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	Department of Mathematics & Statistics
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NORMALITY AND λ-EXTENDABLE PROPERTIES IN $\Sigma_\kappa(F)$-PRODUCTS

J. ANGOA, Y. F. ORTIZ-CASTILLO, AND Á. TAMARIZ-MASCARÚA

Abstract. For a cardinal number $\kappa \geq \omega_1$, a family $\{X_s : s \in S\}$ of topological spaces and a point $p \in \prod_{s\in S} X_s = X$, let $\Sigma_\kappa(p) = \{x \in X : |\{s \in S : x_s \neq p_s\}| < \kappa\}$ where for a point $y \in X$, y_s is the projection of y to X_s. For $F = \{x_j : j \in J\} \subseteq X$ and a collection $\Gamma = \{\gamma_j : j \in J\}$ of uncountable cardinal numbers, we define $\Sigma_\Gamma(F) = \bigcup_{j\in J} \Sigma_{\gamma_j}(x_j)$. In this article we analyze the normality and compact-like properties in spaces of the type $\Sigma_\Gamma(F)$. In particular, we prove:

(1) Let X_s be a compact space for each $s \in S$ such that for every $J \in [S]^{<\omega}$, $hd(\prod_{s\in J} X_s) \leq \omega$. Let $F \subseteq \prod_{s\in S} X_s = X$ and let Γ be a family of uncountable cardinals. Assume that $\Sigma_\Gamma(F) \neq X$. Then $\Sigma_\Gamma(F)$ is normal if and only if there is a regular cardinal κ and a point $p \in X$ such that $\Sigma_\Gamma(F) = \Sigma_\kappa(p)$.

(2) Let X_s be a compact space for each $s \in S$, κ a regular cardinal and α an infinite cardinal such that $\alpha < \kappa$. Then $\Sigma_\kappa(F)$ is initially α-compact (respectively, α-bounded) if and only if there is $F^* \subseteq X$ such that F^* is initially α-compact (respectively, α-bounded) and $\Sigma_\kappa(F) = \Sigma_\kappa(F^*)$.

Introduction

It is known that every Σ-product (that is a space of the form $\Sigma_{\omega_1}(p)$) of compact spaces possesses strong compactness-like properties. In particular, these spaces are ω-bounded. With respect to normality, H.H. Corson [4] proved that every Σ-product of complete metric spaces is normal. A.P. Kombarov and V.I. Malyhin [10] showed that every Σ-product of separable metric spaces is normal by proving the following result:

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Theorem 0.1. *Let $\{X_s : s \in S\}$ be a family of spaces such that for each $F \in [S]^{\aleph_0}$, $\prod_{s\in F} X_s$ is hereditarily separable and normal. Then for every $p \in \prod_{s\in S} X_s$, $\Sigma_{\omega_1}(p)$ is normal.*

Finally, S.P. Gul'ko [8] and M.E. Rudin [15] independently proved that every Σ-product of metric spaces is normal.

In this article we study some compact-like properties and the normality in Σ_κ-products when $\kappa \geq \omega_1$. In Section 1, we introduce the notion of λ-extendable property and we prove that if $X = \prod_{s\in S} X_s$ is initially $<\tau$-compact (resp., p-compact, $<\tau$-bounded), then $\Sigma_\kappa(p)$ is initially $<\tau$-compact (resp., p-compact, $<\tau$-bounded) when $\omega < \kappa \leq |S|$, $\tau = cof(\kappa)$ and $p \in X$. Also, we generalize the Kombarov-Malyhin's theorem, showing in particular that every Σ_κ-product $\Sigma_\kappa(p)$ of a product of compact metric spaces is normal if and only if κ is regular.

In Section 2, we introduce the $\Sigma_\kappa(F)$-products and $\Sigma_\Gamma(F)$-products, and we prove that for a compact product X, $\Sigma_\kappa(F)$ is initially α-compact (resp., p-compact, α-bounded) if and only if there exists $F^* \subseteq X$ such that F^* is initially α-compact (resp., p-compact, α-bounded) and $\Sigma_\kappa(F) = \Sigma_\kappa(F^*)$, when $\omega \leq \alpha < cof(\kappa) \leq \kappa \leq |S|$.

In the last section, Section 3, we study the normality of $\Sigma_\Gamma(F)$-products and we conclude that every proper $\Sigma_\Gamma(F)$-product which is normal must be equal to a Σ_κ-product where κ is regular. In particular, a proper $\Sigma_\Gamma(F)$-product of a product of compact metric spaces is normal if and only if it coincides with a Σ_κ-product where κ is regular.

Every space in this article is a Tychonoff space with more than one point. The letters ξ, ζ and η represent ordinal numbers and the letters α, γ, κ, λ and τ represent cardinal numbers; ω is the first infinite cardinal and ω_1 is the first uncountable cardinal. Given a set X and a cardinal number κ, $[X]^{\leq\kappa} = \{A \subseteq X : |A| \leq \kappa\}$, $[X]^{<\kappa} = \{A \subseteq X : |A| < \kappa\}$, $[X]^{\kappa} = \{A \subseteq X : |A| = \kappa\}$. The letters $\mathbb{N}$, $\mathbb{R}$, I, represent the Natural Numbers, the space of the Real Numbers and the interval $[0,1]$, respectively. Given two spaces X, Y, $C(X,Y)$ denotes the set of continuous functions from X to Y; if $Y = \mathbb{R}$ we write $C(X)$ and $C^*(X)$ for the set of continuous functions with real values and bounded continuous functions with real values, respectively. βX is the Stone-Čech compactification of the space X and X^* denotes the set $\beta X \setminus X$. If $X = \prod_{s\in S} X_s$, $s \in S$ and $F \subseteq S$, π_s is the projection from X onto X_s and π_F is the projection from X onto $X_F = \prod_{s\in F} X_s$. Given a canonical open set U of a topological product $X = \prod_{s\in S} X_s$, the support of U is the set $\{s \in S : \pi_s[U] \neq X_s\}$ and it will be denoted by $Supp[U]$. For a cardinal number τ, we use the symbol $[0,\tau)$ to denote the space of ordinals which are strictly less than τ provided with its order topology.

A collection $\{S_i : i \in J\}$ of subsets of a set S is a *partition* of S if $\bigcup_{i\in J} S_i = S$ and $S_i \cap S_j = \emptyset$ for all $i, j \in J$ with $i \neq j$. And a collection $\{S_i : i \in J\}$ of S is a *κ-partition*, where κ is a cardinal number, if $\{S_i : i \in J\}$ is a partition of S and $|S_i| = \kappa$ for every $i \in J$. For those concepts which appear in this article without definition, consult [5].

1. Σ_κ-Products

Given a family of spaces $\{X_s : s \in S\}$ and a fixed point $x \in \prod_{s\in S} X_s = X$, for each $z \in X$, the support z with respect to x is the set

$$supp_x(z) = \{s \in S : \pi_s(x) \neq \pi_s(z)\}.$$

For an infinite cardinal κ, we define the Σ_κ-product of X with respect to x to be the subspace

$$\Sigma_\kappa(x, X) = \{z \in X : |supp_x(z)| < \kappa\}$$

of X which can be denoted also by $\Sigma_\kappa(x)$ when it is clear which the space X is. When $\kappa = \omega$, $\Sigma_\kappa(x, X)$ is usually denoted by $\sigma(x, X)$, and when $\kappa = \omega_1$,
we simply use the notation $\Sigma(x, X)$. If $A \subseteq \Sigma_\kappa(x)$ we write $supp_x(A)$ to denote the set $\cup_{a\in A} supp_x(a)$.

Note that the collection of the Σ_κ-products form a partition of X; that is, if $\omega \leq \alpha \leq \kappa \leq |S|$ are cardinal numbers and $x, y \in X$, then $x \in \Sigma_\kappa(y)$ if and only if $\Sigma_\alpha(x) \cap \Sigma_\kappa(y) \neq \emptyset$ if and only if $\Sigma_\alpha(x) \subseteq \Sigma_\kappa(y)$. It is also known that (1) every Σ_ω-product (and so every Σ_κ-product) is dense in X, (2) $\pi_F[\S igma_\kappa(x)] = X_F$ for every $F \in [S]^{<\kappa}$ and $x \in X$, and (3) every Σ_κ-product is *$<\kappa$-closed* if κ is regular, that is, for every subset A of a Σ_κ-product with $|A| < \kappa$, we have that $Cl_X(A)$ remains included in the Σ_κ-product.

Recall that *a net* in X is a family $\mathcal{N}$ of subsets of X such that for each $x \in X$ and each open subset U with $x \in U$, there is an element $N \in \mathcal{N}$ such that $x \in N \subseteq U$. The *net weight* of a space X is the least cardinality of a net for X, and it is denoted by $nw(X)$. Also recall the following known result (a proof can be seen in [2], page 65).

Lemma 1.1. (A.V. Arhangel'skii) *Let Z be a topological space and let $\{X_s : s \in S\}$ be a family of spaces such that $nw(X_s) = \omega$ for each $s \in S$, and let X be the product of this family. If $C \subseteq X$ is dense and $f : C \longrightarrow Z$ is a continuous function, then there exists a set $S_0 \in [S]^{\leq w(Z)}$ and a continuous function $\varphi : \pi_{S_0}[C] \longrightarrow Z$ such that $f = \varphi \circ \pi_{S_0}$.*

The following result gives us sufficient conditions under which a Σ_κ-product is C-embedded in its product; part (1) is a consequence of Lemma 1.1 and part (2) is already known (see [11], Problem 4R).

Lemma 1.2. *Let $X = \prod_{s\in S} X_s$ and let κ be a cardinal number such that $\omega < \kappa \leq |S|$.*

(1) *If for each $s \in S$, $nw(X_s) = \omega$, then every Σ_κ-product in X is C-embedded in X.*
(2) *If for each $s \in S$, X_s is compact, then every Σ_κ-product in X is C-embedded in X.*

Now, we introduce the concept of λ-extendability.

Definition 1.3. Let λ be an infinite cardinal and P a topological property. We say that P is *λ-extendable* if in order for a space X to have P it is sufficient that each $A \in [X]^{<\lambda}$ be contained in some subspace $Y \subseteq X$ satisfying P.

It happens that if $\lambda \leq \kappa$ and P is λ-extendable, then P is κ-extendable.

Recall that a topological property P is *closed hereditary* if every closed subset of a space with P has P. P is *finite productive* (resp., productive) if the product of a finite collection (resp., an arbitrary collection) of spaces satisfying P, has P.

Also, recall the following definitions.

Definition 1.4. Let X be a topological space, α an infinite cardinal number and $U(\alpha)$ the set of uniform ultrafilters on α.

(1) Let $(x_\xi)_{\xi<\alpha}$ be a transfinite sequence in X and $p \in U(\alpha)$. A point $z \in X$ is called *p-limit* of the α-sequence $(x_\xi)_{\xi<\alpha}$ if for each neighborhood W of z, $\{\xi < \alpha : x_\xi \in W\} \in p$.
(2) Let p be a free ultrafilter on $\mathbb{N}$. X is *p-compact* if every sequence in X has a p-limit point.

Definition 1.5. Let X be a space and α an infinite cardinal number.

(1) X is initially α-compact if every open cover of X of cardinality $\leq \alpha$ contains a finite subcover.
(2) For each infinite cardinal number $\theta \leq \alpha$, let $p_\theta \in U(\theta)$. We say that X is *$\{p_\theta : \theta \leq \alpha\}$-compact* if for each $\theta \leq \alpha$ and each θ-sequence $(x_\xi)_{\xi<\theta}$ of points in X, $(x_\xi)_{\xi<\theta}$ has a p_θ-limit.
(3) X is *α-bounded* if every subset of cardinality $\leq \alpha$ of X is contained in a compact subset of X.

The concepts given in Definitions 1.4 and 1.5 were introduced and studied in [3] and [17]. It is known that for every infinite cardinal number α, every α-bounded space is $\{p_\theta : \theta \leq \alpha\}$-compact, and every $\{p_\theta : \theta \leq \alpha\}$-compact space is initially α-compact. Also, all these properties are inherited by closed subsets and they are invariant under continuous functions. Furthermore, p-compactness, $\{p_\theta : \theta \leq \alpha\}$-compactness and α-boundedness are productive properties, and the product of an initially α-compact space with a compact space is initially α-compact.

It is worth mentioning that a space X is initially α-compact if and only if every subset of X of cardinality $\leq \alpha$ has a complete accumulation point.

If for every cardinal number α, α-P is a topological property, we say that X is $<\alpha$-P if X is τ-P for all $\tau < \alpha$; and X is strictly $<\alpha$-P if it is $<\alpha$-P and it is not α-P.

