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A NOTE ON SEPARATION OF DIAGONAL

LIANG–XUE PENG AND JING LI

Abstract. In this note, we point out that a Δ-paracompact normal space is functionally Δ-paracompact. As a corollary, we have that regular Δ-paracompact spaces are functionally Δ-paracompact and functionally Δ-normal. This gives a positive answer to two questions of Burke and Buzyakova, which appear in Topology and Applications (**157** (2010), 2261–2270).

In the last part of this note, we show that a space X is a paracompact T_2-space if and only if X is a submetacompact Δ-paracompact regular space. We also introduce a concept of Δ-metacompact, and point out that Δ-metacompactness implies neither metacompactness, nor Δ-paracompactness.

Introduction

In [2] and [1], it is investigated when and how the diagonal of a space X can be separated from any closed subset of the square X^2 that lies off the diagonal. Let X be a space; the diagonal of X is $\Delta_X = \{(x,x) : x \in X\}$.

By conclusions which appear in [2], we know that a paracompact T_2-space is functionally Δ-paracompact, a functionally Δ-paracompact space is regular Δ-paracompact, and a regular Δ-paracompact space is Δ-paracompact. In [2] it is also pointed out that there is a Δ-paracompact space which is neither functionally Δ-paracompact, nor regular Δ-paracompact. It is proved in [2] that every functionally Δ-paracompact space is functionally Δ-normal. The following questions appear in [1].

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Question 1. Does regular Δ-paracompactness imply functional Δ-paracompactness?

Question 2. Does regular Δ-paracompactness imply functional Δ-normality?

These questions also appear in Table 1 in [2].

In fact, by some known conclusions we can prove that every Δ-paracompact normal space is functionally Δ-paracompact, and hence regular Δ-paracompactness implies functional Δ-paracompactness and functional Δ-normality. Thus the two questions of Burke and Buzyakova are answered. In last part of this note, we show that a space X is a paracompact T_2-space if and only if X is a submetacompact Δ-paracompact regular space. In this note, we also introduce the concept of Δ-metacompact. We point out that Δ-metacompactness implies neither metacompactness, nor Δ-paracompactness.

All spaces in this note are assumed to be T_1-spaces. In notation and terminology we will follow [3].

Main results

Definition 1. ([5]) A space X is Δ-*normal* if for every $A \subset X^2 \setminus \Delta_X$ closed in X^2 there exist disjoint open sets U and V in X^2 such that $A \subset U$ and $\Delta_X \subset V$.

Definition 2. ([2]) A space X is *functionally* Δ-*normal* if for every $A \subset X^2 \setminus \Delta_X$ closed in X^2 there exists a continuous function $f : X^2 \to [0,1]$ such that $f(A) \subset \{1\}$ and $f(\Delta_X) = \{0\}$.

Definition 3. ([2]) A space X is Δ-*paracompact* if for every $A \subset X^2 \setminus \Delta_X$ closed in X^2 there exists a locally finite open cover $\mathcal{U}$ of X such that $\bigcup\{U \times U : U \in \mathcal{U}\}$ does not meet A.

Definition 4. ([2]) A space X is *regular* Δ-*paracompact* if for every $A \subset X^2 \setminus \Delta_X$ closed in X^2 there exists a locally finite open cover $\mathcal{U}$ of X such that $\bigcup\{\overline{U} \times \overline{U} : U \in \mathcal{U}\}$ does not meet A.

Definition 5. ([2]) A space X is *functionally* Δ-*paracompact* if for every $A \subset X^2 \setminus \Delta_X$ closed in X^2 there exists a locally finite open cover $\mathcal{U}$ of X by functionally open sets (i.e. cozero sets) such that $\bigcup\{U \times U : U \in \mathcal{U}\}$ does not meet A.

In [2] the author states that all spaces in that paper are Tychonoff. In fact, to get the following Lemma 6, the T_1 separation axiom is enough.

