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THE CARTESIAN PRODUCT OF A PSEUDO-ARC AND A PSEUDO-SOLENOID IS FACTORWISE RIGID

KEVIN B. GAMMON

Abstract. The Cartesian product of two spaces is called factorwise rigid if every self homeomorphism can be written as a composition of a product of homeomorphisms on the individual coordinates with a permutation of the coordinates. In 1983 D. Bellamy and J. Lysko proved that the Cartesian product of two pseudo-arcs is factorwise rigid. This result was extended by the author to the Cartesian product of a pseudo-arc and a pseudo-circle. We extend the result to the Cartesian product of a pseudo-arc and any pseudo-solenoid.

1. Introduction

A *continuum* is a nonempty, nondegenerate, compact, and connected metric space. Given a continuum X and a point $x \in X$, $K(x)$ will denote the composant of X which contains x. A continuum is *indecomposable* if it is not the union of two proper subcontinuum. A continuum is *hereditarily indecomposable* if every subcontinuum of it is indecomposable. Let X be a continuum and let x and y be elements of X. X is *chainable between x and y* provided that, given $\epsilon > 0$, there is an ϵ-chain covering X so that x is in the first link and y is in the last link. For more background on chains see, for example, [1] and [2].

This paper focuses on the Cartesian product of two types of well known hereditarily indecomposable continua. The first, the pseudo-arc, was originally discovered by B. Knaster in 1922 [14]. In 1948, E. E. Moise constructed a family of plane indecomposable continua with the property that each was homeomorphic to each of its subcontinua [19]. Because an arc also has the property that it is homeomorphic to each of its subcontinua, he used the term pseudo-arc to define a continuum obtained from his construction. He noted that the continua he constructed were similar

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to the example constructed by Knaster in 1922. In 1948, R. H. Bing constructed an indecomposable homogeneous plane continuum and showed that his example was homeomorphic to each of the continua constructed by Moise [1]. Moise independently proved in 1949 that his examples were homogeneous [20]. In 1951, Bing proved that any two hereditarily indecomposable chainable continua are homeomorphic [3]. The term *pseudo-arc* is now used for any hereditarily indecomposable chainable continuum. The pseudo-arc has been the subject of many interesting research questions and results. A survey of many other aspects of the pseudo-arc can be found in survey papers by W. Lewis ([17] and [18]). Throughout this paper, P will denote a pseudo-arc which is chainable between two points α and β.

A pseudo-solenoid is an inverse limit of pseudo-circles where finite covering maps are used as bonding maps. An alternative characterization is a hereditarily indecomposable continuum which is circularly chainable but not chainable. In 1972, L. Fearnley proved that two indecomposable circularly chainable continua are homeomorphic if and only if they have isomorphic first Čech cohomology groups [8]. In this argument L. Fearnley used Čech cohomology with integer coefficients. Since a pseudo-solenoid is closely related to a particular solenoid, $S(\Sigma)$ will denote a pseudo-solenoid whose Čech cohomology is isomorphic to the Čech cohomology of the solenoid Σ. Given a pseudo-solenoid $S(\Sigma)$, L. Fearnley defined a map $\phi : S(\Sigma) \to \Sigma$ which preserves the first Čech cohomology group. Using this map, we define a *pseudo-composant* to be the inverse image of an arc component of the solenoid Σ.

The Cartesian product $X \times Y$ of two spaces is called *factorwise rigid* provided that if $h : X \times Y \to X \times Y$ is a self homeomorphism, then h is a composition of a product of homeomorphisms on the coordinates with a permutation of the coordinates.

It is known that the Cartesian product of two pseudo-arcs is factorwise rigid. This result is due to D. Bellamy and J. Lysko in [6]. Since the pseudo-arc and pseudo-circle share many properties, it was suspected that the result could be generalized to include pseudo-circles. In 2008, the author extended the result to include the Cartesian product of a pseudo-arc and a pseudo-circle by the use of a covering space of the Cartesian product of a pseudo-arc and pseudo-circle. In this paper the result will be extended to the Cartesian product of a pseudo-arc and a pseudo-solenoid. This work originated out of an attempt to answer the open question published by D. Bellamy and J. Kennedy in [4] which asked if the arbitrary product of hereditarily indecomposable continua is factorwise rigid.

2. Preliminary Information

In the following, the projection from a Cartesian product $A \times B$ to the first factor space will be denoted by π_1. Likewise, π_2 will denote the projection to the second factor space. If A is a space, then $\check{H}_1(A)$ will denote the first Čech cohomology of the space A with integer coefficients.