Theorem 1.6. *Let α be an infinite cardinal, and for each infinite cardinal $\theta \leq \alpha$ let $p_\theta \in U(\theta)$. Then the properties of being initially α-compact, $\{p_\theta : \theta \leq \alpha\}$-compact and α-bounded are α^+-extendable.*

Proof. Let X be a space and $P =$ initial α-compactness (resp., $P = \{p_\theta : \theta \leq \alpha\}$-compactness). Let $A \subseteq X$ such that $|A| \leq \alpha$ (resp., let $\theta \leq \alpha$ and $(x_\xi)_{\xi<\theta}$ be a θ-sequence of points in X). Let Y be a subset of X with property P such that $A \subseteq Y$. Then there is a point $x \in Y$ such that x is a complete accumulation point of A (resp., x is a p_θ-limit of A) in Y. It is clear that x is a complete accumulation point of A (resp., x is a p_θ-limit of A) in X.

Now assume that $P = \alpha$-boundedness. Let $A \subseteq X$ such that $|A| \leq \alpha$, and let Y be a subset of X with property P such that $A \subseteq Y$. We have that $cl_Y(A)$ is compact, and so $cl_X(A) = cl_Y(A)$ is compact. □

Corollary 1.7. *Let $p \in \omega^*$. The countable compactness, p-compactness and ω-boundedness are ω_1-extendable.*

Lemma 1.8. *Let $X = \prod_{s\in S} X_s$ and $\omega < \kappa \leq |S|$. If $\tau = cof(\kappa)$, then for every $p \in X$, $\Sigma_\kappa(p)$ is strictly $<\tau$-closed in X.*

Proof. First we will show that $\Sigma_\kappa(p)$ is $<\tau$-closed. Let $A \in [\Sigma_\kappa(p)]^{<\tau}$ and $supp_p(A) = \cup_{a\in A} supp_p(a)$. It is clear that for each $a \in A$, $|supp_p(a)| < \kappa$, thus $|supp_p(A)| < \kappa$; this implies that

$$Y = \{y \in X : supp_p(y) \subseteq supp_p(A)\} \subseteq \Sigma_\kappa(p).$$

Furthermore, Y is closed in X and $A \subseteq Y$; hence, $Cl_X(A) \subseteq Y \subseteq \Sigma_\kappa(p)$ and so $\Sigma_\kappa(p)$ is $<\tau$-closed.

Now, we will see that $\Sigma_\kappa(p)$ is not τ-closed. Take $S' \in [S]^\kappa$ and $q \in X$ such that $supp_p(q) = S'$; also, let $(S_\xi)_{\xi<\tau}$ be a strictly increasing sequence of subsets of S such that $S' = \cup_{\xi<\tau} S_\xi$ and for each $\xi < \tau$, $|S_\xi| < \kappa$. Furthermore, for each $\xi < \tau$, let $x_\xi \in X$ be such that $\pi_{S_\xi}(x_\xi) = \pi_{S_\xi}(q)$ and $\pi_{S\setminus S_\xi}(x_\xi) = \pi_{S\setminus S_\xi}(p)$.

Since for each $\xi < \tau$, $|supp_p(x_\xi)| = |S_\xi| < \kappa$, then $B = \{x_\xi : \xi < \tau\} \subseteq \Sigma_\kappa(p)$. Moreover, it is clear that $|B| = \tau$ and $q \in Cl_X(B) \setminus \Sigma_\kappa(p)$. □

Theorem 1.9. *Let $X = \prod_{s\in S} X_s$. Let λ and κ be two cardinal numbers such that $\kappa \geq \omega_1$ and $\lambda \leq cof(\kappa) \leq \kappa \leq |S|$. Let P be a λ-extendable and closed hereditary property. If X has P, then $\Sigma_\kappa(x)$ has property P for each $x \in X$.*

Proof. Let $A \subseteq \Sigma_\kappa(x)$ be such that $|A| < \lambda$. Then $|A| < cof(\kappa)$ and, by Lemma 1.8, $Cl_X(A) \subseteq \Sigma_\kappa(x)$. Since P is hereditary with respect to closed subsets, $Cl_X(A)$ has P and since P is λ-extendable and $A \subseteq Cl_X(A) \subseteq \Sigma_\kappa(x)$, we obtain that $\Sigma_\kappa(x)$ has property P. □

The next lemma is well known. For a proof of the case $\kappa = \omega_1$, see Proposition 7.2 of [12].

Lemma 1.10. *Let $X = \prod_{s\in S} X_s$ and κ be a cardinal such that $\omega \leq \kappa \leq |S|$. Let $\tau = cof(\kappa)$ and suppose that $|X_s| > 1$ for every $s \in S$. Then, for each $p \in X$ there exists a closed copy of $[0, \tau)$ in $\Sigma_\kappa(p)$.*

In [16], V. Saks and R. M. Stephenson, Jr. stated that if $X = \prod_{s\in S} X_s$ is a product of compact spaces, each of them with at least two points, then for every uncountable cardinal κ and every $p \in X$, $\Sigma_{\kappa^+}(p)$ is initially κ-compact and it is not initially κ^+-compact. We complete this result with the following theorem.

Theorem 1.11. *Let $X = \prod_{s\in S} X_s$ and κ be a cardinal number such that $\omega < \kappa \leq |S|$. Suppose that $|X_s| > 1$ for every $s \in S$ and assume that X is initially $<\tau$-compact (resp., $\{p_\theta : p_\theta \in U_\theta, \theta < \tau\}$-compact, $<\tau$-bounded) where $\tau = cof(\kappa)$. Then for each $p \in X$, $\Sigma_\kappa(p)$ is initially $<\tau$-compact (resp., $\{p_\theta : p_\theta \in U_\theta, \theta < \tau\}$-compact, $<\tau$-bounded) and it is not initially τ-compact.*

Proof. By Corollary 1.9, $\Sigma_\kappa(p)$ is initially $<\tau$-compact ($\{p_\theta : p_\theta \in U_\theta,\ \theta < \tau\}$-compact, $<\tau$-bounded) because these properties are inherited by closed subsets.

On the other hand, by Lemma 1.10, $\Sigma_\kappa(p)$ is not initially τ-compact because the space $[0, \tau)$ is not initially τ-compact. □

Recall that a space Z is τ-paracompact if every open cover of cardinality τ has an open refinement which is locally finite in Z; Z is $<\kappa$-paracompact if it is τ-paracompact for all $\tau < \kappa$. A space Z is strictly τ-paracompact if it is τ-paracompact and it is not τ^+-paracompact. Z is strictly $<\kappa$-paracompact if it is $<\kappa$-paracompact and it is not κ-paracompact. Note that every initially α-compact space is α-paracompact; so, for every product $X = \prod_{s\in S} X_s$ of α-bounded spaces, every $a \in X$ and every regular cardinal number $\kappa \leq \alpha$, $\Sigma_\kappa(a)$ is $<\kappa$-paracompact. By Theorem 1.11 and Lemma 1.10, we obtain the following corollary.

Corollary 1.12. *Let $X = \prod_{s\in S} X_s$ where for each $s \in S$, X_s is a compact space with more than one point. Then, if $|S|$ is regular and uncountable, for every $p \in X$, $\Sigma_{|S|}(p)$ is strictly $<|S|$-paracompact.*

Lemma 1.13. ([12], Corollary 3.4) *The following conditions are equivalent for a space Y:*

(1) *Y is paracompact,*
(2) *$Y \times \beta Y$ is normal, and*
(3) *$Y \times \alpha Y$ is normal for some (every) compactification αY of Y.*

Corollary 1.14. *Let $X = \prod_{s\in S} X_s$ be a product of compact spaces with more than one point. Then, for every $p \in X$, and every κ with $\omega < \kappa \leq |S|$, $\Sigma_\kappa(p, X) \times X$ is not normal.*

Proof. We know that $\Sigma_\kappa(p, X)$ is dense and C^*-embedded in X, so $X = \beta\Sigma_\kappa(p, X)$. If $\Sigma_\kappa(p, X) \times X$ were normal, by Lemma 1.13, $\Sigma_\kappa(p, X)$ would be paracompact, which is a contradiction. So $\Sigma_\kappa(p, X) \times X$ is not normal. □

Our first important result concerning normality (see Theorem 1.19, below) generalizes the known fact that Σ-products of separable metric spaces are normal. Our proof is based on that given by A.P. Kombarov and V.I. Malyhin to Theorem 0.1. First, we establish some previous results. We denote by $hd(X)$ the hereditary density of a space X.

Lemma 1.15. (Theorem 1.7, [6]) *Let α be an infinite cardinal. Then, every α-pseudocompact, $<\alpha$-bounded normal space is initially α-compact.*

Theorem 1.16. *Let $X = \{0, 1\}^S$ and let κ be a singular cardinal such that $\kappa \leq |S|$. Then, for every $p \in X$, $\Sigma_\kappa(p)$ is not normal.*

Proof. Take $\tau = cof(\kappa)$. By Lemma 1.15, it is enough to show that $\Sigma_\kappa(p)$ is τ-pseudocompact, $<\tau$-bounded and it is not initially τ-compact.

Since κ is singular, $\tau < \kappa$, and since τ^+ is regular and $\tau^+ \leq \kappa$, we must have $\tau^+ < \kappa$. Thus, $\Sigma_{\tau^+}(p)$ is a τ-pseudocompact subspace which is dense in $\Sigma_\kappa(p)$; so $\Sigma_\kappa(p)$ is τ-pseudocompact.

Moreover, by Theorem 1.11, $\Sigma_\kappa(p)$ is $<\tau$-bounded and it is not initially τ-compact. □

For a topological space T and an element $a \in T$, we denote by $\underline{a}$ the point belonging to T^S such that for each $s \in S$ $\pi_s(\underline{a}) = a$. The following theorem is a consequence of some remarks made to the authors by O. Okunev.

Theorem 1.17. *If $X = \prod_{s\in S} X_s$ and κ is a singular cardinal with $\kappa \leq |S|$, then for every $p \in X$, $\Sigma_\kappa(p)$ is not normal.*

Proof. Let $p, q \in X$ such that for all $s \in S$, $\pi_s(p) \neq \pi_s(q)$. Let $f : \{0, 1\}^S \longrightarrow X$ such that for each $x \in \{0, 1\}^S$ and each $s \in S$, $\pi_s(f(x)) = \pi_s(p)$ if $\pi_s(x) = 0$ and $\pi_s(f(x)) = \pi_s(q)$ otherwise. The function f is injective. Furthermore, for each $s \in S$, $\pi_s \circ f$ is continuous; so f is a continuous function. Since $\{0, 1\}^S$ is compact, f is a closed embedding of $\{0, 1\}^S$ in X.

Claim: $f^{-1}[\Sigma_\kappa(p)] = \Sigma_\kappa(\underline{0})$.

Indeed, it is clear that $f(\underline{0}) = p$. Thus, $y \in \Sigma_\kappa(\underline{0})$ if and only if $|supp_{\underline{0}}(y)| < \kappa$, if and only if $|\{s \in S : \pi_s(y) = 1\}| < \kappa$ and for all $s \in (S \setminus supp_{\underline{0}}(y))$, $\pi_s(y) = 0$, if and only if $|\{s \in S : \pi_s(f(y)) = \pi_s(q)\}| < \kappa$ and for all $s \in (S \setminus supp_{\underline{0}}(y))$, $\pi_s(f(y)) = \pi_s(p)$, if and only if $|supp_p(f(y))| < \kappa$, if and only if $f(y) \in \Sigma_\kappa(p)$. In this manner we have finished the proof of the Claim.

By Theorem 1.16, $\Sigma_\kappa(\underline{0})$ is not normal, so $\Sigma_\kappa(p) \cap f[\{0,1\}^S] = f[\Sigma_\kappa(\underline{0})]$ is not normal, and since $f[\{0,1\}^S]$ is closed in X, $\Sigma_\kappa(p) \cap f[\{0,1\}^S]$ is closed in $\Sigma_\kappa(p)$ and so $\Sigma_\kappa(p)$ is not normal. □

Lemma 1.18. *For $X = \prod_{s \in S} X_s$ it happens that*

$$hd(X) = max\{|S|, sup_{F \in [S]^{<\omega}} hd(X_F)\}.$$

Recall that if Y is a metric space, then $hd(Y) = d(Y)$.

Theorem 1.19. *Let $\{X_s : s \in S\}$ be a family of spaces and κ a cardinal number such that $\omega < \kappa \leq |S|$ and for each $F \in [S]^{<\omega}$, $hd(X_F) < \kappa$. Then, for every $p \in X$, $\Sigma_\kappa(p)$ is normal if and only if κ is regular and for each $F \in [S]^{<\kappa}$, X_F is normal.*

Proof. ($\Rightarrow$) It is clear that for each $F \in [S]^{<\kappa}$, X_F is homeomorphic to a closed subspace of $\Sigma_\kappa(p)$. Then, if $\Sigma_\kappa(p)$ is normal, X_F is normal too. On the other hand, if κ is singular, by Theorem 1.17, $\Sigma_\kappa(p)$ is not normal.