Lemma 6. ([2]) *The following holds:*

(1) *A functionally* Δ-*paracompact space is functionally* Δ-*normal and regular* Δ-*paracompact.*

(2) *A regular Δ-paracompact space is Δ-normal, normal, and Δ-paracompact.*

Since every paracompact T_2-space is normal, we have:

Lemma 7. ([2]) *A paracompact T_2-space is functionally Δ-paracompact.*

Lemma 8. ([3, Theorem 1.5.18]) *For every point-finite open cover $\{U_s : s \in S\}$ of a normal space X there exists an open cover $\{V_s : s \in S\}$ of X such that $\overline{V_s} \subset U_s$ for each $s \in S$.*

Theorem 9. *If X is a Δ-paracompact normal space, then X is functionally Δ-paracompact.*

Proof. Let $A \subset X^2 \setminus \Delta_X$ be a closed subset of X^2. The space X is Δ-paracompact, there exists a locally finite open cover $\mathcal{U}$ of X such that $(U \times U) \cap A = \emptyset$ for each $U \in \mathcal{U}$. The open cover $\mathcal{U}$ is locally finite, and hence it is point-finite. If we let $\mathcal{U} = \{U_s : s \in S\}$, then there is an open cover $\mathcal{V} = \{V_s : s \in S\}$ of X such that $\overline{V_s} \subset U_s$ for each $s \in S$ by Lemma 8. By Urysohn's Lemma, we have a continuous function $f_s : X \to [0,1]$ such that $f_s(X \setminus U_s) \subset \{0\}$ and $f(\overline{V_s}) \subset \{1\}$ for each $s \in S$. Thus $\overline{V_s} \subset f_s^{-1}((0,1]) \subset U_s$. If $\mathcal{U}^* = \{f_s^{-1}((0,1]) : s \in S\}$, then $\mathcal{U}^*$ is a locally finite cover of X by functionally open sets of X such that $\bigcup\{O \times O : O \in \mathcal{U}^*\}$ does not meet A. Thus X is functionally Δ-paracompact. □

By Lemma 6 and Theorem 9, we have:

Corollary 10. *A space X is functionally Δ-paracompact if and only if X is regular Δ-paracompact.*

Corollary 11. *If X is regular Δ-paracompact, then X is functionally Δ-normal.*

Corollary 12. *Let X be a normal space. The following are equivalent.*

(1) *X is functionally Δ-paracompact;*
(2) *X is regular Δ-paracompact;*
(3) *X is Δ-paracompact.*

By Corollary 12, the Theorem 2.15 and Corollary 2.16 which appear in [2] are generalized.

By Theorem 2.5 in [2] we know that every generalized ordered space is Δ-paracompact. Recall that every generalized ordered space is monotonically normal ([6]), and hence it is collectionwise normal. The space ω_1 is a Δ-paracompact monotonically normal space which is not paracompact. So we would like to know what property $\mathcal{P}$ such that a Δ-paracompact space which has property $\mathcal{P}$ is paracompact. In what follows, we will discuss this question.

Recall that a space is called *collectionwise normal* if for every discrete family $\{F_s : s \in S\}$ of closed subsets of X there exists a discrete collection $\{U_s : s \in S\}$ such that $F_s \subset U_s$ for each $s \in S$. In [1, Theorem 2.8], it is proved that if X is a regular Δ-paracompact space then X is collectionwise normal. By Corollary 2.13 which appears in [2], we know that Δ-paracompactness neither implies normality, nor collectionwise normality.

Recall that a space X is called *submetacompact* if for any open cover $\mathcal{U}$ of X there is an open refinement $\mathcal{V} = \bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ such that for each $x \in X$ there is $n_x \in \mathbb{N}$ such that $ord(x, \mathcal{V}_{n_x}) < \omega$ and $\bigcup \mathcal{V}_n = X$ for each $n \in \mathbb{N}$, where $ord(x, \mathcal{V}_{n_x}) = |\{V : x \in V \text{ and } V \in \mathcal{V}_{n_x}|$. We know that metacompact spaces and subparacompact spaces are submetacompact.

Theorem 13. *If X is a submetacompact Δ-paracompact regular space, then X is collectionwise normal.*

Proof. Since Δ-paracompact normal spaces are regular Δ-paracompact by Corollary 12 and every regular Δ-paracompact space is collectionwise normal ([1, Theorem 2.8]), it suffices to prove normality of X.

