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**SOME COMPACTNESS PROPERTIES
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AND ULTRAFILTER CONVERGENCE**

PAOLO LIPPARINI

ABSTRACT. We discuss some notions of compactness and convergence relative to a specified family \mathcal{F} of subsets of some topological space X . The two most interesting cases of our construction appear to be

- (1) the case in which \mathcal{F} is the family \mathcal{S} of all singletons of X , in which case we get back the more usual notions;
- (2) the case in which \mathcal{F} is the family \mathcal{O} of all nonempty open subsets of X , in which case we get notions related to pseudocompactness.

A large part of the results in this paper are known for case (1); the results are, in general, new in case (2). As an example, we characterize those spaces which are D -pseudocompact for some ultrafilter D uniform over λ .

1. INTRODUCTION

In this paper, we study various compactness and convergence properties relative to a family \mathcal{F} of subsets of some topological space. In particular, we relativize to \mathcal{F} the notions of D -compactness, CAP_λ , and $[\mu, \lambda]$ -compactness. The two cases which motivate our treatment are when \mathcal{F} is either (1) the family \mathcal{S} of all singletons of X , or (2) the family \mathcal{O} of all nonempty open sets of X . As far as case (2) is concerned, we can equivalently consider nonempty elements of some base, and we can also

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equivalently consider nonempty regular closed sets (sets which are the closure of some nonempty open set).

Throughout the paper, \mathcal{S} and \mathcal{O} will denote the above families.

Our results concern the mutual relationship among the above compactness properties and their behavior with respect to products. Some results which are known for case (1) are generalized to the case of an arbitrary family \mathcal{F} . Apparently, a few results are new even for case (1).

Already for case (2), our results appear to be new. For example, we get characterizations of those spaces which are D -pseudocompact for some ultrafilter D uniform over λ (Corollary 5.5).

Similarly, we get equivalent conditions for the weaker local form asserting that, for every λ -indexed sequence of nonempty open sets of X , there exists some uniform ultrafilter D over λ such that the sequence has some D -limit point in X (Theorem 4.4). When $\lambda = \omega$, we get nothing but more conditions equivalent to pseudocompactness (for Tychonoff spaces).

At first reading, the reader might consider only cases (1) and (2) and look at this paper as a generalization to pseudocompactness-like notions of results already known about ultrafilter convergence and complete accumulation points. Of course, it might be that our definitions and results can be applied to other situations, apart from the two mentioned ones; however, we have not worked out the details yet. Following a suggestion by the referee, we mention that it might be of some interest to study the case when \mathcal{F} is taken to be the family of the zero-sets of some Tychonoff space.

Further elaboration on the notions introduced here is presented in [15] and [17]. The latter manuscript solves a problem by Teklehaimanot Retta [19].

No separation axiom is assumed in the present paper, unless explicitly mentioned.

1.1. SOME HISTORY AND OUR MAIN AIM.

The notion of (pointwise) ultrafilter convergence has proven particularly useful in topology, especially in connection with the study of compactness properties and existence of complete accumulation points, not excluding many other kinds of applications. In particular, ultrafilter convergence is an essential tool in studying compactness properties of products. In a sense made precise in [11], the use of ultrafilters is unavoidable in this situation.

John Ginsburg and Victor Saks' 1975 paper [9] is a pioneering work in applications of pointwise ultrafilter convergence. In addition, it introduces a fundamental new tool, the idea of considering ultrafilter limits of subsets (rather than points) of a topological space. In particular, taking

into consideration ultrafilter limits of nonempty open sets provides deep applications to pseudocompactness, as well as the possibility of introducing further pseudocompactness-like notions. Some analogies, as well as some differences between the two cases were already discussed in [9]. Subsequently, Salvador Garcia-Ferreira [7] analyzed some analogies in more detail.

Ginsburg and Saks' work concentrated on ultrafilters uniform over ω . Generalizations and improvements for ultrafilters over larger cardinals appeared in [20], in the case of pointwise D -convergence, and in [8], in the case of D -pseudocompactness.

A new wave of results, partially inspired by seemingly unrelated problems in mathematical logic, arose when Xavier Caicedo [2] and [3], using ultrafilters, proved some two-cardinals transfer results for compactness of products. For example, among many other things, Caicedo proved that if all powers of some topological space X are $[\lambda^+, \lambda^+]$ -compact, then all powers of X are $[\lambda, \lambda]$ -compact. Subsequently, further results along this line appeared in [11], [12], [13], and [14].