($\Leftarrow$) Let A, B be two disjoint closed subsets of $\Sigma_\kappa(p)$. We will construct increasing sequences

$$\mathcal{A} = \{A_n \in [A]^{<\kappa} : n \in \mathbb{N}\}, \quad \mathcal{B} = \{B_n \in [B]^{<\kappa} : n \in \mathbb{N}\},$$
$$\mathcal{H} = \{H_n \in [S]^{<\kappa} : n \in \mathbb{N}\}$$

such that for each $n \in \mathbb{N}$,

$$\pi_{H_n}[A] \subseteq Cl_{X_{H_n}}(\pi_{H_n}[A_{n+1}]), \quad \pi_{H_n}[B] \subseteq Cl_{X_{H_n}}(\pi_{H_n}[B_{n+1}])$$

and

$$supp_p(A_n) \cup supp_p(B_n) \subseteq H_{n+1}.$$

Take $s_0 \in S$ and put $H_0 = \{s_0\}$. Observe that, for each $H \in [S]^{<\kappa}$, either $|[H]^{<\omega}| < \omega$ or $|[H]^{<\omega}| = |H|$, and since κ is regular, by Lemma 1.18, for each $H \in [S]^{<\kappa}$,

$$hd(X_H) = max\{|H|, sup_{F \in [H]^{<\omega}} hd(X_F)\} < \kappa.$$

In particular, $hd(X_{s_0}) < \kappa$; so, there exist sets

$$C_1 \in [\pi_{s_0}[A]]^{<\kappa} \quad \text{and} \quad D_1 \in [\pi_{s_0}[B]]^{<\kappa}$$

which are dense. Let

$$A_1 \in [A]^{<\kappa} \quad \text{and} \quad B_1 \in [B]^{<\kappa}$$

such that

$$\pi_{s_0}[A_1] = C_1 \quad \text{and} \quad \pi_{s_0}[B_1] = D_1.$$

It is clear that

$$\pi_{s_0}[A] \subseteq Cl_{X_{s_0}}(\pi_{s_0}[A_1]) \quad \text{and} \quad \pi_{s_0}[B] \subseteq Cl_{X_{s_0}}(\pi_{s_0}[B_1]).$$

Finally, we set

$$H_1 = supp_p(A_1) \cup supp_p(B_1) \cup \{s_0\}.$$

Note that

$$|A_1| < \kappa, \quad |B_1| < \kappa, \quad \text{and} \quad |H_1| < \kappa$$

because κ is regular.

Assume that for each $n \in \mathbb{N}$ we have already constructed the sets A_n, B_n and H_n which satisfy the required conditions. Since $|H_n| < \kappa$ and $hd(X_{H_n}) < \kappa$, we can find sets

$$C_{n+1} \in [\pi_{H_n}[A]]^{<\kappa} \quad \text{and} \quad D_{n+1} \in [\pi_{H_n}[B]]^{<\kappa}$$

such that

$$\pi_{H_n}[A] \subseteq Cl_{X_{H_n}}(C_{n+1}) \quad \text{and} \quad \pi_{H_n}[B] \subseteq Cl_{X_{H_n}}(D_{n+1}).$$

Let

$$A^*_{n+1} \in [A]^{<\kappa} \quad \text{and} \quad B^*_{n+1} \in [B]^{<\kappa}$$

such that

$$\pi_{H_n}[A^*_{n+1}] = C_{n+1} \quad \text{and} \quad \pi_{H_n}[B^*_{n+1}] = D_{n+1};$$

and put

$$A_{n+1} = A_n \cup A^*_{n+1} \quad B_{n+1} = B_n \cup B^*_{n+1}$$

and

$$H_{n+1} = H_n \cup supp_p(A_{n+1}) \cup supp_p(B_{n+1}).$$

It is clear that this concludes our construction and that each family satisfies the above mentioned conditions.

Let

$$S' = \bigcup \mathcal{H}, \quad A' = \bigcup \mathcal{A} \quad \text{and} \quad B' = \bigcup \mathcal{B}.$$

Again, because κ is an uncountable regular cardinal,

$$|A'| < \kappa, \quad |B'| < \kappa, \quad \text{and} \quad |S'| < \kappa.$$

Claim 1: $\pi_{S'}[A] \subseteq Cl_{X_{S'}}(\pi_{S'}[A'])$ and $\pi_{S'}[B] \subseteq Cl_{X_{S'}}(\pi_{S'}[B'])$.

Let $z \in A$ and $V = \cap_{s\in F}\pi_s^{-1}[V_s]$, where $F \in [S']^{<\omega}$, $\pi_{S'}(z) \in V$ and each V_s is open in X_s. Since F is finite, there exists $n \in \mathbb{N}$ such that $F \subseteq H_n$. Since $z \in A$, we have that

$$\pi_{H_n}(z) \in Cl_{X_{H_n}}(\pi_{H_n}[A_{n+1}]);$$

so

$$\pi_{H_n}[V] \cap \pi_{H_n}[A_{n+1}] \neq \emptyset.$$

Thus, there is $a \in A_{n+1} \subseteq A'$ such that $\pi_s(\pi_{H_n}(a)) \in V_s$ for each $s \in F$. In this way we obtain that $\pi_{S'}(a) \in V \cap \pi_{S'}[A']$; that is,

$$\pi_{S'}(z) \in Cl_{X_{S'}}(\pi_{S'}[A]).$$

In an analogous way we obtain

$$\pi_{S'}[B] \subseteq Cl_{X_{S'}}(\pi_{S'}[B']),$$

finishing the proof of Claim 1.

Claim 2: $Cl_{X_{S'}}(\pi_{S'}[A'])$ and $Cl_{X_{S'}}(\pi_{S'}[B'])$ are disjoint in $X_{S'}$.

Suppose that there exists a point

$$z \in (Cl_{X_{S'}}(\pi_{S'}[A']) \cap Cl_{X_{S'}}(\pi_{S'}[B'])).$$

Define a point $y \in X$ such that $\pi_s(y) = \pi_s(z)$ if $s \in S'$ and $\pi_s(y) = \pi_s(p)$ if $s \notin S'$.

Let

$$V = \cap_{s\in F}\pi_s^{-1}[V_s]$$

where $F \in [S]^{<\omega}$, $y \in V$ and each V_s is open in X_s; and take $L = F \cap S'$ and $M = F \setminus S'$.

Note that for each $a \in A'$, $b \in B'$ and $s \in (S \setminus S')$,

$$\pi_s(a) = \pi_s(b) = \pi_s(p) = \pi_s(y).$$

Then, if $L = \emptyset$, $A' \cup B' \subseteq V$ and

$$V \cap A' \neq \emptyset \neq V \cap B'.$$

Assume that $L \neq \emptyset$. Since $\pi_s(y) = \pi_s(z)$ for every $s \in S'$, we have that $z \in \pi_{S'}[V]$; and since

$$z \in Cl_{X_{S'}}(\pi_{S'}[A'])$$

it happens that

$$\pi_{S'}[V] \cap \pi_{S'}[A'] \neq \emptyset.$$

In a similar way

$$\pi_{S'}[V] \cap \pi_{S'}[B'] \neq \emptyset.$$

Take $a \in A'$, $b \in B'$ such that

$$\pi_{S'}(a) \in \pi_{S'}[V] \cap \pi_{S'}[A'] \quad \text{and} \quad \pi_{S'}(b) \in \pi_{S'}[V] \cap \pi_{S'}[B'].$$

Then, for each $s \in L$, $\pi_s(a)$, $\pi_s(b) \in V_s$. Moreover,

$$supp_p(a) \cup supp_p(b) \subseteq S';$$

then, for each $s \in M \subseteq (S \setminus S')$,

$$\pi_s(a) = \pi_s(b) = \pi_s(p) = \pi_s(y);$$

this implies that a, $b \in V$, and we again obtain

$$V \cap A' \neq \emptyset \neq V \cap B'.$$

In any case,

$$y \in Cl(A') \cap Cl(B')$$

and since $|S'| < \kappa$, $y \in \Sigma_\kappa(p)$. Then

$$y \in Cl_{\Sigma_\kappa(p)}(A') \cap Cl_{\Sigma_\kappa(p)}(B') \subseteq A \cap B.$$

So, we conclude the proof of Claim 2.

Because of the relation $|S'| < \kappa$, $X_{S'}$ is normal. By Claim 2, there exist two disjoint open subsets U, V of $X_{S'}$ such that

$$Cl_{X_{S'}}(\pi_{S'}[A']) \subseteq U \quad \text{and} \quad Cl_{X_{S'}}(\pi_{S'}[B']) \subseteq V.$$

By Claim 1,

$$\pi_{S'}[A] \subseteq Cl_{X_{S'}}(\pi_{S'}[A']) \subseteq U \ \text{ and } \ \pi_{S'}[B] \subseteq Cl_{X_{S'}}(\pi_{S'}[B']) \subseteq V.$$

Therefore, $A \subseteq \pi_{S'}^{-1}[U]$ and $B \subseteq \pi_{S'}^{-1}[V]$. It is apparent that $\pi_{S'}^{-1}[U]$ and $\pi_{S'}^{-1}[V]$ are disjoint open subsets of X, so we conclude our proof. □

Remark 1.20. Let α be an infinite cardinal and $\{X_s : s \in S\}$ a family of metric spaces such that $\alpha < |S|$ and $d(X_s) \leq \alpha$ for each $s \in S$. Then, by Corollary 6.5 of [12], for every regular cardinal κ which satisfies $max\{\omega_1, \alpha\} < \kappa \leq |S|$ and every $p \in X$, $\Sigma_\kappa(p)$ is normal if and only if all, except perhaps a countable subcollection of the spaces X_s, are compact.

2. λ-EXTENDABLE PROPERTIES IN $\Sigma_\Gamma(F)$-PRODUCTS

Definition 2.1. Let $\{X_s : s \in S\}$ be a family of spaces and let X be the topological product of this collection. Let $F = \{x_i : i \in J\}$ be a set of elements in X and $\Gamma = \{\gamma_i : i \in J\}$ a set of infinite cardinal numbers.

(1) We define the $\Sigma_\Gamma(F)$-product of X as

$$\Sigma_\Gamma(F) = \cup_{i \in J} \Sigma_{\gamma_i}(x_i).$$

(2) Given a $\Sigma_\Gamma(F)$-product in X, we will say that

$$F^* = \{y_k : k \in J^*\} \subseteq X \text{ and } \Gamma^* = \{\gamma_k : k \in J^*\}$$

depurate $\Sigma_\Gamma(F)$ if

$$\Sigma_\Gamma(F) = \Sigma_{\Gamma^*}(F^*)$$

and for each pair y_i, $y_j \in F^*$ it happens that $i \neq j$ if and only if $\Sigma_{\gamma_i}(y_i) \cap \Sigma_{\gamma_j}(y_j) = \emptyset$.

(3) We will say that $\Sigma_\Gamma(F)$ is depurated if F and Γ depurate $\Sigma_\Gamma(F)$.

Note that if in the previous definition $\Gamma = \{\kappa\}$ then, for each $i \in J$, $\gamma_i = \kappa$; in this situation, we will write $\Sigma_\kappa(F)$ instead of $\Sigma_{\{\kappa\}}(F)$.

Proposition 2.2. *Let $\{X_s : s \in S\}$ be a family of topological spaces and let X be the product of such a family. Moreover, let $\Gamma = \{\gamma_i : i \in J\}$ be a collection of infinite cardinals and $F = \{x_i : i \in J\} \subseteq X$. If $\kappa = min(\Gamma)$, then there exists $F^* \subseteq X$ such that $\Sigma_\Gamma(F) = \Sigma_\kappa(F^*)$.*

Proof. Let $F^* = \Sigma_\Gamma(F)$. It is clear that $\Sigma_\Gamma(F) \subseteq \Sigma_\kappa(F^*)$. Besides, for each $x \in F^*$, there is $i \in J$ such that $x \in \Sigma_{\gamma_i}(x_i)$. Since $\kappa \leq \gamma_i$,

$$\Sigma_\kappa(x) \subseteq \Sigma_{\gamma_i}(x_i) \subseteq \Sigma_\Gamma(F).$$

Thus, $\Sigma_\kappa(F^*) \subseteq \Sigma_\Gamma(F)$. □

Definition 2.3. Let $\{X_s : s \in S\}$ be a family of spaces and $X = \prod_{s\in S} X_s$. Let $\Gamma = \{\gamma_i : i \in J\}$ be a set of infinite cardinal numbers and $F = \{x_i : i \in J\} \subseteq X$ such that $\Sigma_\Gamma(F)$ is a proper subset of X. We will say that Γ is *maximal for* $\Sigma_\Gamma(F)$, (or simply *maximal*) if for each $i \in J$,

$$\Sigma_{\gamma_i^+}(x_i) \setminus \Sigma_\Gamma(F) \neq \emptyset.$$

Note that if $\Sigma_\Gamma(F)$ is a proper subset of X, Γ is maximal and $x \in \Sigma_{\gamma_i}(x_i)$ for some $i \in J$, then $\Sigma_{\gamma_i^+}(x) \setminus \Sigma_\Gamma(F) \neq \emptyset$; this is a consequence of the equality $\Sigma_{\gamma_i^+}(x) = \Sigma_{\gamma_i^+}(x_i)$. Moreover, if $\Gamma = \{\kappa\}$ is maximal, then for each cardinal γ it happens that: $\gamma > \kappa$ if and only if for every $x \in \Sigma_\kappa(F)$, $\Sigma_\gamma(x) \setminus \Sigma_\kappa(F) \neq \emptyset$, if and only if there exists $x \in \Sigma_\kappa(F)$ such that $\Sigma_\gamma(x) \setminus \Sigma_\kappa(F) \neq \emptyset$.