Let F_1 and F_2 be any two disjoint closed subsets of X. For each $x \in F_1$, there is an open neighborhood V_x of x such that $x \in V_x \subset \overline{V_x} \subset X \setminus F_2$. If $\mathcal{U} = \{V_x : x \in F_1\} \cup \{X \setminus F_1\}$, then $\mathcal{U}$ is an open cover of X. Since X is submetacompact, the open cover $\mathcal{U}$ has an open refinement $\mathcal{V} = \bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ such that for each $x \in X$ there is $n_x \in \mathbb{N}$ such that $ord(x, \mathcal{V}_{n_x}) < \omega$ and $\bigcup \mathcal{V}_n = X$ for each $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, denote $\mathcal{V}_n^* = \{V : V \in \mathcal{V}_n \text{ and } V \cap F_1 \neq \emptyset\} \cup \{X \setminus F_1\}$. If $B_n = X^2 \setminus \bigcup\{V \times V : V \in \mathcal{V}_n^*\}$, then B_n is a closed subset of X^2 and $B_n \cap \Delta_X = \emptyset$. Since X is Δ-paracompact, there is a locally finite open cover $\mathcal{U}_n$ of X such that $(U \times U) \cap B_n = \emptyset$ for each $U \in \mathcal{U}_n$. If $F_{1n} = \{x : x \in F_1 \text{ and } ord(x, \mathcal{V}_n^*) < \omega\}$ for each $n \in \mathbb{N}$, then $F_1 = \bigcup\{F_{1n} : n \in \mathbb{N}\}$.

Claim. Let $n \in \mathbb{N}$. If $U \in \mathcal{U}_n$ and $U \cap F_{1n} \neq \emptyset$, then $\overline{U} \cap F_2 = \emptyset$.

Proof of the Claim. Suppose there are some $n \in \mathbb{N}$ and $U \in \mathcal{U}_n$ such that $U \cap F_{1n} \neq \emptyset$ and $\overline{U} \cap F_2 \neq \emptyset$. Let $x \in U \cap F_{1n}$. Thus $ord(x, \mathcal{V}_n^*) < \omega$. If $V \in \mathcal{V}_n^*$ and $x \in V$, then there is some $y \in F_1$ such that $V \subset V_y$. Since $\overline{V_y} \cap F_2 = \emptyset$, we have that $\overline{V} \cap F_2 = \emptyset$. So $U \not\subset \overline{V}$ for each $V \in \{V : x \in V \text{ and } V \in \mathcal{V}_n^*\}$. Since the family $\{V : x \in V \text{ and } V \in \mathcal{V}_n^*\}$ is a finite family of X, we have $\overline{\bigcup\{V : x \in V, V \in \mathcal{V}_n^*\}} = \bigcup\{\overline{V} : x \in V, V \in \mathcal{V}_n^*\}$. Since $\overline{U} \cap F_2 \neq \emptyset$, $\overline{U} \setminus \bigcup\{\overline{V} : x \in V, V \in \mathcal{V}_n^*\} \neq \emptyset$ and hence $U \setminus \overline{\bigcup\{V : x \in V, V \in \mathcal{V}_n^*\}} \neq \emptyset$. So $U \setminus \bigcup\{\overline{V} : x \in V, V \in \mathcal{V}_n^*\} \neq \emptyset$.

If $z \in U \setminus \bigcup\{\overline{V} : x \in V, V \in \mathcal{V}_n^*\}$, then $(x, z) \in U \times U$ and $(x, z) \notin V \times V$ for each $V \in \mathcal{V}_n^*$. Thus $(x, z) \in (U \times U) \cap B_n$. This is a contradiction with $(U \times U) \cap B_n = \emptyset$. So we have proved the Claim.

If $P_n = \bigcup\{U : U \in \mathcal{U}_n \text{ and } U \cap F_{1n} \neq \emptyset\}$ for each $n \in \mathbb{N}$, then $F_{1n} \subset P_n$ and $\overline{P_n} \cap F_2 = \emptyset$ by the Claim and locally finite property of $\mathcal{U}_n$. Thus we have a countable family $\{P_n : n \in \mathbb{N}\}$ of open subsets of X such that $F_1 \subset \bigcup\{P_n : n \in \mathbb{N}\}$ and $\overline{P_n} \cap F_2 = \emptyset$ for each $n \in \mathbb{N}$.