The aim of this paper is twofold. First, we provide analogues for pseudocompactness-like notions of results previously proved only for pointwise convergence; in particular, we provide versions of many results which appeared in [2], [3], [11], and [12].

Our second aim is to insert the two above-mentioned kinds of results into a more general framework. Apart from the advantage of a unified treatment of both cases, we hope that this abstract approach will contribute to clarifying the methods and notions used in the more familiar case of pointwise convergence. Moreover, as we mentioned, Ginsburg and Saks [9] noticed certain analogies between the two cases, but noticed also that there are asymmetries. In our opinion, our treatment provides a very neat explanation for such asymmetries. See the discussion below in subsection 1.2 relative to section 5.

Finally, let us mention that for case (1), a large part of the results presented here are well known; however, even in this particular and well-studied case, we provide a couple of results which might be new: see, e. g., propositions 3.3 and 3.5, and Remark 5.4.

1.2. SYNOPSIS.

Section 2 introduces the notion of D -compactness relative to some family \mathcal{F} of subsets of some topological space X . This provides a common generalization of both pointwise D -compactness and D -pseudocompactness, as introduced by [9] and [8]. Some trivial facts hold about this notion; for example, we can equivalently consider the family of all the closures of elements of \mathcal{F} .

In section 3, we discuss the notion of a complete accumulation point relative to \mathcal{F} . In fact, two versions are presented: the first one, starred, deals with *sequences* of subsets, and the second one, unstarred, deals with *sets* of subsets. That is, in the starred case, repetitions are allowed, while in the unstarred case, they are not allowed. The difference between the two cases is essential only when dealing with singular cardinals (Proposition 3.3). In the classical case when $\mathcal{F} = \mathcal{S}$, the unstarred notion is most used in the literature; however, we show that the exact connection between the notion of a D -limit point and the existence of a complete accumulation point holds only for the starred variant (Proposition 4.1).

In section 4, we introduce a generalization of $[\mu, \lambda]$ -compactness which also depends on \mathcal{F} , and in Theorem 4.4, we prove the equivalence among many of the \mathcal{F} -dependent notions defined before.

Section 5 discusses the behavior of the above notions in connection with (Tychonoff) products. Actually, for the sake of simplicity only, we deal mostly with powers. Since, in our notions, a topological space X comes equipped with a family \mathcal{F} of subsets attached to it, we have to specify which family should be attached to the power X^δ . In order to get significant results, the right choice is to attach to X^δ the family \mathcal{F}^δ consisting of all products of δ members of \mathcal{F} (some variations are possible). In the case when \mathcal{F} is the family of all the singletons of X , then \mathcal{F}^δ turns out to be the family of all singletons of X^δ again; thus, we get back the classical results about ultrafilter convergence in products. On the other hand, when \mathcal{F} is the family of all nonempty open subsets of X , then \mathcal{F}^δ , in general, contains certain sets which are not open in X^δ ; in fact, \mathcal{F}^δ is a base for the box topology on X^δ , a topology generally strictly finer than the Tychonoff topology.

The above fact explains the reason why, in the case of products, there is not a total symmetry between results on compactness and results about pseudocompactness. For example, as already noticed in [9], it is true that all powers of some topological space X are countably compact if and only if X is D -compact, for some ultrafilter D uniform over ω . On the other hand, Ginsburg and Saks [9] constructed a Tychonoff space X such that all powers of X are pseudocompact, but there exists no uniform ultrafilter D over ω such that X is D -pseudocompact. Our framework not only explains the reason for this asymmetry, but can be used in order to provide a characterization of D -pseudocompact spaces, a characterization parallel to that of D -compact spaces. Indeed, we do find versions for D -pseudocompactness of the classical results about D -convergence (Corollary 5.5). Though statements become a little more involved, we believe that these results have some intrinsic interest.

