_1 \rightarrow \{p_1^1, \dots, p_{n_1}^1\} \setminus \{p_1^2, \dots, p_{n_2}^2\}$  and  $\phi_2 : J_2 \rightarrow \{p_1^2, \dots, p_{n_2}^2\} \setminus \{p_1^1, \dots, p_{n_1}^1\}$  be bijections where  $J_0, J_1$  and  $J_2$  are disjoint subsets of  $\mathbb{N}$  such that  $J_0 \cup J_1 \cup J_2 = \{1, \dots, n\}$  for some  $n$ . Then  $\max\{n_1, n_2\} \leq n \leq n_1 + n_2$ .

For  $m = 1, 2, 3$  and for each  $j \in J_m$ , let  $p_j = \phi_m(j)$ . Then let  $p = \langle p_1, \dots, p_n \rangle$ ; so that the coordinates of  $p$  are the distinct elements of  $\{p_1^1, \dots, p_{n_1}^1\} \cup \{p_1^2, \dots, p_{n_2}^2\}$ .

We start by choosing any  $\alpha \in C^n(X)$  such that  $|\alpha(p)| \neq 0$ . Now let us abbreviate  $\alpha(p)$ ,  $\alpha^1(p^1)$  and  $\alpha^2(p^2)$  as  $A$ ,  $A^1$  and  $A^2$ , respectively. Also let  $A_{ij}$ ,  $A_{ij}^1$  and  $A_{ij}^2$  be the  $ij$  cofactors of  $A$ ,  $A^1$  and  $A^2$ , respectively. Finally, thinking of  $\beta \in C^n(X)$  as our unknown, let  $\delta_j = \beta_j - \alpha_j$  for  $1 \leq j \leq n$ , let  $\delta_j^1 = \beta_j^1 - \alpha_j^1$  for  $1 \leq j \leq n_1$ , and let  $\delta_j^2 = \beta_j^2 - \alpha_j^2$  for  $1 \leq j \leq n_2$ .

If  $\sigma = \langle n, \alpha, \beta, x \rangle$  for our unknown  $\beta$ , then for each  $f \in C(X)$ , we can write

$$\hat{\sigma}(f) = f + \sum_{j=1}^n f(x_j) \sum_{i=1}^n \frac{A_{ij}}{|A|} \delta_i.$$

Now let us consider the product  $\hat{\sigma}_2 \hat{\sigma}_1$ . For each  $f \in C(X)$ ,

$$\begin{aligned} \hat{\sigma}_2 \hat{\sigma}_1(f) &= \hat{\sigma}_2\left(f + \frac{1}{|A^1|} \sum_{j=1}^{n_1} f(p_j^1) \sum_{i=1}^{n_1} A_{ij}^1 \delta_i^1\right) \\ &= f + \frac{1}{|A^1|} \sum_{j=1}^{n_1} f(p_j^1) \sum_{i=1}^{n_1} A_{ij}^1 \delta_i^1 \\ &\quad + \frac{1}{|A^2|} \sum_{l=1}^{n_2} [f(p_l^2) + \frac{1}{|A^1|} \sum_{j=1}^{n_1} f(p_j^1) \sum_{i=1}^{n_1} A_{ij}^1 \delta_i^1(p_l^2)] \sum_{k=1}^{n_2} A_{kl}^2 \delta_k^2 \\ &= f + \frac{1}{|A^1|} \sum_{j=1}^{n_1} f(p_j^1) \sum_{i=1}^{n_1} A_{ij}^1 \delta_i^1 + \frac{1}{|A^2|} \sum_{l=1}^{n_2} f(p_l^2) \sum_{k=1}^{n_2} A_{kl}^2 \delta_k^2 \\ &\quad + \frac{1}{|A^1||A^2|} \sum_{j=1}^{n_1} f(p_j^1) \sum_{l=1}^{n_2} \sum_{i=1}^{n_1} \sum_{k=1}^{n_2} A_{ij}^1 A_{kl}^2 \delta_i^1(p_l^2) \delta_k^2. \end{aligned}$$

The last expression can be written as

$$f + \sum_{j=1}^{n_1} f(p_j^1) \sum_{i=1}^{n_1} \left[ \frac{A_{ij}^1}{|A^1|} \delta_i^1 + \sum_{l=1}^{n_2} \sum_{k=1}^{n_2} \frac{A_{ij}^1 A_{kl}^2}{|A^1||A^2|} \delta_i^1(p_l^2) \delta_k^2 \right] + \sum_{j=1}^{n_2} f(p_j^2) \sum_{k=1}^{n_2} \frac{A_{kj}^2}{|A^2|} \delta_k^2.$$