Similarly, we can get a countable family $\{Q_n : n \in \mathbb{N}\}$ of open subsets of X such that $F_2 \subset \bigcup\{Q_n : n \in \mathbb{N}\}$ and $\overline{Q_n} \cap F_1 = \emptyset$ for each $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, we let $P_n^* = P_n \setminus \bigcup\{\overline{Q_m} : m \leq n\}$ and let $Q_n^* = Q_n \setminus \bigcup\{\overline{P_m} : m \leq n\}$. If $P = \bigcup\{P_n^* : n \in \mathbb{N}\}$ and $Q = \bigcup\{Q_n^* : n \in \mathbb{N}\}$, then we can easily prove that P and Q are disjoint open subsets of X such that $F_1 \subset P$ and $F_2 \subset Q$. Thus X is a normal space. □

Corollary 14. *If X is a metacompact (or subparacompact) Δ-paracompact regular space, then X is collectionwise normal.*

Since a submetacompact (metacompact or subparacompact) collectionwise normal space is a paracompact space, we have the following theorem by Theorem 13, Lemma 6 and 7.

Theorem 15. *A space X is a paracompact T_2-space if and only if X is a submetacompact (metacompact or subparacompact) Δ-paracompact regular space.*

Recall that if S is the Sorgenfrey line, then the space S is a paracompact space and hence S is Δ-paracompact.

Corollary 16. *S^2 is not Δ-paracompact.*

Proof. The space S^2 is a subparacompact regular space (cf. [7]). Since S^2 is not normal, it is not paracompact. Thus S^2 is not Δ-paracompact by Theorem 15. □

Corollary 16 shows that the product of Δ-paracompact spaces (or generalized ordered spaces) may not be Δ-paracompact.

By Theorem 15, we know that if X is a metacompact non-paracompact regular space then X is not Δ-paracompact.

It is pointed out ([Page 266, 4]) that the following space is a metacompact non-paracompact regular space:

$$X = \{(\nu, \delta + 1) : \nu \leq \omega \wedge \delta < \omega_1\} \cup \{(n, \omega_1) : n < \omega\}.$$

The space X is a regular metacompact space by Theorem 5.1 which appears in [4]. Since X is not normal, X is not paracompact. Thus the space X is a metacompact non-paracompact regular space, and hence the space X is not Δ-paracompact by Theorem 15.

A space X is Δ-*metacompact* if for every $A \subset X^2 \setminus \Delta_X$ closed in X^2 there exists a point-finite open cover $\mathcal{U}$ of X such that $\bigcup\{U \times U : U \in \mathcal{U}\}$ does not meet A.

Theorem 17. *Every metacompact space is Δ-metacompact*

Proof. Let $A \subset X^2 \setminus \Delta_X$ be a closed subset of X^2. For each $x \in X$, let V_x be an open neighborhood of x such that $(V_x \times V_x) \cap A = \emptyset$. Since X is metacompact, the open cover $\{V_x : x \in X\}$ has a point-finite open refinement $\mathcal{U}$. Thus $(U \times U) \cap A = \emptyset$ for each $U \in \mathcal{U}$. □

It is obvious that every Δ-paracompact space is Δ-metacompact. The space $X = \{(\nu, \delta+1) : \nu \leq \omega \wedge \delta < \omega_1\} \cup \{(n, \omega_1) : n < \omega\}$ is metacompact, and hence it is Δ-metacompact. But the space X is not Δ-paracompact by Theorem 15. We have mentioned that the space ω_1 is a Δ-paracompact and hence it is Δ-metacompact, but it is not metacompact. Thus we have:

Proposition 18. *Δ-metacompactness implies neither metacompactness, nor Δ-paracompactness.*

By Lemma 8 and the proof of Theorem 9, we have:

Theorem 19. *Let X be a normal space. The following are equivalent.*

(1) *X is a Δ-metacompact space;*
(2) *For every $A \subset X^2 \setminus \Delta_X$ closed in X^2 there exists a point-finite open cover $\mathcal{U}$ of X by functionally open sets such that $\bigcup\{U \times U : U \in \mathcal{U}\}$ does not meet A;*
(3) *For every $A \subset X^2 \setminus \Delta_X$ closed in X^2 there exists a point-finite open cover $\mathcal{U}$ of X such that $\bigcup\{\overline{U} \times \overline{U} : U \in \mathcal{U}\}$ does not meet A.*

In [1], it is proved that if X is a Δ-paracompact regular space then X is collectionwise Hausdorff. A space is *collectionwise Hausdorff* ([1]) if for every closed discrete subset $A \subset X$ there exists a discrete collection $\{U_a : a \in A\}$ such that $U_a \cap A = \{a\}$ for each $a \in A$. We have the following more general conclusion.