In section 6, we show that cardinal transfer results for decomposable ultrafilters deeply affect compactness properties relative to these cardinals. More exactly, if λ and μ are cardinals such that every uniform ultrafilter over λ is μ -decomposable, then every topological space X which is \mathcal{F} - D -compact for some ultrafilter D uniform over λ , is also \mathcal{F} - D' -compact for some ultrafilter D' uniform over μ . Of course, this result applies also to all the equivalent notions discussed in the preceding sections. Since there are highly nontrivial set theoretical results on transfer of ultrafilter decomposability, our theorems provide deep unexpected applications of set theory to compactness properties of products. We appropriately reformulate known results about decomposability of ultrafilters in such a way that they can be applied to the present context. We also get a few new results about the relation $\lambda \overset{\infty}{\cong} K$: “every uniform ultrafilter over λ is μ -decomposable for some $\mu \in K$,” where K is a class of infinite cardinals. The results here generalize some results which appear in [11] and [16]. At the end of the section, we show that the relation $\lambda \overset{\infty}{\cong} K$ can be equivalently expressed in terms of pseudocompactness properties of box products (Corollary 6.8).

Finally, in section 7, we discuss still another generalization of $[\lambda, \mu]$ -compactness. Again, there are relationships with the other compactness properties which were introduced before, as well as with further variations on pseudocompactness. The notions introduced here are probably worthy of further study.

2. D -COMPACTNESS RELATIVE TO SOME FAMILY \mathcal{F}

Suppose that D is an ultrafilter over some set Z and X is a topological space.

A sequence $(x_z)_{z \in Z}$ of (not necessarily distinct) elements of X is said to D -converge to some point $x \in X$ if and only if $\{z \in Z \mid x_z \in U\} \in D$ for every neighborhood U of x in X .

The space X is said to be D -compact if and only if every sequence $(x_z)_{z \in Z}$ of elements of X converges to some point of X .

If $(Y_z)_{z \in Z}$ is a sequence of (not necessarily distinct) subsets of X , then x is called a D -limit point of $(Y_z)_{z \in Z}$ if and only if $\{z \in Z \mid Y_z \cap U \neq \emptyset\} \in D$ for every open neighborhood U of x in X .

Since $Y_z \cap U \neq \emptyset$ if and only if $\bar{Y}_z \cap U \neq \emptyset$, we have that x is a D -limit point of $(Y_z)_{z \in Z}$ if and only if x is a D -limit point of $(\bar{Y}_z)_{z \in Z}$.

The space X is said to be D -pseudocompact if and only if every sequence $(O_z)_{z \in Z}$ of nonempty open subsets of X has some D -limit point in X . The above notion is due to [9, Definition 4.1] for non-principal ultrafilters over ω and appears in [8] for uniform ultrafilters over arbitrary cardinals. In [9] and [8], the above definitions are given for Tychonoff spaces. When

X is not assumed to be Tychonoff, R. M. Stephenson, Jr., [23] uses the term *D-feebly compact*.

The above notions can be simultaneously generalized as follows.

Definition 2.1. Suppose that D is an ultrafilter over some set Z , X is a topological space, and \mathcal{F} is a specified family of subsets of X . We say that the space X is *\mathcal{F} -D-compact* if and only if every sequence $(F_z)_{z \in Z}$ of members of \mathcal{F} has some D -limit point in X .

Thus, we get the notion of D -compactness when $\mathcal{F} = \mathcal{S}$, and we get the notion of D -pseudocompactness when $\mathcal{F} = \mathcal{O}$.

If \mathcal{G} is another family of subsets of X , let us write $\mathcal{F} \triangleright \mathcal{G}$ to mean that, for every $F \in \mathcal{F}$, there is $G \in \mathcal{G}$ such that $F \supseteq G$.

With this notation, it is trivial to show that if $\mathcal{F} \triangleright \mathcal{G}$ and X is \mathcal{G} - D -compact, then X is \mathcal{F} - D -compact.

If \mathcal{F} is a family of nonempty subsets of X , and \mathcal{F} contains all singletons, then both $\mathcal{F} \triangleright \mathcal{S}$ and $\mathcal{S} \triangleright \mathcal{F}$; hence, \mathcal{F} - D -compactness is the same as D -compactness.

If \mathcal{F} is a family of subsets of X , let $\overline{\mathcal{F}} = \{\overline{F} \mid F \in \mathcal{F}\}$ be the set of all closures of elements of \mathcal{F} . With this notation, it is trivial to show that X is \mathcal{F} - D -compact if and only if X is $\overline{\mathcal{F}}$ - D -compact.