Proposition 2.4. *Let $\Gamma = \{\gamma_i : i \in J\}$ be a set of infinite cardinals, $X = \prod_{s\in S} X_s$ and $F = \{x_i : i \in J\} \subseteq X$. Assume that $\Sigma_\Gamma(F)$ is a proper subset of X. For each $i \in J$, define*

$$\gamma_i' = sup\{\gamma \leq |S| : \Sigma_\gamma(x_i) \subseteq \Sigma_\Gamma(F)\}$$

and put

$$\Gamma' = \{\gamma_i' : i \in J\}.$$

Then $\Sigma_{\Gamma'}(F) = \Sigma_\Gamma(F)$ and Γ' is maximal for $\Sigma_{\Gamma'}(F)$.

Proof. For each $i \in J$, $\Sigma_{\gamma_i}(x_i) \subseteq \Sigma_\Gamma(F)$, so for each $i \in J$, $\gamma_i \leq \gamma'_i$. Then, it is obvious that $\Sigma_\Gamma(F) \subseteq \Sigma_{\Gamma'}(F)$.

On the other hand, let $i \in J$ and $x \in \Sigma_{\gamma'_i}(x_i)$. We know that $|supp_x(x_i)| < \gamma'_i$ and because of the definition of γ'_i, there exists a cardinal γ such that $|supp_x(x_i)| < \gamma \leq \gamma'_i$ and $\Sigma_\gamma(x_i) \subseteq \Sigma_\Gamma(F)$. Therefore, $x \in \Sigma_\gamma(x_i)$ and so $\Sigma_{\Gamma'}(F) \subseteq \Sigma_\Gamma(F)$.

In order to finish our proof, if $i \in J$ and $\Sigma_{\gamma_i'^+}(x_i) \subseteq \Sigma_{\Gamma'}(F) = \Sigma_\Gamma(F)$ then, because of the definition of γ'_i, either $\gamma'_i \geq \gamma_i'^+$ or $\gamma_i'^+ > |S|$. Of course it is not possible that $\gamma'_i \geq \gamma_i'^+$, so $\gamma_i'^+ > |S|$, but then $X = \Sigma_{\gamma_i'^+}(x_i) \subseteq \Sigma_\Gamma(F)$ which is a contradiction obtained by assuming that Γ' is not maximal. Therefore, Γ' is maximal. □

Proposition 2.5. *Let $X = \prod_{s\in S} X_s$. For every set $F = \{x_i : i \in J\} \subseteq X$ and every set of infinite cardinals $\Gamma = \{\gamma_i : i \in J\}$ such that $\Sigma_\Gamma(F)$ is a proper subset of X, there exist $F^* \subseteq F$, $J^* \subseteq J$ and $\Gamma^* = \{\gamma'_i : i \in J^*\}$ such that $\Sigma_\Gamma(F) = \Sigma_{\Gamma^*}(F^*)$, $\Sigma_{\Gamma^*}(F^*)$ is depurated and Γ^* is maximal for $\Sigma_{\Gamma^*}(F^*)$.*

Proof. By Proposition 2.4, there exists $\Gamma' = \{\gamma'_i : i \in J\}$ such that $\Sigma_\Gamma(F) = \Sigma_{\Gamma'}(F)$ and Γ' is maximal for $\Sigma_{\Gamma'}(F)$. Since Γ' is maximal, for each i, $j \in J$, if $\Sigma_{\gamma'_i}(x_i) \cap \Sigma_{\gamma'_j}(x_j) \neq \emptyset$ then $\gamma'_i = \gamma'_j$. Indeed, suppose that $\gamma'_i \leq \gamma'_j$. Since $\Sigma_{\gamma'_i}(x_i) \cap \Sigma_{\gamma'_j}(x_j) \neq \emptyset$ then $\Sigma_{\gamma'_i}(x_i) \subseteq \Sigma_{\gamma'_j}(x_j)$. But Γ' is maximal, so we must have $\gamma'_j \leq \gamma'_i$.

Therefore, if for each $\alpha \in \Gamma'$ we take $J_\alpha = \{i \in J : \gamma_i = \alpha\}$, and if J^* is a subset of J constituted by exactly one element of each J_α with $\alpha \in \Gamma'$, then
the sets $F^* = \{x_i : i \in J^*\}$ and $\Gamma^* = \{\gamma_i : i \in J^*\}$ satisfy the requested condi- tions. □

We now begin with some results about λ-extendable properties in $\Sigma_\Gamma(F)$-products.

Theorem 2.6. *Let $X = \prod_{s\in S} X_s$, $\omega \leq \lambda \leq \mathrm{cof}(\kappa) \leq \kappa \leq |S|$ with $\kappa \geq \omega_1$ and $F = \{x_i : i \in J\} \subseteq X$. Assume that X satisfies a λ-extendable property P which is invariant under continuous functions and finite productive. Then, $\Sigma_\kappa(F)$ has property P if and only if there is a set $F^* \subseteq X$ such that F^* satisfies P and*

$$\Sigma_\kappa(F) = \Sigma_\kappa(F^*).$$

Proof. ($\Leftarrow$) Without loss of generality, we can assume that F satisfies P. Since P is λ-extendable, it is enough to show that for each $A \in [\Sigma_\kappa(F)]^{<\lambda}$ there is a subspace Y of $\Sigma_\kappa(F)$ such that $A \subseteq Y$ and Y satisfy P.

Let $A \in [\Sigma_\kappa(F)]^{<\lambda}$, and for each $a \in A$ pick $i(a) \in J$ such that $a \in \Sigma_\kappa(x_{i(a)})$. Take $s(A) = \cup_{a\in A} supp_a(x_{i(a)})$. Note that for each $a \in A$, $|supp_a(x_{i(a)})| < \kappa$; so $|s(A)| < \kappa$.

Define $X_{s(A)} = \pi_{s(A)}[X]$ and $F_{S\setminus s(A)} = \pi_{S\setminus s(A)}[F]$, and put

$$Y = \{g \in X : \exists\ f \in F \text{ with } g \restriction_{S\setminus s(A)} = f \restriction_{S\setminus s(A)}\}.$$

$X_{s(A)}$ and $F_{S\setminus s(A)}$ have property P because this property is invariant under continuous functions. Then, $X_{s(A)} \times F_{S\setminus s(A)}$ has property P. Since Y is homeomorphic to $X_{s(A)} \times F_{S\setminus s(A)}$, we conclude that Y satisfies property P.

It is clear that $A \subseteq Y$ because if $a \in A$ and $supp_a(x_{i(a)}) \subseteq s(A)$, then for every $s \in S \setminus s(A)$, $\pi_s(x_{i(a)}) = \pi_s(a)$ and so $a \in Y$.

Moreover, if $y \in Y$, $\pi_{S\setminus s(A)}(y) \in F_{S\setminus s(A)}$, so there is $x \in F$ such that $\pi_{S\setminus s(A)}(y) = \pi_{S\setminus s(A)}(x)$. This implies that $supp_x(y) \subseteq s(A)$. Since $|s(A)| < \kappa$, $y \in \Sigma_\kappa(x) \subseteq \Sigma_\kappa(F)$. So, we conclude that $\Sigma_\kappa(F)$ has property P.

($\Rightarrow$) Now, suppose that $\Sigma_\kappa(F)$ has P. The proof is immediate because

$$\Sigma_\kappa(F) = \Sigma_\kappa(\Sigma_\kappa(F)),$$

so it is enough to take $F^* = \Sigma_\kappa(F)$. □

Corollary 2.7. *Under the same conditions of Theorem 2.6, if P is closed hereditary and F is closed in $\Sigma_\kappa(F)$, then $\Sigma_\kappa(F)$ satisfies property P if and only if F satisfies property P.*

Theorem 2.8. *Let $X = \prod_{s\in S} X_s$, $\omega \le \lambda \le \text{cof}(\kappa) \le \kappa \le |S|$ with $\kappa \ge \omega_1$ and $F = \{x_i : i \in J\} \subseteq X$. Assume that X is compact and P is a property which is invariant under continuous functions, λ-extendable and such that the product of a compact space with a space having P has P. Then, $\Sigma_\kappa(F)$ has property P if and only if there is a set $F^* \subseteq X$ such that F^* has property P and*

$$\Sigma_\kappa(F) = \Sigma_\kappa(F^*).$$

Furthermore, if P is closed hereditary and F is a closed subset of $\Sigma_\kappa(F)$, then $\Sigma_\kappa(F)$ satisfies P if and only if F satisfies P.

Proof. Take A and Y as in the proof of Theorem 2.6 and note that it is enough to show that Y has property P. Observe that $F_{S\setminus s(A)} = \pi_{S\setminus s(A)}[F]$ satisfies P because F has P and this property is invariant under every projection. Moreover, $X_{s(A)} = \pi_{s(A)}[X]$ is compact, so $X_{s(A)} \times F_{S\setminus s(A)}$ has property P. We conclude that Y has property P. The rest of the proof is similar to that given for Theorem 2.6. The second part of the Theorem is obvious. □

Recall that α-boundedness and p-compactness satisfy the hypotheses of Theorem 2.6, and initial α-compactness and countable compactness satisfy the hypotheses of Theorem 2.8. So, we have:

Corollary 2.9. *Let $X = \prod_{s\in S} X_s$, $\omega \leq \alpha < \mathrm{cof}(\kappa) \leq \kappa \leq |S|$ and $F = \{x_i : i \in J\} \subseteq X$.*

(1) *If X_s is α-bounded ($\{p_\theta : \theta \leq \alpha\}$-compact), then $\Sigma_\kappa(F)$ is α-bounded ($\{p_\theta : \theta \leq \alpha\}$-compact) if and only if there exists a set $F^* \subseteq X$ such that F^* is α-bounded ($\{p_\theta : \theta \leq \alpha\}$-compact) and*

$$\Sigma_\kappa(F) = \Sigma_\kappa(F^*).$$

(2) *Assume that X is compact. Then $\Sigma_\kappa(F)$ is initially α-compact if and only if there is a set $F^* \subseteq X$ such that F^* is initially α-compact and*

$$\Sigma_\kappa(F) = \Sigma_\kappa(F^*).$$

Examples 2.10. (1) Let α and τ be infinite cardinals. Then, for every finite set $F \subseteq [0, \alpha^+)^\tau = X$, the space $\Sigma_{\alpha^+}(F, X)$ is α-bounded because $[0, \alpha^+)$ is α-bounded and because of (1) in Corollary 2.9.

(2) Let S be an infinite set such that $\omega < |S|$, and let $\{S_n : n \in \mathbb{N}\}$ be a partition of S such that for each $n \in \mathbb{N}$, $|S_n| = |S|$. Let $p \in \mathbb{N}^*$ and $\beta_p(\mathbb{N})$ be the maximal p-compact extension of $\mathbb{N}$ (see [7]). For each $n \in \mathbb{N}$, take $x_n \in \beta_p(\mathbb{N})^S = X$ such that $\pi_s(x_n) = 1$ if $s \in \cup_{m\leq n} S_m$ and $\pi_s(x_n) = 2$ if $s \in (S \setminus \cup_{m\leq n} S_m)$. Take $F = \{\underline{1}\} \cup \{x_n : n \in \mathbb{N}\}$. Since $\beta_p(\mathbb{N})$ is p-compact and F is compact, for each regular cardinal $\kappa > \omega$, $\Sigma_\kappa(F, X)$ is p-compact ((1) in Corollary 2.9).

(3) Given a Tychonoff space Y, we can assume that $Y \subseteq [0, 1]^{w(Y)}$ where $w(Y)$ is the weight of Y. Given a cardinal $\kappa \leq w(Y)$, we define the space $\Sigma_\kappa(Y) = \Sigma_\kappa(Y, [0, 1]^{w(Y)})$. Then, by Corollary 2.9.(2), for every infinite cardinal $\alpha < \mathrm{cof}(\kappa)$ and every initially α-compact space Y, $\Sigma_\kappa(Y)$ is initially α-compact.

Corollary 2.11. *Let T be a space, τ an uncountable cardinal, $X = T^\tau$ and $F = \{\underline{x}_i : i \in J\} \subseteq X$ a set of constant functions. Then, for each regular cardinal κ with $\omega < \kappa \leq \tau$ and every κ-extendable property P, closed hereditary, invariant under continuous functions and finite productive, the following assertions are equivalent:*

(1) *$\Sigma_\kappa(F)$ satisfies property P,*
(2) *F satisfies property P and*
(3) *$F_0 = \{x \in T : \underline{x} \in F\}$ satisfies property P.*

Proof. (1) $\Leftrightarrow$ (2). Take $\Delta = \{\underline{z} : z \in T\}$. It happens that $F = \Delta \cap \Sigma_\kappa(F)$. Since Δ is closed in X, the result follows from Corollary 2.7.