Let G be a relation on $P \times P$ which collapses the fiber $P \times \{\alpha\}$ to a single point and $P \times \{\beta\}$ to a single point. Let $q : P \times P \to (P \times P)/G$ be the quotient map. Throughout the paper, Φ will be the set $P \times \{\alpha\} \cup P \times \{\beta\}$. It is useful to notice that if $W \subset (P \times P)/G$ such that $q^{-1}(W)$ intersects $P \times \{\alpha\}$ (or $P \times \{\beta\}$), then $q^{-1}(W)$ contains $P \times \{\alpha\}$ (or $P \times \{\beta\}$.)

Lemma 2.1. *If $B \subset (P \times P)/G$ is a continuum, then $q^{-1}(B)$ is a continuum.*

Proof. $q^{-1}(B)$ is closed and hence compact because q is continuous. The fact that $q^{-1}(B)$ is connected is proven in Theorem 9, page 131 of [16]. □

Let $X = P \times S(\Sigma)$. Then X can be essentially embedded in the Cartesian product, Y, of a disk D^2 and a solid torus T. Let $\widetilde{Y}$ denote the universal covering space of Y with covering map p, which contains an infinite, connected covering space $\widetilde{X}$ of X. $\widehat{Y}$ will denote the two-point compactification of $\widetilde{Y}$ by adding points $\bar{a}$ and $\bar{b}$. Likewise, $\widehat{X}$ will denote the two-point compactification of $\widetilde{X}$ contained in $\widehat{Y}$.

We will also briefly discuss the covering space $\widetilde{S(\Sigma)}$ of $S(\Sigma)$ contained in the universal covering space $\widetilde{T}$ of T. Let $p_T : \widetilde{T} \to T$ denote the covering map and let $p_{T'}$ denote the restriction of the covering map to $\widetilde{S(\Sigma)}$.

Lemma 2.2. *Let C_λ be a pseudo-composant of $S(\Sigma)$. The two-point compactification of C_λ is a pseudo-arc.*

Proof. This follows from the fact that C_λ is an infinite, connected covering space of the pseudo-circle. From a result originally due to Bellamy and Lewis in [5] and re-proven by the author in [12], the two point compactification of an infinite, connected covering space of the pseudo-circle is a pseudo-arc. □

Corollary 2.3. *Let C_λ be a pseudo-composant of $S(\Sigma)$. Then the two-point compactification of the lift $p_{T'}^{-1}(C_\lambda)$ is a pseudo-arc P.*

Corollary 2.4. *Let C_λ be a pseudo-composant of $S(\Sigma)$ and let $x \in P$. Then the two-point compactification of the lift $p^{-1}(\{x\} \times C_\lambda)$, is a pseudo-arc P_λ.*

Lemma 2.5. *Let C_λ be a pseudo-composant of $S(\Sigma)$. Then the two-point compactification of the lift $p^{-1}(P \times C_\lambda)$ is homeomorphic to the quotient space G.*

Proof. From Lemma 2.3, there is a homeomorphism f_1 from C_λ onto $P - \{\alpha, \beta\}$. The map $h_1(x, y) = (id_P(x), f_1(y))$, where id_P is the identity map on P, is a homeomorphism from $p^{-1}(P \times C_\lambda)$ to $(P \times P) - (P \times \{\alpha\} \cup P \times \{\beta\})$. This map extends uniquely to a homeomorphism $H : \widehat{X} \to (P \times P)/G$. □

In the following, if C_λ is a pseudo-composant, the two-point compactification $p^{-1}(P \times C_\lambda) \cup \{\bar{a}, \bar{b}\}$ will be denoted as G_λ. Let $q_\lambda : P \times P \to G_\lambda$ be the map which consists of composition of the quotient map q and the homeomorphism described in the proof of the previous lemma.

Let $g : X \to X$ be a homeomorphism. There exists a lift $\tilde{g}$ such that the following diagram commutes:

$$\begin{array}{ccc} & \widetilde{g} & \\ \widetilde{X} & \rightarrow & \widetilde{X} \\ \downarrow p & & \downarrow p \\ X & \rightarrow & X \\ & g & \end{array}$$

The argument that such a lift exists is similar to the lifting argument used by K. Kuperberg and the author in [15]. First note that since the solid torus T is an absolute neighborhood retract, g extends to a continuous map f from a closed, connected neighborhood of X homeomorphic to $D^2 \times T$ into $D^2 \times T$. Then $D^2 \times T$ can be retracted to this neighborhood of X. The composition of these maps has a lift. The appropriate restriction of this lift provides the lift of g.