The most interesting cases in Definition 2.1 appear to be the two mentioned ones, that is, when either $\mathcal{F} = \mathcal{S}$ or $\mathcal{F} = \mathcal{O}$.

When $\mathcal{F} = \mathcal{S}$, most of the results we prove here are essentially known, except for the technical difference that we deal with sequences rather than subsets. The difference is substantial only when dealing with singular cardinals. See Remark 3.2 and Proposition 3.3.

When $\mathcal{F} = \mathcal{O}$, most of our results appear to be new.

Remark 2.2. Notice that if \mathcal{B} is a base (consisting of nonempty sets) of the topological space X , then both $\mathcal{O} \triangleright \mathcal{B}$ and $\mathcal{B} \triangleright \mathcal{O}$. Hence, \mathcal{O} - D -compactness is the same as \mathcal{B} - D -compactness. A similar remark applies to all compactness properties we shall introduce later (except for those introduced in section 7).

3. COMPLETE ACCUMULATION POINTS RELATIVE TO \mathcal{F}

We are now going to generalize the notion of an accumulation point.

Definition 3.1. If λ is an infinite cardinal and $(Y_\alpha)_{\alpha \in \lambda}$ is a sequence of subsets of some topological space X , we say that $x \in X$ is a *λ -complete accumulation point* of $(Y_\alpha)_{\alpha \in \lambda}$ if and only if $|\{\alpha \in \lambda \mid Y_\alpha \cap U \neq \emptyset\}| = \lambda$, for every neighborhood U of x in X .

In case $\lambda = \omega$, we get the usual notion of a *cluster point*.

Notice that x is a λ -complete accumulation point of $(Y_\alpha)_{\alpha \in \lambda}$ if and only if x is a λ -complete accumulation point of $(\overline{Y}_\alpha)_{\alpha \in \lambda}$.

If \mathcal{F} is a family of subsets of X , we say that X satisfies \mathcal{F} -CAP $^*_\lambda$ if and only if every sequence $(F_\alpha)_{\alpha \in \lambda}$ of members of \mathcal{F} has a λ -complete accumulation point.

If $\mathcal{F} \triangleright \mathcal{G}$ and X satisfies \mathcal{G} -CAP $^*_\lambda$, then X satisfies \mathcal{F} -CAP $^*_\lambda$. Moreover, X satisfies \mathcal{F} -CAP $^*_\lambda$ if and only if it satisfies $\overline{\mathcal{F}}$ -CAP $^*_\lambda$.

Notice that if X is a Tychonoff space, then a result by Irving Glicksberg [10], when reformulated in the present terminology, asserts that \mathcal{O} -CAP $^*_\omega$ is equivalent to pseudocompactness. See also [9, §4], [8], and [23]. Without assuming X to be Tychonoff, \mathcal{O} -CAP $^*_\omega$ turns out to be equivalent to a condition which is usually called *feeble compactness*. See Remark 4.5.

More generally, \mathcal{O} -CAP $^*_\kappa$ is called *pseudo- (κ, κ) -compactness* in [4]. See [6], [19], and [17] for the study of related notions.

Remark 3.2. When each Y_α is a singleton in Definition 3.1 and all such singletons are distinct, we get back the usual notion of a complete accumulation point.

A point $x \in X$ is said to be a *complete accumulation point* of some infinite subset $Y \subseteq X$ if and only if $|Y \cap U| = |Y|$ for every neighborhood U of x in X .

A topological space X satisfies CAP $_\lambda$ if and only if every subset $Y \subseteq X$ with $|Y| = \lambda$ has a complete accumulation point.

When λ is a singular cardinal, there is some difference between the classic notion of a complete accumulation point and the notion of a λ -complete accumulation point, as introduced in Definition 3.1. This happens because, for our purposes, it is more convenient to deal with sequences rather than subsets, that is, we allow repetitions. This is the reason for the $*$ in \mathcal{F} -CAP $^*_\lambda$ in Definition 3.1.

As pointed out in [14, Proposition 1], if $\mathcal{F} = \mathcal{S}$, then, for λ regular, \mathcal{F} -CAP $^*_\lambda$ is equivalent to CAP $_\lambda$, and, for λ singular, \mathcal{F} -CAP $^*_\lambda$ is equivalent to the conjunction of CAP $_\lambda$ and CAP $_{\text{cf } \lambda}$.