Theorem 20. *Let X be a Δ-paracompact regular space. If $\mathcal{F} = \{F_s : s \in S\}$ is a discrete collection of compact sets of X, then there is a discrete collection $\{W_s : s \in S\}$ of open sets of X such that $F_s \subset W_s$ for each $s \in S$.*

Proof. The proof is similar to the proof of Theorem 2.3 which appears in [1]. To assist the reader, we give the proof.

If $A_s = \bigcup\{F_{s'} : s' \in S \setminus \{s\}\}$, then A_s is closed in X for each $s \in S$. By regularity of X and compactness of F_s, there is an open set U_s of X such that $F_s \subset U_s \subset \overline{U_s} \subset X \setminus A_s$. If $\mathcal{U} = \{U_s : s \in S\} \cup \{X \setminus \bigcup \mathcal{F}\}$, then $\mathcal{U}$ is an open cover of X. Thus the set $F = X^2 \setminus \bigcup\{U \times U : U \in \mathcal{U}\} \subset X^2 \setminus \Delta_X$ is closed in X^2.

Since X is Δ-paracompact, there exists a locally finite open cover $\mathcal{V}$ of X such that $\bigcup\{V \times V : V \in \mathcal{V}\}$ misses F. For each $s \in S$, we have $|\{V : V \in \mathcal{V}, V \cap F_s \neq \emptyset\}| < \omega$ by compactness of F_s and locally finite property of $\mathcal{V}$. For each $s \in S$, let $\mathcal{V}_s = \{V : V \in \mathcal{V}, V \cap F_s \neq \emptyset\}$.

For each $V \in \mathcal{V}_s$, we have $V \subset U_s$.

Suppose there is some $V \in \mathcal{V}_s$ such that $V \not\subset U_s$. We let $x \in V \cap F_s$ and let $y \in V \setminus U_s$. Thus the point $(x, y) \in V \times V$. Since for each $U \in \mathcal{U}$, $x \notin U$ if $U \neq U_s$, we have $(x, y) \in F$. This is a contradiction with $(V \times V) \cap F = \emptyset$. Thus $V \subset U_s$ for each $V \in \mathcal{V}_s$, and hence $\overline{V} \cap A_s = \emptyset$. So $\mathcal{V}_{s_1} \cap \mathcal{V}_{s_2} = \emptyset$ if $s_1 \in S, s_2 \in S$ and $s_1 \neq s_2$.

For each $s \in S$, let $O_s = \bigcup \mathcal{V}_s \setminus \overline{\bigcup\{\bigcup \mathcal{V}_{s'} : s' \in S \setminus \{s\}\}}$. Since $\mathcal{V}$ is locally finite, we have $F_s \subset O_s$. We can see that $O_{s_1} \cap O_{s_2} = \emptyset$ if $s_1 \in S, s_2 \in S$ and $s_1 \neq s_2$. For each $x \in X$, there is an open neighborhood M_x of x such that $\{V : V \in \mathcal{V}, V \cap M_x \neq \emptyset\}| < \omega$. Since $O_s \subset \bigcup \mathcal{V}_s$ and $\mathcal{V}_{s_1} \cap \mathcal{V}_{s_2} = \emptyset$ if $s_1 \in S, s_2 \in S$ and $s_1 \neq s_2$, we have $|\{s : M_x \cap O_s \neq \emptyset, s \in S\}| < \omega$. Thus $\{O_s : s \in S\}$ is a locally finite family of open sets of X. For each $s \in S$, there exists an open set W_s of X such that $F_s \subset W_s \subset \overline{W_s} \subset O_s$ by compactness of F_s and regularity of X. Thus $\{M_s : s \in S\}$ is a discrete collection of open sets of X such that $F_s \subset W_s$ for each $s \in S$. □

The following question which appears in [1] is still open.

Question 21. *Does Δ-normality imply functional Δ-normality?*

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