(2) $\Leftrightarrow$ (3). This is immediate because F_0 and F are homeomorphic (see Corollary 2.3.21 in [5]) □

Corollary 2.12. *Let $X = I^\tau$. Let F_0 be a closed subset of I, and put $F = \{\underline{x} : x \in F_0\}$. Then, for each regular cardinal κ with $\omega < \kappa \leq \tau$, $\Sigma_\kappa(F)$ is $<\kappa$-bounded.*

It is possible to have a countably compact $\Sigma_\kappa(F)$-product where F is not countably compact. Indeed:

Proposition 2.13. *Let $\{X_s : s \in S\}$ be a family of topological spaces, X the product of such a family and κ a cardinal such that $\omega < \kappa \leq |S|$. Assume that at least the space X_{s_0} is infinite for a $s_0 \in S$. Let $F \subseteq X$ be an infinite set such that $\Sigma_\kappa(F, X)$ is depurated. Then, there is a set $F^* \subseteq X$ which depurates $\Sigma_\kappa(F, X)$ and it is not countably compact.*

Proof. Let $\{x_n\}_{n<\omega}$ be a discrete countable subset of X_{s_0} and let $\{a_n\}_{n<\omega}$ be a countable subset of F. For each $a \in F$, take $b(a) \in X$ such that

$$\pi_{S\setminus\{s_0\}}(b(a)) = \pi_{S\setminus\{s_0\}}(a),$$

$$\pi_{s_0}(b(a_n)) = x_n$$

for each $n < \omega$ and

$$\pi_{s_0}(b(a)) = x_0$$

for each $a \in F \setminus \{a_n : 0 < n < \omega\}$.

Consider the set $F^* = \{b(a) \in X : a \in F\}$. It is clear that F^* depurates $\Sigma_\kappa(F, X)$. Now, we are going to prove that the sequence $\{b(a_n) : n < \omega\} \subseteq F^*$ is closed and discrete in F^*. We know that for each $n < \omega$, there is an open subset V_n of X_{s_0} such that $V_n \cap \{x_n : n < \omega\} = \{x_n\}$. By definition, for each $b(a) \in F^*$ there is a unique $k < \omega$ such that $b(a) \in \pi_{s_0}^{-1}[V_k]$. Thus,

$$\pi_{s_0}^{-1}[V_k] \cap \{b(a_n) : n < \omega\} = \{b(a_k)\}.$$

This proves that $\{b(a_n) : n < \omega\} \subseteq F^*$ is closed and discrete in F^*. Therefore, F^* is not countably compact. □

By Proposition 2.13, for each of the examples (2) and (3) in 2.10 there exists F^* which depurates the corresponding $\Sigma_\Gamma(F)$-product and which is not even countably compact (and so, it is neither p-compact nor initially α-compact). On the other hand, the following example shows that it is not possible to obtain a similar result to Theorem 2.6 for $\Sigma_\Gamma(F)$-products when $|\Gamma| > 1$.

Example 2.14. A $\Sigma_\Gamma(F)$-product which is not countably compact where F is compact.

In fact, let $X = I^S$, $x_0 = \underline{0}$, $\gamma_0 = \omega_1$ and for each $0 < n < \omega$, let $x_n = \underline{\frac{1}{n}}$ and $\gamma_n = \kappa^+$ where $\omega_1 \le \kappa < |S|$. Take $F = \{x_n : n < \omega\}$ and $\Gamma = \{\gamma_n : n < \omega\}$. It is clear that F is compact. Now, take $T \in [S]^\kappa$, and for each $n \in \mathbb{N}$, let y_n be such that $\pi_s(y_n) = 1$ if $s \in T$ and $\pi_s(y_n) = \frac{1}{n}$ if $s \notin T$. It is clear that for each $n \in \mathbb{N}$, $y_n \in \Sigma_{\gamma_n}(x_n) \subseteq \Sigma_\Gamma(F)$. Now it is not difficult to prove that the set $N = \{y_n : n \in \mathbb{N}\}$ is closed and discrete in $\Sigma_\Gamma(F)$. □

3. Normality of $\Sigma_\Gamma(F)$-Products.

In this section we will study normality in $\Sigma_\Gamma(F)$-products; in particular, we will prove that every proper $\Sigma_\Gamma(F)$-product of a product of compact metric spaces is normal if and only if $\Sigma_\Gamma(F)$ is equal to some $\Sigma_\kappa(x)$ where κ is regular.

Recall that a point x in a space X is a *butterfly point* of X ([14]) if there are two disjoint subsets A and B of $X\setminus\{x\}$ such that $\{x\} = Cl(A)\cap Cl(B)$. This kind of point has been useful in solving problems about normality as we can see in the following known result.

Lemma 3.1. *Let Y be a C^*-embedded subspace of X. If there is a point $x \in X \setminus Y$ which is a butterfly point of $Y \cup \{x\}$, then Y is not normal.*

Notation 3.2. *In this section we will use the following notation: for $x, y, z \in X = \prod_{s\in S} X_s$ and for an infinite cardinal number κ, we define the sets:*

$$T_{x,y} = \{s \in S : \pi_s(x) = \pi_s(y)\} = S \setminus supp_x(y),$$

$$T_{x,y}(z) = T_{x,z} \cup T_{y,z},$$

$$Z_{x,y}(\kappa) = \{w \in X : |S \setminus T_{x,y}(w)| < \kappa\}.$$

Theorem 3.3. *Let $X = \prod_{s\in S} X_s$, $F = \{x_i : i \in J\} \subseteq X$ and let $\Gamma = \{\gamma_i : i \in J\}$ be a collection of infinite cardinals such that the space $\Sigma_\Gamma(F)$ is depurated and C^*-embedded in X. If there are i, $j \in J$ and a point $z \in X \setminus \Sigma_\Gamma(F)$ such that $i \ne j$ and*

$$|S \setminus T_{x_i,x_j}(z)| < min\{\gamma_i, \gamma_j\},$$

then $\Sigma_\Gamma(F)$ is not normal.

Proof. Let $z \in X \setminus \Sigma_\Gamma(F)$ and x_i, $x_j \in F$ such that $|S \setminus T| < min\{\gamma_i, \gamma_j\}$, where $T = T_{x_i,x_j}(z)$. We define the set

$$A_i = \{x \in \Sigma_{\gamma_i}(x_i) : \pi_s(x) = \pi_s(z) \text{ if } s \notin T \text{ and } \pi_s(x) = \pi_s(x_i) \text{ if } s \in T_{x_i,z}\}.$$

In a similar way we define A_j. These are non-empty subsets of their corresponding Σ-products because

$$|S \setminus T| < min\{\gamma_i, \gamma_j\}.$$

Furthermore, it is clear that $A_i \cap A_j = \emptyset$ because $\Sigma_\Gamma(F)$ is depurated. We are going to show that $Cl_X(A_i) \cap Cl_X(A_j) = \{z\}$. By Lemma 3.1, this will be enough in order to conclude that $\Sigma_\Gamma(F)$ is not normal because $z \notin \Sigma_\Gamma(F)$.

For each $x \in A_i$ and $s \in S \setminus T_{x_j,z}$, $\pi_s(x) = \pi_s(z)$. Besides, for each $R \in [T_{x_j,z}]^{<\omega}$ there exists $x \in A_i$ such that $\pi_r(x) = \pi_r(z)$ if $r \in R$. Then, for each canonical open set U of X containing z, there exists an element $x \in A_i$ such that for all $s \in Supp[U]$, $\pi_s(x) = \pi_s(z)$. This means that $z \in Cl_X(A_i)$. In a similar way, we can prove that $z \in Cl_X(A_j)$.

Moreover, if $y \neq z$ then there is a $s \in S$ such that $\pi_s(y) \neq \pi_s(z)$ and so, there are disjoint open subsets U, V of X_s such that $\pi_s(y) \in U$ and $\pi_s(z) \in V$.

Observe that $y \in \pi_s^{-1}[U]$ and

$$\pi_s^{-1}[U] \cap \pi_s^{-1}[V] = \emptyset.$$

If $s \in T_{x_i,z}$ then

$$A_i \subseteq \pi_s^{-1}[V],$$

that is $y \notin Cl_X(A_i)$. Analogously, if $s \in T_{x_j,z}$ then $y \notin Cl_X(A_j)$. Finally, if $s \in S \setminus T$ then

$$(A_i \cup A_j) \subseteq \pi_s^{-1}[V],$$

hence

$$y \notin Cl(A_i) \cup Cl(A_j).$$

In any case, if $y \neq z$

$$y \notin Cl(A_i) \cap Cl(A_j)$$

and this completes our proof. □

Corollary 3.4. *Let $\{X_s : s \in S\}$ be a family of spaces and let X be the product of such a family. Let $F = \{x_i : i \in J\} \subseteq X$ and let κ be an infinite cardinal such that the space $\Sigma_\kappa(F)$ is depurated and C^*-embedded in X. If x, $y \in \Sigma_\kappa(F)$, $|supp_x(y)| \geq \kappa$ and $\Sigma_\kappa(F)$ is normal, then for each $z \in X$ such that $|S \setminus T_{x,y}(z)| < \kappa$, z must belong to $\Sigma_\kappa(F)$.*

Proof. Let x, $y \in \Sigma_\kappa(F)$ and $z \in X$ such that $|supp_x(y)| \geq \kappa$ and $|S \setminus T_{x,y}(z)| < \kappa$. Assume that $z \notin \Sigma_\kappa(F)$. Moreover, let i, $j \in J$ be such that $x \in \Sigma_\kappa(x_i)$ and $y \in \Sigma_\kappa(x_j)$. Since $|supp_x(y)| \geq \kappa$ and $\Sigma_\kappa(F)$ is depurated, $i \neq j$; and since

$$T_{x_i,x_j}(z) \supseteq T_{x,y}(z) \setminus (supp_x(x_i) \cup supp_y(x_j)),$$

then

$$|S \setminus T_{x_i,x_j}(z)| \le |S \setminus (T_{x,y}(z) \setminus (supp_x(x_i) \cup supp_y(x_j))| =$$
$$|(S \setminus T_{x,y}(z)) \cup (supp_x(x_i) \cup supp_y(x_j))| \le$$
$$|(S \setminus T_{x,y}(z))| + |supp_x(x_i)| + |supp_y(x_j))| < \kappa.$$

Thus, by Theorem 3.3, $\Sigma_\kappa(F)$ is not normal, which is a contradiction obtained by assuming that $z \notin \Sigma_\kappa(F)$. □

Corollary 3.5. *Let $\{X_s : s \in S\}$ be a family of topological spaces. Let $X = \prod_{s\in S} X_s$, $F = \{x_i : i \in J\} \subseteq X$ and let $\Gamma = \{\gamma_i : i \in J\}$ be a family of cardinal numbers. Assume that $\Sigma_\Gamma(F)$ is a proper, depurated and C^*-embedded subset of X, Γ is maximal and $|\Gamma| > 1$. Then, $\Sigma_\Gamma(F)$ is not normal.*

Proof. Let γ_i and γ_j be two elements in Γ with $\gamma_i < \gamma_j$. Since Γ is maximal, there is $z \in \Sigma_{\gamma_j}(x_i) \setminus \Sigma_\Gamma(F)$. It happens then that

$$\gamma_i \le |supp_z(x_i)| < \gamma_j \le \kappa = |S|.$$

Let $y_j \in X$ be defined by $y_j(s) = z(s)$ if $s \in supp_z(x_i)$, and $y_j(s) = x_j(s)$ if $s \in S \setminus supp_z(x_i)$. Hence, $y_j \in \Sigma_{\gamma_j}(x_j)$. Therefore, $\Sigma_{\gamma_j}(x_j) = \Sigma_{\gamma_j}(y_j)$. So, $\Sigma_\Gamma(F) = \Sigma_\Gamma((F \cup \{y_j\}) \setminus \{x_j\})$. Let $F' = (F \cup \{y_j\}) \setminus \{x_j\})$.

Then, we have $z \in X \setminus \Sigma_\Gamma(F')$ and

$$\{s \in S : \pi_s(z) = \pi_s(x_i) \text{ or } \pi_s(z) = \pi_s(y_i)\} = S.$$

Because of Theorem 3.3, $\Sigma_\Gamma(F') = \Sigma_\Gamma(F)$ is not normal. □

Lemma 3.6. *Let $\{X_s : s \in S\}$ be a family of spaces and let $X = \prod_{s\in S} X_s$. If x, $y \in X$ are two points satisfying $|supp_x(y)| = \kappa \ge \omega$, then there exists a set $F \subseteq X$ such that x, $y \in F$, $|F| = 2^\kappa$, $\Sigma_\kappa(F)$ is depurated and for all $z \in F$, $T_{x,y}(z) = S$.*

Proof. Let $\{S_\xi : \xi < \kappa\}$ be a κ-partition of $supp_x(y)$. Let $\{A_\eta : \eta < 2^\kappa\}$ be the collection of all the subsets of κ. For each $\eta < 2^\kappa$, take $x_\eta \in X$ such that $\pi_s(x_\eta) = \pi_s(y)$ for each $s \in \cup_{\xi\in A_\eta} S_\xi$ and $\pi_s(x_\eta) = \pi_s(x)$ otherwise. Now, take $F = \{x_\eta : \eta < 2^\kappa\}$. Observe that if $A_\eta = \emptyset$ then $x_\eta = x$ and if $A_\eta = \kappa$ then $x_\eta = y$; so, x, $y \in F$. Moreover, by definition we have that for each $\eta < 2^\kappa$ and each $s \in S$, $\pi_s(x_\eta) \in \{\pi_s(x), \pi_s(y)\}$. Therefore, for each $\eta < 2^\kappa$, $T_{x,y}(x_\eta) = S$.