We need to find  $\delta_1, \dots, \delta_n$  such that for each  $m = 1, 2, 3$  and for each  $j \in J_m$ ,

$$\sum_{i=1}^n \frac{A_{ij}}{|A|} \delta_i = \Delta_j^m,$$

where

$$\begin{aligned}\Delta_j^0 &= \sum_{i=1}^{n_1} \left[ \frac{A_{i\phi_0(j)}^1}{|A^1|} \delta_i^1 + \sum_{l=1}^{n_2} \sum_{k=1}^{n_2} \frac{A_{i\phi_0(j)}^1 A_{kl}^2}{|A^1||A^2|} \delta_i^1 (p_l^2) \delta_k^2 \right] + \sum_{k=1}^{n_2} \frac{A_{k\phi_0(j)}^2}{|A^2|} \delta_k^2, \\ \Delta_j^1 &= \sum_{i=1}^{n_1} \left[ \frac{A_{i\phi_1(j)}^1}{|A^1|} \delta_i^1 + \sum_{l=1}^{n_2} \sum_{k=1}^{n_2} \frac{A_{i\phi_1(j)}^1 A_{kl}^2}{|A^1||A^2|} \delta_i^1 (p_l^2) \delta_k^2 \right], \\ \Delta_j^2 &= \sum_{k=1}^{n_2} \frac{A_{k\phi_2(j)}^1}{|A^1|} \delta_k^1.\end{aligned}$$

Let  $\delta$  be the column matrix whose  $i$ th element is  $\delta_i$ , and let  $\Delta$  be the column matrix such that for each  $j = 1, \dots, n$ , the  $j$ th element of  $\Delta$  is  $\Delta_j^m$  where  $j \in J_m$ . Then the system of equations we need to solve for  $\delta$  becomes  $\alpha(p)^{-1}\delta = \Delta$ , which has solution  $\delta = \alpha(p)\Delta$ . We now define  $\beta = \alpha + \delta$  and let  $\sigma = \langle n, \alpha, \beta, p \rangle$ . Then we have  $\hat{\sigma} = \hat{\sigma}_2 \hat{\sigma}_1$ . Since  $\hat{\sigma}_2 \hat{\sigma}_1$  is a bijection, so is  $\hat{\sigma}$ . If we can show that  $|\beta(p)| \neq 0$ , then we know that  $\sigma \in Q(X)$ , and it would follow that we could define  $a = \hat{\sigma}$  and have  $a_2 a_1 = a \in \mathcal{A}(C(X))$ .

To show that  $|\beta(p)| \neq 0$ , it actually suffices to know that  $\hat{\sigma}$  is a surjection, which we do. Suppose, by way of contradiction, that  $|\beta(p)| = 0$ . For each  $j = 1, \dots, n$ , let  $\beta(p_j)$  denote the  $j$ th column of  $\beta(p)$ . Then for some  $m = 1, \dots, n$ , we can write  $\beta(p_m)$  as a linear combination of the other columns; say

$$\beta(p_m) = \sum_{\substack{k=1, \dots, n \\ k \neq m}} c_k \beta(p_k).$$

For each  $f \in C(X)$ ,

$$\begin{aligned}\hat{\sigma}(f) &= f + \sum_{i=1}^n \frac{|\alpha(p, f, i)|}{|\alpha(p)|} (\beta_i - \alpha_i) \\ &= f + \sum_{i=1}^n \sum_{j=1}^n f(p_j) \frac{A_{ij}}{A} (\beta_i - \alpha_i) \\ &= f + \sum_{j=1}^n f(p_j) \sum_{i=1}^n \frac{A_{ij}}{A} (\beta_i - \alpha_i).\end{aligned}$$

So for each  $f \in C(X)$  and each  $k = 1, \dots, n$ ,

$$\begin{aligned}\hat{\sigma}(f)(p_k) &= f(p_k) + \sum_{j=1}^n f(p_j) \sum_{i=1}^n \frac{A_{ij}}{A} (\beta_i(p_k) - \alpha_i(p_k)) \\ &= f(p_k) + \sum_{j=1}^n f(p_j) \left[ \sum_{i=1}^n \frac{A_{ij}}{A} \beta_i(p_k) - \sum_{i=1}^n \frac{A_{ij}}{A} \alpha_i(p_k) \right] \\ &= \sum_{j=1}^n f(p_j) \sum_{i=1}^n \frac{A_{ij}}{A} \beta_i(p_k).\end{aligned}$$

Then for each  $f \in C(X)$ ,

$$\begin{aligned}\hat{\sigma}(f)(p_m) &= \sum_{j=1}^n f(p_j) \sum_{i=1}^n \frac{A_{ij}}{A} \beta_i(p_m) \\ &= \sum_{j=1}^n f(p_j) \sum_{i=1}^n \frac{A_{ij}}{A} \sum_{\substack{k=1, \dots, n \\ k \neq m}} c_k \beta_i(p_k) \\ &= \sum_{\substack{k=1, \dots, n \\ k \neq m}} c_k \sum_{j=1}^n f(p_j) \sum_{i=1}^n \frac{A_{ij}}{A} \beta_i(p_k) \\ &= \sum_{\substack{k=1, \dots, n \\ k \neq m}} c_k \hat{\sigma}(f)(p_k).\end{aligned}$$

Since the  $p_1, \dots, p_n$  are distinct, there exists a  $g \in C(X)$  such that

$$g(p_m) \neq \sum_{\substack{k=1, \dots, n \\ k \neq m}} c_k g(p_k).$$

Then  $g$  is not in the image of  $\hat{\sigma}$ , which contradicts  $\hat{\sigma}$  being a surjection.  $\square$

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