Moreover, $\widetilde{g}$ extends uniquely to a map $H : \widehat{X} \to \widehat{X}$. This map is a continuous bijection and hence a homeomorphism. Any such homeomorphism has the property that the set $\{\bar{a}, \bar{b}\}$ is invariant.

Lemma 2.6. *Let C_λ be a pseudo-composant of $S(\Sigma)$. Then $H(C_\lambda) \subset G_\gamma$ for some pseudo-composant G_γ.*

Proof. Suppose that $H(G_\lambda)$ contained points in G_γ and G_δ where $\gamma \neq \delta$. Since $\{\bar{a}, \bar{b}\}$ is an invariant set, $\{\bar{a}, \bar{b}\}$ is a subset of $H(G_\lambda)$. Then $H(G_\lambda) - \{\bar{a}, \bar{b}\}$ is disconnected. However, this is impossible since $G_\lambda - \{\bar{a}, \bar{b}\}$ is connected. □

The following lemma in [6] will be needed:

Lemma 2.7. [Bellamy and Lysko, [6], Lemma 6] *Suppose X and Y are indecomposable continua, $a \in X$ and $h : X \times Y \to X \times Y$ is a homeomorphism. Then either $\pi_1(h(\{a\} \times Y)) = X$ or $\pi_2(h(\{a\} \times Y) = Y$.*

The following theorem of J. T. Rogers, Jr. will also be used.

Theorem 2.8. [Rogers, [21], Theorem 14] *No pseudo-solenoid is the continuous image of the pseudo-arc.*

Lemma 2.9. *Let $a \in S(\Sigma)$. Then $\pi_1(g(P \times \{a\})) = P$.*

Proof. Notice that $\pi_2 \circ g(P \times \{a\}))$ is a continuous mapping of a pseudo-arc into a pseudo-solenoid. From Theorem 2.8, the pseudo-solenoid cannot be the continuous image of a pseudo-arc. Therefore, that $\pi_2 \circ g(P \times \{a\}))$ must be a proper subcontinuum of $S(\Sigma)$. Thus, from Lemma 2.7, it follows that $\pi_1(g(P \times \{a\})) = P$. □

Lemma 2.10. *Let $a \in P$. Then $\pi_2(g(\{a\} \times S(\Sigma))) = S(\Sigma)$.*

Proof. Since $\check{H}_1(P)$ is trivial, the restriction $g|_{\{a\} \times S(\Sigma)} : \{a\} \times S(\Sigma) \to P \times S(\Sigma)$ induces an isomorphism between the groups $\check{H}_1(\{a\} \times S(\Sigma))$ and $\check{H}_1(P \times S(\Sigma))$. Likewise, since $\check{H}_1(P)$ is trivial, $\pi_2 : P \times S(\Sigma) \to S(\Sigma)$ induces an isomorphism between $\check{H}_1(P \times S(\Sigma))$ and $\check{H}_1(S(\Sigma))$. Therefore, the composition of these two maps induces an isomorphism between $\check{H}_1(\{a\} \times S(\Sigma))$ and $\check{H}_1(S(\Sigma))$. In particular, this implies that $\pi_2 \circ g(\{a\} \times S(\Sigma))$ must be onto. □

For the following proofs it will be necessary to adapt a lemma of Bellamy and Lysko in [6]:

Lemma 2.11. [Bellamy and Lysko, [6], Corollary 3] *Let X and Y be chainable continua and suppose W and M are subcontinua of $X \times Y$ such that $\pi_1(W) \subset \pi_1(M)$ while $\pi_2(M) \subset \pi_2(W)$. Then $W \cap M \neq \emptyset$.*

Lemma 2.12. *Suppose that W and M are subcontinua of $(P \times P)/G$ such that $\pi_1 \circ q^{-1}(W) \subset \pi_1 \circ q^{-1}(M)$ and $\pi_2 \circ q^{-1}(M) \subset \pi_2 \circ q^{-1}(W)$, then $M \cap N \neq \emptyset$.*

Proof. Since the inverses image under q of a continuum is a continuum, the inverse image satisfies the conditions of Lemma 2.11. □

With the previous lemmas in mind, it will be proven that the induced homeomorphism $H : \widehat{X} \to \widehat{X}$ has the additional properties that for any point $p \in P$, if H maps G_λ into G_γ then

(1) $[q_\gamma^{-1} \circ H \circ q_\lambda](P \times \{p\}) = P \times \{a\}$ for some $a \in P$ and
(2) $[q_\gamma^{-1} \circ H \circ q_\lambda](\{p\} \times P) = \Phi \cup (\{b\} \times P)$ for some $b \in P$.