Now, we only have to prove that $\Sigma_\kappa(F)$ is depurated. Let η, ζ be two different elements in 2^τ; thus $A_\eta \ne A_\zeta$. Without loss of generality we can assume that there is $\xi \in A_\eta \setminus A_\zeta$, then $S_\xi \subseteq supp_{x_\eta}(x_\zeta) \subseteq supp_x(y)$ which implies that $\kappa = |supp_x(y)| = |supp_{x_\eta}(x_\zeta)| \ge |S_\xi| = \kappa$. Therefore, $\Sigma_\kappa(x_\eta) \cap \Sigma_\kappa(x_\zeta) = \emptyset$, so $\Sigma_\kappa(F)$ is depurated and $|F| = 2^\kappa$. □

As a corollary we obtain the following result. Recall that a space X is τ-resolvable if there exists a family $\mathcal{D}$ of cardinality τ constituted by disjoint dense subsets of X. (W.W. Comfort communicated to the third author that this result was proved by O. Masaveu with a similar proof to that given here.)

Corollary 3.7. *Let $X = \prod_{s\in S} X_s$ with $|S| \geq \omega$. Then X is $2^{|S|}$-resolvable.*

Proof. Let x, $y \in X$ such that for each $s \in S$, $\pi_s(x) \neq \pi_s(y)$. Then $supp_x(y) = S$; so $|supp_x(y)| = |S|$.

Because of Lemma 3.6, there is a set F such that $|F| = 2^{|S|}$ and $\Sigma_{|S|}(F)$ is depurated; that is, $\{\Sigma_{|S|}(z) : z \in F\}$ is a family of cardinality $|F| = 2^{|S|}$ of disjoint dense subsets of X. □

The following result is already known (see, for instance, [1], Theorem 1.4).

Lemma 3.8. *Let $\{X_s : s \in S\}$ be a family of topological spaces and X its topological product. Let κ be a cardinal such that $\omega \leq \kappa \leq |S|$. Let $\tau < cof(\kappa)$ be an infinite cardinal and $\{S_\xi : \xi < \tau\}$ a partition of S such that for each $\xi < \tau$, $|S_\xi| = |S|$. Then, for every $p \in X$, $\Sigma_\kappa(p, X)$ is homeomorphic to the space*

$$\prod_{\xi<\tau} \Sigma_\kappa(\pi_{S_\xi}(p), X_{S_\xi}).$$

Corollary 3.9. *Let $X = \prod_{s\in S} X_s$ with $|S| \geq \omega$ and $F = \{x_i : i \in J\} \subseteq X$. Let $\Gamma = \{\gamma_i : i \in J\}$ be a family of infinite cardinals and put $\kappa = sup\{cof(\gamma_i) : i \in J\}$. Then, $\Sigma_\Gamma(F)$ is 2^τ-resolvable for all $\tau < \kappa$.*

Proof. Let $\tau < \kappa$. Then there is $i \in J$ such that $\tau < cof(\gamma_i) \leq \kappa$. By Lemma 3.8, $\Sigma_{\gamma_i}(x_i)$ is homeomorphic to

$$\prod_{\xi<\tau} \Sigma_{\gamma_i}(\pi_{S_\xi}(x_i), X_{S_\xi})$$

where $\{S_\xi : \xi < \tau\}$ is a $|S|$-partition of S. By Corollary 3.7, $\Sigma_{\gamma_i}(x_i)$ is 2^τ-resolvable. On the other hand, $\Sigma_{\gamma_i}(x_i)$ is a dense subset of $\Sigma_\Gamma(F)$; so, we conclude that $\Sigma_\Gamma(F)$ is 2^τ-resolvable. □

Observe that, by Corollary 3.9, for every topological product X and every $x \in X$, $\sigma(x, X)$ is ω-resolvable.

Now, we are going to characterize the normality of $\Sigma_\Gamma(F)$-products when $X = \{0, 1\}^S$. We will use the conventions established in Notation 3.2.

Theorem 3.10. *Let $X = \{0,1\}^S$ and $F \subseteq X$. Suppose that $|F| > 1$ and let κ be a cardinal such that $\omega \le \kappa \le |S|$. If*

$$Z = \cup_{x \ne y \in F} Z_{x,y}(\kappa) \ne \Sigma_\kappa(F),$$

then $\Sigma_\kappa(F)$ is not normal.

Proof. Suppose that $\Sigma_\kappa(F)$ is depurated (Proposition 2.5) and normal. It is apparent that for each $x, y \in F$ with $x \ne y$, $\Sigma_\kappa(\{x,y\}) \subseteq Z_{x,y}(\kappa)$, so we always have that $\Sigma_\kappa(F) \subseteq Z$.

Furthermore, if $z \in Z$, then there are $x, y \in F$ with $x \ne y$ such that $z \in Z_{x,y}(\kappa)$. This means that $|S \setminus T_{x,y}(z)| < \kappa$. Because of Corollary 3.4, since we are assuming that $\Sigma_\kappa(F)$ is normal, we conclude that $z \in \Sigma_\kappa(F)$. □

Lemma 3.11. *Let $X = \{0,1\}^S$, $\omega \le \kappa \le |S|$ and $F \subseteq X$. Given a point $x \in \Sigma_\kappa(F)$ and a set $R \subseteq S$, let*

$$Z = \{z \in X : supp_x(z) \subseteq R\}$$

and let $f : Z \longrightarrow Y = \{0,1\}^R$ be the natural homeomorphism between Z and Y. Then, the subspace $\Sigma_\kappa(F) \cap Z$ is a closed subset of $\Sigma_\kappa(F)$ such that

$$f[\Sigma_\kappa(F) \cap Z] = \Sigma_\kappa(F', Y)$$

for some $F' \subseteq \{0,1\}^R$.

Proof. Let $f : Z \longrightarrow Y = \{0,1\}^R$ be the natural homeomorphism between Z and Y; that is, for each $z \in Z$, $f(z) = \pi_R(z)$. If $|R| < \kappa$, then there is nothing to prove because, in general, for each $T \in [S]^{<\kappa}$, $\{z \in X : supp_x(z) \subseteq T\} \subseteq \Sigma_\kappa(F)$ and $\pi_T[\Sigma_\kappa(F)] = \{0,1\}^T$.

Assume that $|R| \ge \kappa$. It is clear that $x \in \Sigma_\kappa(F) \cap Z$ and $\Sigma_\kappa(F) \cap Z$ is homeomorphic to $\Sigma = f(\Sigma_\kappa(F) \cap Z)$. It suffices to prove that $\Sigma = \Sigma_\kappa(\Sigma, Y)$.

It is evident that $\Sigma \subseteq \Sigma_\kappa(\Sigma, Y)$. Let $p \in \Sigma_\kappa(\Sigma, Y)$ and let $q \in \Sigma_\kappa(F) \cap Z$ such that $p \in \Sigma_\kappa(f(q))$. Thus $|supp_q(f^{-1}(p)) \cap R| < \kappa$, and since q and $f^{-1}(p)$ belong to Z, then for each $s \notin R$, $\pi_s(q) = \pi_s(f^{-1}(p)) = \pi_s(x)$; so, $supp_q(f^{-1}(p)) \cap R = supp_q(f^{-1}(p))$ which implies that $f^{-1}(p) \in (\Sigma_\kappa(q) \cap Z) \subseteq (\Sigma_\kappa(F) \cap Z)$ and $p \in \Sigma$.

Now, since Z is closed in X, $\Sigma_\kappa(F) \cap Z$ is closed in $\Sigma_\kappa(F)$. □

Theorem 3.12. *For every $F \subseteq X = \{0,1\}^S$ where $|S| > \omega$, and every family Γ of uncountable cardinals, $\Sigma_\Gamma(F)$ is normal if and only if either $\Sigma_\Gamma(F) = X$ or $\Sigma_\Gamma(F) = \Sigma_\kappa(x)$ for some $x \in X$ and some regular cardinal κ with $\omega < \kappa \le |S|$.*

Proof. If $\Sigma_\Gamma(F) = X$, then $\Sigma_\Gamma(F)$ is normal. If $\Sigma_\Gamma(F) = \Sigma_\kappa(x)$ where $x \in X$, κ is regular and $\omega < \kappa \leq |S|$, then $\Sigma_\kappa(x)$ is normal according to Theorem 1.19.

Now, assume that $\Sigma_\Gamma(F)$ is depurated (Proposition 2.5), normal and $|F| > 1$. It is clear that if $\Sigma_\Gamma(F) = X$, then there is nothing to prove, so we will suppose that $\Sigma_\Gamma(F)$ is a proper subset of X. Because of Corollary 3.5, we can assume that $\Gamma = \{\kappa\}$ for a cardinal number $\kappa > \omega$ which is maximal; that is, $\Sigma_\Gamma(F) = \Sigma_\kappa(F)$ and if $p \in F$ and $\gamma > \kappa$, then $\Sigma_\gamma(p) \setminus \Sigma_\Gamma(F) \neq \emptyset$. If x, $y \in F$, according to Theorem 3.10, $Z_{x,y}(\kappa) \subseteq \Sigma_\kappa(F)$ because $\Sigma_\kappa(F)$ is normal; and if in addition $x \neq y$, then $|S \setminus T_{x,y}| = |supp_x(y)| \geq \kappa$ because $\Sigma_\Gamma(F)$ is depurated. So, since $|F| > 1$, we can fix two points $x, y \in F$ such that $x \neq y$.

Since $\Sigma_\kappa(F)$ is a proper subset of X and κ is maximal, there exists a point $p \in \Sigma_{\kappa^+}(x) \setminus \Sigma_\kappa(F)$. Moreover, $Z_{x,y}(\kappa) \subseteq \Sigma_\kappa(F)$, hence $|supp_x(p) \setminus supp_x(y)| = \kappa$ because $p \notin Z_{x,y}(\kappa)$. Let $x_0 \in Z_{x,y}(\kappa)$ such that $\pi_{supp_x(y)}(x_0) = \pi_{supp_x(y)}(p)$ and $\pi_{T_{x,y}}(x_0) = \pi_{T_{x,y}}(x)$; and let

$$R_0 = supp_x(p) \setminus supp_x(y), \quad R_1 \in [supp_x(y)]^\kappa, \quad R = R_0 \cup R_1,$$
$$K_0 = \{z \in X : supp_{x_0}(z) \subseteq R_0\}, \quad K_1 = \{z \in X : supp_{x_0}(z) \subseteq R_1\}$$
$$\text{and} \quad K = \{z \in X : supp_{x_0}(z) \subseteq R\}.$$

Observe that $R_0 = T_{x,y} \cap supp_x(p)$ and $|R_0| = \kappa = |R_1|$. Let

$$f_0 : K_0 \longrightarrow Y_0 = \{0,1\}^{R_0}, \quad f_1 : K_1 \longrightarrow Y_1 = \{0,1\}^{R_1}$$
$$\text{and} \quad f : K \longrightarrow Y = \{0,1\}^R$$

be the natural homeomorphisms between all these spaces. By Lemma 3.11, the subspaces $Q = \Sigma_\kappa(F) \cap K$ and $Q_0 = \Sigma_\kappa(F) \cap K_0$ are closed in $\Sigma_\kappa(F)$ and such that

$$f[Q] = \Sigma_\kappa(F', Y) \quad \text{and} \quad f_0[Q_0] = \Sigma_\kappa(F_0, Y_0),$$

where $F' \subseteq Y = \{0,1\}^R$ and $F_0 \subseteq Y_0 = \{0,1\}^{R_0}$. We can choose F' and F_0 in such a manner that $\Sigma_\kappa(F', Y)$ and $\Sigma_\kappa(F_0, Y_0)$ are depurated. On the other hand, it is clear that $K_1 \subseteq Z_{x,y}(\kappa)$, so $K_1 = \Sigma_\kappa(F) \cap K_1$. Also note, from definition of x_0, that $supp_{x_0}(p) = T_{x,y} \cap supp_x(p) = R_0$; then $p \in K_0$, and since $p \notin \Sigma_\kappa(F)$, $p \in K_0 \setminus Q_0$.

Claim: $\Sigma_\kappa(F', Y)$ is homeomorphic to $\Sigma_\kappa(F_0, Y_0) \times Y_1$.