In fact, a more general result holds for families of nonempty sets. In order to clarify the situation, let us introduce the following unstarred variant of \mathcal{F} -CAP $^*_\lambda$. If \mathcal{F} is a family of subsets of X , we say that X satisfies \mathcal{F} -CAP $_\lambda$ if and only if every sequence $(F_\alpha)_{\alpha \in \lambda}$ of *distinct* members of \mathcal{F} has a λ -complete accumulation point.

Then we have the following proposition.

Proposition 3.3. *Suppose that X is a topological space and \mathcal{F} is a family of nonempty subsets of X .*

- (a) *If λ is a regular cardinal, then X satisfies \mathcal{F} -CAP $^*_\lambda$ if and only if X satisfies \mathcal{F} -CAP $_\lambda$.*

- (b) *If λ is a singular cardinal, then X satisfies \mathcal{F} -CAP $^*_\lambda$ if and only if X satisfies both \mathcal{F} -CAP $_\lambda$ and \mathcal{F} -CAP $_{cf\lambda}$.*

Proof. It is obvious that \mathcal{F} -CAP $^*_\lambda$ implies \mathcal{F} -CAP $_\lambda$ for every cardinal λ .

Suppose that λ is regular, that \mathcal{F} -CAP $_\lambda$ holds, and that $(F_\alpha)_{\alpha \in \lambda}$ is a sequence of elements of \mathcal{F} . If some subsequence consists of λ -many distinct elements, then, by \mathcal{F} -CAP $_\lambda$, this subsequence has some λ -complete accumulation point which necessarily is also a λ -complete accumulation point for $(F_\alpha)_{\alpha \in \lambda}$. Otherwise, since λ is regular, there exists some $F \in \mathcal{F}$ which appears λ -many times in $(F_\alpha)_{\alpha \in \lambda}$. Since, by assumption, F is nonempty, just take some $x \in F$ to get a λ -complete accumulation point for $(F_\alpha)_{\alpha \in \lambda}$. Thus, we have proved that \mathcal{F} -CAP $_\lambda$ implies \mathcal{F} -CAP $^*_\lambda$ for λ regular.

Now suppose that λ is singular and that both \mathcal{F} -CAP $_\lambda$ and \mathcal{F} -CAP $_{cf\lambda}$ hold. We are going to show that \mathcal{F} -CAP $^*_\lambda$ holds. Let $(F_\alpha)_{\alpha \in \lambda}$ be a sequence of elements of \mathcal{F} . There are three cases.

Case i: There exists some $F \in \mathcal{F}$ which appears λ -many times in $(F_\alpha)_{\alpha \in \lambda}$. In this case, as above, it is enough to choose some element from F .

Case ii: Some subsequence of $(F_\alpha)_{\alpha \in \lambda}$ consists of λ -many distinct elements. Then, as above, apply \mathcal{F} -CAP $_\lambda$ to this subsequence.

Case iii: Otherwise, $(F_\alpha)_{\alpha \in \lambda}$ consists of $< \lambda$ different elements, each one appearing $< \lambda$ times. Moreover, if $(\lambda_\beta)_{\beta \in cf\lambda}$ is a sequence of cardinals $< \lambda$ whose supremum is λ , then, for every $\beta \in cf\lambda$, there is $F_\beta \in \mathcal{F}$ appearing at least λ_β -many times. Since, for each β , F_β appears $< \lambda$ times, we can choose $cf\lambda$ -many distinct F_β 's as above. Applying \mathcal{F} -CAP $_{cf\lambda}$ to those F_β 's, we get a λ -complete accumulation point for $(F_\alpha)_{\alpha \in \lambda}$.