Theorem 2.13. *For every $x \in P$ and for any λ, $[q_\gamma^{-1} \circ H \circ q_\lambda](P \times \{x\}) = P \times \{b\}$ for some $b \in P$ and for some γ.*

Proof. If $x \in \{\alpha, \beta\}$, the result follows because the set $q_\lambda(\Phi)$ is invariant under the homeomorphism H.

If $x \notin \{\alpha, \beta\}$, then the observations of the previous lemmas allow the use of the proof of the main theorem in [6] developed by Bellamy and Lysko. Suppose that $\pi_2(q_\gamma^{-1} \circ H \circ q_\lambda(P \times \{x\}))$ is nondegenerate. Let $\mathbb{N}$ denote the set of positive integers and let $< W_n >_{n \in \mathbb{N}}$ be a sequence of nondegenerate, decreasing subcontinua of P such that $\cap_{n \in \mathbb{N}} W_n = \{x\}$. Since this is a decreasing sequence, assume without loss of generality that $W_n \cap \{\alpha, \beta\} = \emptyset$ for each n. Let $a \in P$ and notice that

$$\bigcap_{n \in \mathbb{N}} (\{a\} \times W_n) = \{(a, x)\} \subset P \times \{x\}.$$

Therefore

$$\bigcap_{n \in \mathbb{N}} [\pi_2 \circ q_\gamma^{-1} \circ H \circ q_\lambda](\{a\} \times W_n) =$$

$$[\pi_2 \circ q_\gamma^{-1} \circ H \circ q_\lambda](a, x) \in [\pi_2 \circ q_\gamma^{-1} \circ H \circ q_\lambda](P \times \{x\}).$$

In particular, $[\pi_2 \circ q_\gamma^{-1} \circ H \circ q_\lambda](a, x)$ is an element of

$$[\pi_2 \circ q_\gamma^{-1} \circ H \circ q_\lambda](P \times \{x\}) \cap [\pi_2 \circ q_\gamma^{-1} \circ H \circ q_\lambda](\{a\} \times W_n)$$

for each n. Since P is hereditarily indecomposable, this implies that for each n either

(1) $[\pi_2 \circ q_\gamma^{-1} \circ H \circ q_\lambda](\{a\} \times W_n) \subset [\pi_2 \circ q_\gamma^{-1} \circ H \circ q_\lambda](P \times \{x\})$ or
(2) $[\pi_2 \circ q_\gamma^{-1} \circ H \circ q_\lambda](P \times \{x\}) \subset [\pi_2 \circ q_\gamma^{-1} \circ H \circ q_\lambda](\{a\} \times W_n)$.

Since $\cap_{n \in \mathbb{N}}([\pi_2 \circ q_\gamma^{-1} \circ H \circ q_\lambda](\{a\} \times W_n))$ is degenerate, condition (1) cannot be true for each n. Therefore, there exists some N such that $[\pi_2 \circ q_\gamma^{-1} \circ H \circ q_\lambda](\{a\} \times W_N) \subset [\pi_2 \circ q_\gamma^{-1} \circ H \circ q_\lambda](P \times \{x\})$.

Let $x_1 \in W_N$ such that $x_1 \neq p$. From the above remarks,

$$[\pi_2 \circ q_\gamma^{-1} \circ H \circ q_\lambda](P \times \{x_1\}) \cap [\pi_2 \circ q_\gamma^{-1} \circ H \circ q_\lambda](P \times \{x\}) \neq \emptyset.$$

This implies that either

(1) $[\pi_2 \circ q_\gamma^{-1} \circ H \circ q_\lambda](P \times \{x_1\}) \subset [\pi_2 \circ q_\gamma^{-1} \circ H \circ q_\lambda](P \times \{x\})$ or
(2) $[\pi_2 \circ q_\gamma^{-1} \circ H \circ q_\lambda](P \times \{x\}) \subset [\pi_2 \circ q_\gamma^{-1} \circ H \circ q_\lambda](P \times \{x_1\})$.