Indeed, let $g_0 : Y \longrightarrow Y_0$ and $g_1 : Y \longrightarrow Y_1$ be the projections, and define $g : Y \longrightarrow Y_0 \times Y_1$ as $g = (g_0, g_1)$. It is obvious that g is a homeomorphism; even more, note that for each $z \in Y$, $g_0(z) = \pi_{R_0}(f^{-1}(z))$ and $g_1(z) = \pi_{R_1}(f^{-1}(z))$. In particular, $g(z) = (\pi_{R_0}(f^{-1}(z)), \pi_{R_1}(f^{-1}(z)))$. We are going to show that

$$g[\Sigma_\kappa(F', Y)] = f_0[f^{-1}[\Sigma_\kappa(F', Y)] \cap K_0] \times f_1[f^{-1}[\Sigma_\kappa(F', Y)] \cap K_1]. \qquad (*)$$

Take $z \in \Sigma_\kappa(F',Y)$, so $g(z) = (\pi_{R_0}(f^{-1}(z)), \pi_{R_1}(f^{-1}(z)))$. Observe that $K_0 \cup K_1 \subseteq K$ and if $z \in \Sigma_\kappa(F',Y)$ and $f^{-1}(z) \in K_0$ ($f^{-1}(z) \in K_1$), then $\pi_{R_0}(f^{-1}(z)) = f_0(f^{-1}(z))$ ($\pi_{R_1}(f^{-1}(z)) = f_1(f^{-1}(z))$). Let z_0, $z_1 \in \Sigma_\kappa(F',Y)$ such that $\pi_{R_0}(z_0) = \pi_{R_0}(z)$, $\pi_{R_1}(z_0) = \pi_{R_1}(x_0)$, $\pi_{R_0}(z_1) = \pi_{R_0}(x_0)$ and $\pi_{R_1}(z_1) = \pi_{R_1}(z)$. Therefore, $f^{-1}(z_0) \in K_0$ and $f^{-1}(z_1) \in K_1$, so

$$(f_0(f^{-1}(z_0)), f_1(f^{-1}(z_1))) \in f_0[f^{-1}[\Sigma_\kappa(F',Y)] \cap K_0] \times f_1[f^{-1}[\Sigma_\kappa(F',Y)] \cap K_1]$$

and

$$\begin{aligned}(f_0(f^{-1}(z_0)), f_1(f^{-1}(z_1))) &= (\pi_{R_0}(f^{-1}(z_0)), \pi_{R_1}(f^{-1}(z_1))) \\ &= (\pi_{R_0}(f^{-1}(z)), \pi_{R_1}(f^{-1}(z))) = g(z).\end{aligned}$$

From this we obtain

$$g[\Sigma_\kappa(F',Y)] \subseteq f_0[f^{-1}[\Sigma_\kappa(F',Y)] \cap K_0] \times f_1[f^{-1}[\Sigma_\kappa(F',Y)] \cap K_1].$$

Besides, take

$$(a,b) \in f_0[f^{-1}[\Sigma_\kappa(F',Y)] \cap K_0] \times f_1[f^{-1}[\Sigma_\kappa(F',Y)] \cap K_1],$$

then there are z_0, $z_1 \in \Sigma_\kappa(F',Y)$ such that $f^{-1}(z_0) \in K_0$, $f^{-1}(z_1) \in K_1$, $f_0(f^{-1}(z_0)) = a$ and $f_1(f^{-1}(z_1)) = b$. Let $z \in Y$ be such that $g_0(z) = g_0(z_0)$ and $g_1(z) = g_1(z_1)$. Observe that, if $|supp_{z_0}(z_1)| < \kappa$, then $|supp_{z_0}(z)| < \kappa$, so $z \in \Sigma_\kappa(F',Y)$. Moreover, note that Q is a closed subset of $\Sigma_\kappa(F)$; hence, Q and $\Sigma_\kappa(F',Y)$ are normal. Furthermore, $R_0 \subseteq T_{z,z_0}$ and $R_1 \subseteq T_{z,z_1}$, thus $R \setminus (T_{z,z_0} \cup T_{z,z_1}) = \emptyset$; so, by Corollary 3.4, since z_0, $z_1 \in \Sigma_\kappa(F',Y)$, if $|supp_{z_0}(z_1)| \geq \kappa$, then $z \in \Sigma_\kappa(F',Y)$. Thus, in any case, $g(z) \in g[\Sigma_\kappa(F',Y)]$ and

$$\begin{aligned}g(z) = (g_0(z), g_1(z)) = (g_0(z_0), g_1(z_1)) &= (\pi_{R_0}(f^{-1}(z_0)), \pi_{R_1}(f^{-1}(z_1))) \\ &= (f_0(f^{-1}(z_0)), f_1(f^{-1}(z_1))) = (a,b).\end{aligned}$$

With all this, we conclude that, in fact, the equality $(*)$ holds.

Then, the function

$$g|_{\Sigma_\kappa(F',Y)} : \Sigma_\kappa(F',Y) \longrightarrow f_0[f^{-1}[\Sigma_\kappa(F',Y)] \cap K_0] \times f_1[f^{-1}[\Sigma_\kappa(F',Y)] \cap K_1]$$

is a homeomorphism and

$$\begin{aligned}&g|_{\Sigma_\kappa(F',Y)}[\Sigma_\kappa(F',Y)] = f_0[f^{-1}[\Sigma_\kappa(F',Y)] \cap K_0] \times f_1[f^{-1}[\Sigma_\kappa(F',Y)] \cap K_1] \\ &= f_0[Q \cap K_0] \times f_1[Q \cap K_1] = f_0[(\Sigma_\kappa(F) \cap K) \cap K_0] \times f_1[(\Sigma_\kappa(F) \cap K) \cap K_1] \\ &= f_0[\Sigma_\kappa(F) \cap K_0] \times f_1[\Sigma_\kappa(F) \cap K_1] = f_0[Q_0] \times f_1[K_1] = \Sigma_\kappa(F_0, Y_0) \times Y_1.\end{aligned}$$

This finishes the proof of the Claim.

Now, since $x \in Q_0$, $f_0(x) \in \Sigma_\kappa(F_0,Y_0)$. From this we have $\Sigma_\kappa(f_0(x)) \subseteq \Sigma_\kappa(F_0,Y_0)$, and so $\Sigma_\kappa(F_0,Y_0)$ is a dense subset of Y_0. Moreover, since Y_0 is compact and $\kappa > \omega$, then, by (2) of Lemma 1.2, $Y_0 = \beta(\Sigma_\kappa(F_0,Y_0))$.

Even more, it is clear that Y_0 is homeomorphic to Y_1 because $|R_0| = |R_1|$; so, because of our Claim we have that $\Sigma_\kappa(F', Y)$ is homeomorphic to $\Sigma_\kappa(F_0, Y_0) \times \beta(\Sigma_\kappa(F_0, Y_0))$.

Since $\Sigma_\kappa(F)$ is normal and since Q and Q_0 are closed subsets of $\Sigma_\kappa(F)$, then Q and Q_0 are normal. Therefore,

$$\Sigma_\kappa(F_0, Y_0) \text{ and } \Sigma_\kappa(F_0, Y_0) \times \beta(\Sigma_\kappa(F_0, Y_0))$$

are normal.

By Lemma 1.13, $\Sigma_\kappa(F_0, Y_0)$ is paracompact; but $\Sigma_\kappa(F_0, Y_0)$ is pseudocompact and normal; this implies that it is countably compact, so it is compact. Then, $\Sigma_\kappa(F_0, Y_0) = \beta\Sigma_\kappa(F_0, Y_0) = Y_0$. Therefore, $Q_0 = f_0^{-1}[\Sigma_\kappa(F_0, Y_0)] = f_0^{-1}[Y_0] = K_0$, but this last equality contradicts the fact that $p \in K_0 \setminus Q_0$.

This contradiction follows from assuming $|F| > 1$. Therefore, it must happen that $\Sigma_\kappa(F)$ is of the form $\Sigma_\kappa(x)$. Moreover, by Theorem 1.16, κ is regular. □

Now, we are ready to characterize normality of C^*-embedded $\Sigma_\Gamma(F)$-products. We are still using the terminology defined in Notation 3.2.

Theorem 3.13. *Let $X = \prod_{s\in S} X_s$. Let Γ be a family of uncountable cardinals and $F = \{x_i : i \in J\} \subseteq X$ such that $\Sigma_\Gamma(F)$ is C^*-embedded in X. If $\Sigma_\Gamma(F)$ is normal, then either $\Sigma_\Gamma(F) = X$ or $\Sigma_\Gamma(F) = \Sigma_\kappa(x)$ for an $x \in X$ and an uncountable regular cardinal number κ.*

Proof. Without loss of generality, we can suppose that $\Gamma = \{\kappa\}$ is maximal and $\Sigma_\kappa(F)$ is depurated (Proposition 2.5). Assume that $\Sigma_\kappa(F)$ is normal and $|F| > 1$. Given points x, $y \in X$ put

$$Q_{x,y} = \{z \in X : (S \setminus T_{x,y}) \subseteq (T_{x,z} \cup T_{y,z})\}.$$

Claim 1: For every x, $y \in \Sigma_\kappa(F)$, if $|supp_x(y)| \geq \kappa$ then $Q_{x,y} \subseteq \Sigma_\kappa(F)$.

Let $x, y \in \Sigma_\kappa(F)$ such that $|supp_x(y)| \geq \kappa$. By Corollary 3.4, $Z_{x,y}(\kappa) \subseteq \Sigma_\kappa(F)$. Let $p \in Q_{x,y} \setminus Z_{x,y}(\kappa)$ and set

$$T = S \setminus (T_{x,p} \cap T_{x,y}).$$

Define $f : \{0,1\}^T \longrightarrow X$ as: for each $a \in \{0,1\}^T$, $\pi_s(f(a)) = \pi_s(x)$ if either $s \in T_{x,p} \cap T_{x,y}$ or $\pi_s(a) = 0$, $\pi_s(f(a)) = \pi_s(p)$ if $s \in T_{x,y} \setminus T_{x,p}$ and $\pi_s(a) = 1$; and $\pi_s(f(a)) = \pi_s(y)$ if $s \in supp_x(y)$ and $\pi_s(a) = 1$.

It is evident that x, $y \in f[\{0,1\}^T]$, so

$$Y = f[\{0,1\}^T] \cap \Sigma_\kappa(F) \neq \emptyset.$$

Moreover, observe that for every $s \in supp_x(y)$, $\pi_s(p) \in \{\pi_s(x), \pi_s(y)\}$, so

$$p \in f[\{0,1\}^T].$$

Also, it is apparent that for each $s \in S$, the function $\pi_s \circ f$ is continuous; hence, f is continuous. On the other hand, for each $s \in T_{x,y} \setminus T_{x,p}$, $\pi_s(x) \neq \pi_s(p)$ and for each $s \in supp_x(y)$, $\pi_s(x) \neq \pi_s(y)$; so, for each $s \in T$, $\pi_s(f(\underline{0})) \neq \pi_s(f(\underline{1}))$. Thus, if a, $b \in \{0,1\}^T$ are different, there exists $s \in T$ such that $\pi_s(a) = 0$ and $\pi_s(b) = 1$ (or $\pi_s(a) = 1$ and $\pi_s(b) = 0$), then $\pi_s(f(a)) \neq \pi_s(f(b))$. So, f is injective and, since $\{0,1\}^T$ is compact, f is a closed embedding from $\{0,1\}^T$ to X. Therefore, function

$$f : \{0,1\}^T \longrightarrow f[\{0,1\}^T]$$

is a homeomorphism.

Set

$$\Sigma = \Sigma_\kappa(f^{-1}[Y], \{0,1\}^T).$$

We are going to show that $f^{-1}[Y] = \Sigma$. It is clear that $f^{-1}[Y] \subseteq \Sigma$. Pick $t \in \Sigma$ and let $r \in Y$ be such that

$$t \in \Sigma_\kappa(f^{-1}(r), \{0,1\}^T).$$

Since f is bijective and $\pi_s(f(t)) = \pi_s(r)$ for each $s \in T_{x,p} \cap T_{x,y}$, we have that

$$|\{s \in S : \pi_s(f(t)) \neq \pi_s(r)\}| = |\{s \in T : \pi_s(t) \neq \pi_s(f^{-1}(r))\}| < \kappa.$$

Then, $f(t) \in \Sigma_\kappa(r)$ and since $r \in \Sigma_\kappa(F)$, $f(t) \in Y$. Therefore, $\Sigma \subseteq f^{-1}[Y]$ and we conclude that $f^{-1}[Y] = \Sigma$.

Without loss of generality, we can assume that $F^* \subseteq f^{-1}[Y]$ depurates Σ and $f^{-1}(x)$, $f^{-1}(y) \in F^*$. Note that $supp_x(y) \subseteq T$, so $supp_{f^{-1}(x)}(f^{-1}(y)) = supp_x(y)$, and then $|F^*| > 1$. If $p \notin \Sigma_\kappa(F)$, then

$$f^{-1}(p) \in \{0,1\}^T \setminus \Sigma_\kappa(F^*),$$

that is

$$\Sigma_\kappa(F^*) \neq \{0,1\}^T.$$

If κ is maximal for $\Sigma_\kappa(F^*, \{0,1\}^T)$ then, by Theorem 3.12, $\Sigma_\kappa(F^*)$ cannot be normal. Since $\Sigma_\kappa(F^*) = f^{-1}[Y]$ and f is a homeomorphism, Y is not normal. Since Y is closed in $\Sigma_\kappa(F)$, this last space is not normal, which is a contradiction obtained by assuming that $p \notin \Sigma_\kappa(F)$.