It remains to show that, for λ singular, \mathcal{F} -CAP $^*_\lambda$ implies \mathcal{F} -CAP $_{cf\lambda}$. Suppose that $(\lambda_\beta)_{\beta \in cf\lambda}$ is a sequence of cardinals $< \lambda$ whose supremum is λ . If $(F_\beta)_{\beta \in cf\lambda}$ is a sequence of distinct members of \mathcal{F} , let $(G_\alpha)_{\alpha \in \lambda}$ be a sequence such that, for every $\beta \in cf\lambda$, $G_\alpha = F_\beta$ for exactly λ_β -many α 's. By \mathcal{F} -CAP $^*_\lambda$, $(G_\alpha)_{\alpha \in \lambda}$ has a λ -complete accumulation point x . It is immediate to show that x is also a $cf\lambda$ -complete accumulation point for $(F_\beta)_{\beta \in cf\lambda}$. \square

If D is an ultrafilter, Y is a D -compact Hausdorff space, and $X \subseteq Y$, then there is the smallest D -compact subspace Z of Y containing X . This is because the intersection of any family of D -compact subspaces of Y is still D -compact, since, in a Hausdorff space, the D -limit of a sequence is unique (if it exists). Such a Z can also be constructed by an iteration procedure in $|I|^+$ stages if D is over I . This is similar to [9, Theorem 2.12].

If X is a Tychonoff space and $Y = \beta(X)$ is the Stone-Ćech compactification of X , the smallest D -compact subspace of $\beta(X)$ containing X is called the D -compactification of X and is denoted by $\beta_D(X)$. See [7, p. 14], [8], or [9] for further references and alternative definitions of the D -compactification (sometimes also called D -compact reflection).

Example 3.4. (a) If λ is singular, then $\text{cf } \lambda$, endowed with either the order topology or the discrete topology, fails to satisfy $\text{CAP}_{\text{cf } \lambda}$, but trivially satisfies CAP_λ .

(b) By propositions 3.3 and 4.1, if D is an ultrafilter uniform over λ and Y is a D -compact topological space, then Y satisfies CAP_λ . In particular, if X is any Tychonoff space and D is uniform over λ , then the D -compactification $\beta_D(X)$ of X satisfies CAP_λ .

(c) If X is λ with the discrete topology, then X does not satisfy CAP_λ . By (b), if D is an ultrafilter uniform over $\text{cf } \lambda$, then the D -compactification $\beta_D(X)$ of X satisfies $\text{CAP}_{\text{cf } \lambda}$. However, if λ is singular, $\beta_D(X)$ does not satisfy CAP_λ . Thus, we have a space satisfying $\text{CAP}_{\text{cf } \lambda}$, but not satisfying CAP_λ .

(d) In order to get an example as in (c), it is not sufficient to take any space X which does not satisfy CAP_λ . Indeed, if X is λ with the order topology, then $\beta_D(X)$ does satisfy CAP_λ whenever D is an ultrafilter uniform over $\text{cf } \lambda$.

The next proposition shows that, when λ is a singular cardinal, $\text{CAP}_{\text{cf } \lambda}$ implies $\mathcal{F}\text{-CAP}_\lambda^*$, provided that $\mathcal{F}\text{-CAP}_\mu$ holds for a set of cardinals unbounded in λ .

Proposition 3.5. *Suppose that X is a topological space, \mathcal{F} is a family of nonempty subsets of X , λ is a singular cardinal, and $(\lambda_\beta)_{\beta \in \text{cf } \lambda}$ is a sequence of cardinals $< \lambda$ such that $\sup_{\beta \in \text{cf } \lambda} \lambda_\beta = \lambda$.*

If X satisfies $\text{CAP}_{\text{cf } \lambda}$, as well as $\mathcal{F}\text{-CAP}_{\lambda_\beta}$, for every $\beta \in \text{cf } \lambda$, then X satisfies $\mathcal{F}\text{-CAP}_\lambda^$.*

In particular, if X satisfies $\text{CAP}_{\text{cf } \lambda}$, and $\text{CAP}_{\lambda_\beta}$, for every $\beta \in \text{cf } \lambda$, then X satisfies CAP_λ^ .¹*

Proof. We first prove that X satisfies $\mathcal{F}\text{-CAP}_\lambda$. The proof takes some ideas from [20, proof of Proposition, p. 94]. So, let $(F_\alpha)_{\alpha \in \lambda}$ be a sequence of distinct elements of \mathcal{F} . For every $\beta \in \text{cf } \lambda$, by $\mathcal{F}\text{-CAP}_{\lambda_\beta}$, we get some element x_β which is a λ_β -complete accumulation point for $(F_\alpha)_{\alpha \in \lambda_\beta}$. By $\text{CAP}_{\text{cf } \lambda}^*$ (which follows from $\text{CAP}_{\text{cf } \lambda}$, by Proposition 3.3(a)), the sequence $(x_\beta)_{\beta \in \text{cf } \lambda}$ has some $\text{cf } \lambda$ -complete accumulation point x . It is now easy to see that x is a λ -complete accumulation point for $(F_\alpha)_{\alpha \in \lambda}$.