We prove the first case. The proof of the second case is similar. Notice from Lemma 2.9, $[\pi_1 \circ q_\gamma^{-1} \circ H \circ q_\lambda](P \times \{x_1\}) = P = [\pi_1 \circ q_\gamma^{-1} \circ H \circ q_\lambda](P \times \{x\})$. Therefore the conditions of Lemma 2.12 are satisfied. Hence $[H \circ q_\lambda](P \times \{x_1\}) \cap [H \circ q_\lambda](P \times \{x\}) \neq \emptyset$. However, this is a contradiction since $[q_\gamma^{-1} \circ H \circ q_\lambda]$ restricted to $(P \times P) - \Phi$ is a homeomorphism. □

Theorem 2.14. *For any* λ, $[q_\gamma^{-1} \circ H \circ q_\lambda](\{a\} \times P) = \Phi \cup (\{b\} \times P)$ *for some* $b \in P$ *and for some* γ.

Proof. Let $x \in P$ such that $K(x)$ does not contain α or β. Such a point exists because an indecomposable continuum has uncountably many pairwise disjoint composants (see, for example, K. Kuratowski, [16], Theorems 5 and 7, p. 212). It will first be shown that $[q_\gamma^{-1} \circ H \circ q_\lambda](\{a\} \times K(x)) \subset \{b\} \times P$ for some $b \in P$.

Let P_1 be a nondegenerate subcontinuum of $K(x)$. Note that P_1 is a pseudo-arc and consider the subcontinuum $P \times P_1$ of $P \times P$. From Theorem 2.13, for each point $x_1 \in P_1$, the set $[q_\gamma^{-1} \circ H \circ q_\lambda](P \times \{x_1\})$ is mapped homeomorphically onto $P \times \{x_2\}$ for some $x_2 \in P$. Note that x_2 cannot equal α or β. In particular, $[q_\gamma^{-1} \circ H \circ q_\lambda](P \times P_1)$ is mapped bijectively onto $P \times P_2$ where P_2 is a proper, nondegenerate subcontinuum of P and therefore a pseudo-arc. Similar to the proof of Theorem 2.13, the proof of the main result by Bellamy and Lysko in [6] can be applied to show that $[q_\gamma^{-1} \circ H \circ q_\lambda]$ restricted to $P \times P_1$ also preserves horizontal fibers. In particular, $[q_\gamma^{-1} \circ H \circ q_\lambda](\{a\} \times P_1) \subset \{b\} \times P$ for some $b \in P$.

Let $p \in P$ and consider $\{p\} \times P_1$. Let $< W_n >_{n\in\mathbb{N}}$ be a sequence of decreasing, nondegenerate subcontinua of P such that $\cap_{n\in\mathbb{N}} W_n = \{p\}$. Next, let $a \in P_1$ and notice that

$$\bigcap_{n\in\mathbb{N}} (W_n \times \{a\}) = \{(p,a)\} \in \{p\} \times P_1.$$

In particular, $\cap_{n\in\mathbb{N}}[\pi_1 \circ q_\gamma^{-1} \circ H \circ q_\lambda](W_n \times \{a\}) = [\pi_1 \circ q_\gamma^{-1} \circ H \circ q_\lambda](p,a)$ is an element of $[\pi_1 \circ q_\gamma^{-1} \circ H \circ q_\lambda](\{p\} \times P_1)$.

Therefore,

$$[\pi_1 \circ q_\gamma^{-1} \circ H \circ q_\lambda](W_n \times \{a\}) \cap [\pi_1 \circ q_\gamma^{-1} \circ H \circ q_\lambda](\{p\} \times P_1) \neq \emptyset$$

for each n. This implies that for each n, either

(1) $[\pi_1 \circ q_\gamma^{-1} \circ H \circ q_\lambda](W_n \times \{a\}) \subset [\pi_1 \circ q_\gamma^{-1} \circ H \circ q_\lambda](\{p\} \times P_1)$ or
(2) $[\pi_1 \circ q_\gamma^{-1} \circ H \circ q_\lambda](\{p\} \times P_1) \subset [\pi_1 \circ q_\gamma^{-1} \circ H \circ q_\lambda](W_n \times \{a\})$.

However, since $\cap([\pi_1 \circ q_\gamma^{-1} \circ H \circ q_\lambda](W_n \times \{a\}))$ is degenerate, condition 2 cannot hold for every n. Thus, there exists some N so that $[\pi_1 \circ q_\gamma^{-1} \circ H \circ q_\lambda](W_N \times \{a\}) \subset [\pi_1 \circ q_\gamma^{-1} \circ H \circ q_\lambda](\{p\} \times P_1)$.