Now, assume that κ is not maximal for $\Sigma_\kappa(F^*, \{0,1\}^T)$. Since κ is maximal for $\Sigma_\kappa(F, X)$, there is a point $z \in X \setminus \Sigma_\kappa(F, X)$ such that $|supp_x(z)| = \kappa$.

Note that, since κ is not maximal for $\Sigma_\kappa(F^*, \{0,1\}^T)$, we have $\Sigma_{\kappa^+}(w, \{0,1\}^T) \subseteq \Sigma_\kappa(F^*, \{0,1\}^T)$ for every $w \in \Sigma_\kappa(F^*, \{0,1\}^T)$. Indeed, assume that there exists $w \in \Sigma_\kappa(F^*, \{0,1\}^T)$ such that

$$\Sigma_{\kappa^+}(w, \{0,1\}^T) \setminus \Sigma_\kappa(F^*, \{0,1\}^T) \neq \emptyset.$$

For each $t \in F^*$, let

$$\gamma_t = sup\{\gamma \leq |T| : \Sigma_\gamma(t, \{0,1\}^T) \subseteq \Sigma_\kappa(F^*, \{0,1\}^T)\},$$

and let $\Gamma^* = \{\gamma_t : t \in F^*\}$. Because of Proposition 2.4, $\Sigma_{\Gamma^*}(F^*, \{0,1\}^T) = \Sigma_\kappa(F^*, \{0,1\}^T)$ and Γ^* is maximal for $\Sigma_{\Gamma^*}(F^*, \{0,1\}^T)$. Let $r \in F^*$ be such that $w \in \Sigma_\kappa(r, \{0,1\}^T)$. Since $\Sigma_{\kappa^+}(w, \{0,1\}^T) = \Sigma_{\kappa^+}(r, \{0,1\}^T)$, $\Sigma_{\kappa^+}(r, \{0,1\}^T) \setminus \Sigma_\kappa(F^*, \{0,1\}^T) \neq \emptyset$. Moreover, since $\Sigma_\kappa(r, \{0,1\}^T) \subseteq \Sigma_\kappa(F^*, \{0,1\}^T)$, κ must be equal to γ_r. Besides, since κ is not maximal for $\Sigma_\kappa(F^*, \{0,1\}^T)$, there is $t \in F^*$ such that $\Sigma_{\kappa^+}(t, \{0,1\}^T) \subseteq \Sigma_\kappa(F^*, \{0,1\}^T)$. Hence, $\gamma_t > \kappa$. So, $\gamma_r, \gamma_t \in \Gamma^*$ and $\gamma_r \neq \gamma_t$; that is, $|\Gamma^*| > 1$. By Corollary 3.5, $\Sigma_{\Gamma^*}(F^*, \{0,1\}^T)$ is not normal. Then, $Y = f[\Sigma_\kappa(F^*, \{0,1\}^T)]$ is not normal. But, as we have already seen, this last assertion produces a contradiction. So, for every $w \in \Sigma_\kappa(F^*, \{0,1\}^T)$, $\Sigma_{\kappa^+}(w, \{0,1\}^T) \subseteq \Sigma_\kappa(F^*, \{0,1\}^T)$. Then,

$$\Sigma_{\kappa^+}(F^*, \{0,1\}^T) = \Sigma_\kappa(F^*, \{0,1\}^T) \neq \{0,1\}^T.$$

Since $\Sigma_{\kappa^+}(F^*, \{0,1\}^T) \neq \{0,1\}^T$, it happens that $|T| \geq \kappa^+$; so, there is a set $R \in [T \setminus supp_x(z)]^\kappa$.

Let $Z = \{0,1\}^{R \cup supp_x(z)}$ and let $g : Z \longrightarrow X$ be such that for each $b \in Z$, $\pi_s(g(b)) = \pi_s(x)$ if either $s \notin R \cup supp_x(z)$ or $\pi_s(b) = 0$, $\pi_s(g(b)) = \pi_s(p)$ if $s \in R \cap (T_{x,y} \setminus T_{x,p})$ and $\pi_s(b) = 1$; $\pi_s(g(b)) = \pi_s(y)$ if $s \in R \cap supp_x(y)$ and $\pi_s(b) = 1$; and $\pi_s(g(b)) = \pi_s(z)$ if $s \in supp_x(z)$ and $\pi_s(b) = 1$. Using similar arguments to those we used before in this same proof, it is possible to show that g is a closed embedding from Z to X such that $g^{-1}[\Sigma_\kappa(F)]$ is equal to $\Sigma_\kappa(F', Z)$ for a non-empty subset F' of Z which depurates $\Sigma_\kappa(F', Z)$.

Let a, $b \in Z$ be such that $\pi_R(a) = \pi_R(\underline{1})$, $\pi_{supp_x(z)}(a) = \pi_{supp_x(z)}(\underline{0})$, $\pi_R(b) = \pi_R(\underline{0})$ and $\pi_{supp_x(z)}(b) = \pi_{supp_x(z)}(\underline{1})$. It is clear that $g(\underline{0}) = x$, so $\underline{0} \in \Sigma_\kappa(F', Z)$. Besides, observe that $supp_x(g(a)) = R$. Since $R \subseteq T$, we can consider the point $q \in \{0,1\}^T$ defined as $\pi_R(q) = \pi_R(\underline{1})$ and $\pi_{T \setminus R}(q) = \pi_{T \setminus R)}(\underline{0})$. Observe that $f(q) = g(a)$ and $supp_{\underline{0}}(q) = R$, then

$$f^{-1}(g(a)) = q \in \Sigma_{\kappa^+}(f^{-1}(x), \{0,1\}^T) \subseteq \Sigma_\kappa(F^*, \{0,1\}^T).$$

So $g(a) \in \Sigma_\kappa(F, X)$ and hence $a \in \Sigma_\kappa(F', Z)$. Since $supp_{\underline{0}}(a) = R$ and $|R| = \kappa$, we can assume, without loss of generality, that $\underline{0}$ and a belong to F'. On the other hand, $g(b) = z \notin \Sigma_\kappa(F, X)$, so $b \notin \Sigma_\kappa(F', Z)$ and since $supp_{\underline{0}}(b) = supp_x(z)$ and $|supp_x(z)| = \kappa$, we can guarantee that κ is maximal for $\Sigma_\kappa(F', Z)$. Then, by Theorem 3.10, $\Sigma_\kappa(F', Z)$ is not normal. In this manner we again obtain a contradiction which is a consequence of assuming that $p \notin \Sigma_\kappa(F)$. So, we have already proved Claim 1.

Claim 2: $X \subseteq \Sigma_\kappa(F)$.

In order to prove Claim 2, first we are going to choose two different points $x, y \in \Sigma_\kappa(F)$ such that $|T_{x,y}| = |S|$. In fact, take $x, x' \in F$ with $x \neq x'$. If $|T_{x,x'}| = |S|$, we simply choose $y = x'$. Suppose now that $|T_{x,x'}| < |S|$. Then $|supp_x(x')| = |S|$. Let $S' \in [supp_x(x')]^{|S|}$ such that $|supp_x(x') \setminus S'| = |S|$ and let $y \in X$ such that $\pi_{S'}(y) = \pi_{S'}(x')$ and $\pi_{S\setminus S'}(y) = \pi_{S\setminus S'}(x)$. It is evident that $|T_{x,y}| = |S|$. Even more, since $y \in Z_{x,x'}(\kappa)$ and $Z_{x,x'}(\kappa) \subseteq \Sigma_\kappa(F)$, $y \in \Sigma_\kappa(F)$.

Now take $z \in X$. We are going to prove that $z \in \Sigma_\kappa(F)$. We have that if $|supp_x(z)| < \kappa$, then $z \in \Sigma_\kappa(F)$. So, assume $|supp_x(z)| \geq \kappa$. Let $x_z, y_z \in X$ be such that:

(i) for each $s \in supp_x(z) \cap supp_x(y)$, $\pi_s(x_z) = \pi_s(y_z) = \pi_s(x)$,
(ii) for each $s \in T_{x,z} \cap supp_x(y)$, $\pi_s(x_z) = \pi_s(y)$ and $\pi_s(y_z) = \pi_s(x)$, and
(iii) for each $s \in T_{x,y}$, $\pi_s(x_z) \neq \pi_s(z)$ and $\pi_s(y_z) = \pi_s(z)$.

By (i) and (ii), it is clear that $x_z, y_z \in Q_{x,y}$, and by (iii) we have that $|supp_{x_z}(y_z)| \geq \kappa$. Then, because of Claim 1, $x_z, y_z \in \Sigma_\kappa(F)$ and $Q_{x_z,y_z} \subseteq \Sigma_\kappa(F)$. Now, it suffices to prove that $z \in Q_{x_z,y_z}$. By (ii) and (iii), we obtain:

(iv) for every $s \notin supp_x(z) \cap supp_x(y)$, $\pi_s(z) = \pi_s(y_z)$.

And by (i), we obtain:

(v) for each $s \in supp_x(z) \cap supp_x(y)$, $\pi_s(x_z) = \pi_s(y_z)$ and
(vi) for each $s \in supp_x(z) \cap supp_x(y)$, $\pi_s(z) \neq \pi_s(y_z)$.

By (v),

$$supp_x(z) \cap supp_x(y) \subseteq T_{x_z,y_z},$$

and by (iv) and (vi),

$$S \setminus (supp_x(z) \cap supp_x(y)) = T_{z,y_z}.$$

Therefore,

$$(S \setminus T_{x_z,y_z}) \subseteq (S \setminus (supp_x(z) \cap supp_x(y))) = T_{z,y_z}.$$

In this way we conclude that $z \in Q_{x_z,y_z}$ which proves Claim 2.

Because of Claim 2, we have $\Sigma_\kappa(F) = X$, and we have finished our proof. □

As a consequence of Lemma 1.2.(2) and Theorems 1.19 and 3.13 we have the following corollaries.

Corollary 3.14. *Let $\{X_s : s \in S\}$ be a family of compact spaces such that for every $J \in [S]^{<\omega}$, $hd(\prod_{s\in J} X_s) \leq \omega$. Let $F \subseteq \prod_{s\in S} X_s = X$ and let Γ be a family of uncountable cardinals. Assume that $\Sigma_\Gamma(F) \neq X$. Then $\Sigma_\Gamma(F)$ is normal if and only if there is a regular cardinal κ and a point $x \in X$ such that $\Sigma_\Gamma(F) = \Sigma_\kappa(x)$.*

Corollary 3.15. *Let* $X = \prod_{s\in S} X_s$ *be a product of compact metric spaces, and let* Γ *be a family of uncountable cardinal numbers and* $F = \{x_i : i \in J\} \subseteq X$*. Then,* $\Sigma_\Gamma(F)$ *is normal if and only if* $\Sigma_\Gamma(F) = X$ *or* $\Sigma_\Gamma(F) = \Sigma_\kappa(x)$ *with* $x \in X$ *and a regular cardinal* κ*.*

Corollary 3.16. *A* $\Sigma_\Gamma(F)$*-product in* $\mathbb{R}^S$ *is normal if and only if either* $\Sigma_\Gamma(F) = \mathbb{R}^S$ *and* $|S| = \omega$ *or* $\Sigma_\Gamma(F) = \Sigma_{\omega_1}(x)$ *for some* $x \in \mathbb{R}^S$*.*

Proof. This follows from Theorems 0.1, 1.19 and 3.13 and Lemma 1.2. □

Corollary 3.17. *Let* $X = \prod_{s\in S} X_s$ *be a product of spaces. Let* Γ *be a family of uncountable cardinal numbers and* $F = \{x_i : i \in J\} \subseteq X$ *such that* $\Sigma_\Gamma(F)$ *is a proper and* C^**-embedded subset of* X*. Then,* $\Sigma_\Gamma(F)$ *is not paracompact.*

Proof. If $\Sigma_\Gamma(F)$ were paracompact, then it would be normal. By Theorem 3.13, $\Sigma_\Gamma(F) = \Sigma_\kappa(x)$ for a $x \in X$ and a regular cardinal κ such that $\omega < \kappa \leq |S|$; but this contradicts Lemma 1.10. □

Questions 3.18.
(1) *Is it possible to find a set* $Z \subseteq [0,1]^S$ *such that* $\Sigma_\kappa(\underline{0}) \cup Z$ *is a non-*$<\kappa$*-bounded (resp., a non-*$<\kappa$*-initially compact) normal space? (see* [6] *y* [9]*).*
(2) *Is it possible to find a set* $Z \subseteq [0,1]^S$ *such that* $\Sigma_\kappa(\underline{0}) \cup Z$ *is a non-p-compact normal space for a* $p \in \omega^*$*? (see* [13]*).*

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Facultad de Ciencias Físico-Matemáticas, Benemerita Universidad Autonoma de Puebla.

E-mail address: jangoa@fcfm.buap.mx

E-mail address: yasserfoc@yahoo.com.mx

Departamento de Matemáticas, Facultad de Ciencias, UNAM.

E-mail address: atamariz@servidor.unam.mx