¹This statement has appeared in [14, p. 2].

Since the members of \mathcal{F} are nonempty, then $\text{CAP}_{\text{cf } \lambda}$ implies $\mathcal{F}\text{-CAP}_{\text{cf } \lambda}$; hence, $\mathcal{F}\text{-CAP}_{\lambda}^*$ follows from $\mathcal{F}\text{-CAP}_{\lambda}$ by Proposition 3.3(b).

The last statement follows by taking $\mathcal{F} = \mathcal{S}$. \square

4. RELATIONSHIP AMONG COMPACTNESS PROPERTIES

In the next proposition, we deal with the fundamental relationship, for a given sequence, between the existence of a λ -complete accumulation point and the existence of a D -limit point for D uniform over λ . Then, in Theorem 4.4, we shall present more equivalent formulations referring to various compactness properties.

Proposition 4.1. *Suppose that λ is an infinite cardinal and that $(Y_{\alpha})_{\alpha \in \lambda}$ is a sequence of subsets of some topological space X .*

Then $x \in X$ is a λ -complete accumulation point of $(Y_{\alpha})_{\alpha \in \lambda}$ if and only if there exists an ultrafilter D uniform over λ such that x is a D -limit point of $(Y_{\alpha})_{\alpha \in \lambda}$.

In particular, $(Y_{\alpha})_{\alpha \in \lambda}$ has a λ -complete accumulation point if and only if $(Y_{\alpha})_{\alpha \in \lambda}$ has a D -limit point for some ultrafilter D uniform over λ .

Proof. If $x \in X$ is a λ -complete accumulation point of $(Y_{\alpha})_{\alpha \in \lambda}$, then the family \mathcal{H} consisting of the sets $\{\alpha \in \lambda \mid Y_{\alpha} \cap U \neq \emptyset\}$ (U a neighborhood of x) and $\lambda \setminus Z$ ($|Z| < \lambda$) has the finite intersection property. Indeed, the intersection of any finite set of members of \mathcal{H} has cardinality λ . Hence, \mathcal{H} can be extended to some ultrafilter D , which is necessarily uniform over λ . It is trivial to see that for such a D , x is a D -limit point of $(Y_{\alpha})_{\alpha \in \lambda}$.

The converse is trivial, since the ultrafilter D is assumed to be uniform over λ . \square

The particular case of Proposition 4.1, in which all Y_{α} 's are distinct one-element sets, is well known. See [20, pp. 80–81].

Definition 4.2. If X is a topological space and \mathcal{F} is a family of subsets of X , we say that X is $\mathcal{F}\text{-}[\mu, \lambda]\text{-compact}$ if and only if the following holds.

For every sequence $(C_{\alpha})_{\alpha \in \lambda}$ of closed sets of X , if, for every $Z \subseteq \lambda$ with $|Z| < \mu$, there exists $F \in \mathcal{F}$ such that $\bigcap_{\alpha \in Z} C_{\alpha} \supseteq F$, then $\bigcap_{\alpha \in \lambda} C_{\alpha} \neq \emptyset$.

Of course, when $\mathcal{F} = \mathcal{S}$, we have that $\mathcal{F}\text{-}[\mu, \lambda]\text{-compactness}$ is the usual notion of $[\mu, \lambda]\text{-compactness}$.

On the other hand, when $\mathcal{F} = \mathcal{O}$, we get notions related to pseudo-compactness. See Remark 4.5. Notice that, by taking complements, we get that $\mathcal{O}\text{-}[\mu, \lambda]\text{-compactness}$ is equivalent to the following statement. For every λ -indexed open cover $(O_{\alpha})_{\alpha \in \lambda}$ of X , there exists $Z \subseteq \lambda$, with $|Z| < \mu$, such that $\bigcup_{\alpha \in Z} O_{\alpha}$ is dense in X .