Let $x_1 \in W_N$ such that $x_1 \neq p$. From the above remarks,

$$[\pi_1 \circ q_\gamma^{-1} \circ H \circ q_\lambda](\{x_1\} \times P_1) \cap [\pi_1 \circ q_\gamma^{-1} \circ H \circ q_\lambda](\{p\} \times P_1) \neq \emptyset.$$

Since the pseudo-arc is hereditarily indecomposable, this implies that either

(1) $[\pi_1 \circ q_\gamma^{-1} \circ H \circ q_\lambda](\{x_1\} \times P_1) \subset [\pi_1 \circ q_\gamma^{-1} \circ H \circ q_\lambda](\{p\} \times P_1)$ or
(2) $[\pi_1 \circ q_\gamma^{-1} \circ H \circ q_\lambda](\{p\} \times P_1) \subset [\pi_1 \circ q_\gamma^{-1} \circ H \circ q_\lambda](\{x_1\} \times P_1)$.

The first case is proven. The second case is similar. Since $[\pi_2 \circ q_\gamma^{-1} \circ H \circ q_\lambda](\{x_1\} \times P_1) = P_2 = [\pi_2 \circ q_\gamma^{-1} \circ H \circ q_\lambda](\{p\} \times P_1)$, the conditions of Lemma 2.12 are satisfied. Therefore, $[H \circ q_\lambda](\{x_1\} \times P_1) \cap [H \circ q_\lambda](\{p\} \times P_1) \neq \emptyset$. This contradicts the fact that $[q_\gamma^{-1} \circ H \circ q_\lambda]$ restricted to $(P \times P) - \Phi$ is a homeomorphism. Therefore $[q_\gamma^{-1} \circ H \circ q_\lambda](\{a\} \times P_1) \subset \{b\} \times P$ for some $b \in P$.

Next, notice that since P is an indecomposable continuum any two points in $K(x)$ can be joined by a proper subcontinuum. Therefore, $[q_\gamma^{-1} \circ H \circ q_\lambda](\{a\} \times K(x)) \subset \{b\} \times P$.

However, note that $H \circ q_\lambda(\{a\} \times P) = H \circ q_\lambda(cl(\{a\} \times K(x))$ because composants in a continuum are dense. From the previous paragraphs, this implies that $H \circ q_\lambda(\{a\} \times P) = q_\gamma(\{b\} \times P)$. Therefore it follows that $[q_\gamma^{-1} \circ H \circ q_\lambda](\{a\} \times P) = (\{b\} \times P) \cup \Phi$. □

3. Factorwise rigidity of $P \times S(\Sigma)$

As in the previous section, let $X = P \times S(\Sigma)$ and let $\widehat{X}$ denote the two-point compactification of the infinite covering space $\widetilde{X}$ of X.

Theorem 3.1. *The Cartesian product $P \times S(\Sigma)$ is factorwise rigid.*

Proof. Let $h : X \to X$ be a homeomorphism. Then there exists a lift $\tilde{h}$ such that the following diagram commutes.

$$\begin{array}{ccc} & \widetilde{h} & \\ \widetilde{X} & \rightarrow & \widetilde{X} \\ \downarrow p & & \downarrow p \\ X & \rightarrow & X \\ & h & \end{array}$$

Then $\widetilde{h}$ extends uniquely to a map $H : \widehat{X} \to \widehat{X}$. This map is a continuous bijection and hence a homeomorphism. Any such homeomorphism has the property that the set $\{\bar{a}, \bar{b}\}$ is invariant. Applying the results in the previous section, for $x \in S(\Sigma)$, $h(P \times \{x\}) = p \circ H \circ p^{-1}(P \times \{x\})$. However, from Theorem 2.13, $H \circ p^{-1}(P \times \{x\}) = p^{-1}(P \times \{y\})$ for some $y \in S(\Sigma)$. Hence $h(P \times \{x\}) = P \times \{y\}$.

Likewise, from Theorem 2.14, it follows that $h(\{r\} \times S(\Sigma)) = p \circ H \circ p^{-1}(\{r\} \times S(\Sigma)) = \{s\} \times S(\Sigma)$ for some $s \in P$.

Therefore, the Cartesian product $P \times S(\Sigma)$ is factorwise rigid. □

This argument utilizes the fact that every subcontinuum of a pseudo-arc or pseudo-solenoid is a pseudo-arc. If X and Y are continua such that every proper subcontinuum is a pseudo-arc, is the product $X \times Y$ factorwise rigid?

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