In the above sense, $\mathcal{O}\text{-}[\omega, \lambda]\text{-compactness}$ has been introduced and studied in [6], where Zdeněk Frolík calls it *almost λ -compactness*. Moreover,

he introduced the notion which corresponds to \mathcal{O} - $[\mu, \lambda]$ -compactness for all cardinals λ , calling it *almost μ -Lindelöfness*. \mathcal{O} - $[\omega, \lambda]$ -compactness has been studied also in [21] under the name *weak- λ - \aleph_0 -compactness* and in [19] and [24] under the name *weak initial λ -compactness*.

Remark 4.3. Trivially, if $\mathcal{F} \triangleright \mathcal{G}$, and X is \mathcal{G} - $[\mu, \lambda]$ -compact, then X is \mathcal{F} - $[\mu, \lambda]$ -compact.

Recall that if \mathcal{F} is a family of subsets of X , we have defined $\overline{\mathcal{F}} = \{\overline{F} \mid F \in \mathcal{F}\}$. It is trivial to observe that X is \mathcal{F} - $[\mu, \lambda]$ -compact if and only if X is $\overline{\mathcal{F}}$ - $[\mu, \lambda]$ -compact.

Theorem 4.4. *Suppose that X is a topological space, \mathcal{F} is a family of subsets of X , and λ is a regular cardinal. Then the following conditions are equivalent.*

- (a) X is \mathcal{F} - $[\lambda, \lambda]$ -compact.
- (b) Suppose that $(C_\alpha)_{\alpha \in \lambda}$ is a sequence of closed sets of X such that $C_\alpha \supseteq C_\beta$ whenever $\alpha \leq \beta < \lambda$. If, for every $\alpha \in \lambda$, there exists $F \in \mathcal{F}$ such that $C_\alpha \supseteq F$, then $\bigcap_{\alpha \in \lambda} C_\alpha \neq \emptyset$.
- (b₁) Suppose that $(C_\alpha)_{\alpha \in \lambda}$ is a sequence of closed sets of X such that $C_\alpha \supseteq C_\beta$ whenever $\alpha \leq \beta < \lambda$. Suppose further that, for every $\alpha \in \lambda$, C_α is the closure of the union of some set of members of \mathcal{F} . If, for every $\alpha \in \lambda$, there exists $F \in \mathcal{F}$ such that $C_\alpha \supseteq F$, then $\bigcap_{\alpha \in \lambda} C_\alpha \neq \emptyset$.
- (b₂) Suppose that $(C_\alpha)_{\alpha \in \lambda}$ is a sequence of closed sets of X such that $C_\alpha \supseteq C_\beta$ whenever $\alpha \leq \beta < \lambda$. Suppose further that, for every $\alpha \in \lambda$, C_α is the closure of the union of some set of $\leq \lambda$ members of \mathcal{F} . If, for every $\alpha \in \lambda$, there exists $F \in \mathcal{F}$ such that $C_\alpha \supseteq F$, then $\bigcap_{\alpha \in \lambda} C_\alpha \neq \emptyset$.
- (c) Every sequence $(F_\alpha)_{\alpha \in \lambda}$ of elements of \mathcal{F} has a λ -complete accumulation point (that is, X satisfies \mathcal{F} -CAP $^*_\lambda$).
- (d) For every sequence $(F_\alpha)_{\alpha \in \lambda}$ of elements of \mathcal{F} , there exists some ultrafilter D uniform over λ such that $(F_\alpha)_{\alpha \in \lambda}$ has a D -limit point.
- (e) For every λ -indexed open cover $(O_\alpha)_{\alpha \in \lambda}$ of X , there exists $Z \subseteq \lambda$, with $|Z| < \lambda$, such that, for every $F \in \mathcal{F}$, $F \cap \bigcup_{\alpha \in Z} O_\alpha \neq \emptyset$.
- (f) For every λ -indexed open cover $(O_\alpha)_{\alpha \in \lambda}$ of X , such that $O_\alpha \subseteq O_\beta$ whenever $\alpha \leq \beta < \lambda$, there exists $\alpha \in \lambda$ such that O_α intersects each $F \in \mathcal{F}$.

In each of the above conditions, we can equivalently replace \mathcal{F} by $\overline{\mathcal{F}}$.

If $\mathcal{F} \triangleright \mathcal{G}$ and $\mathcal{G} \triangleright \mathcal{F}$, then in each of the above conditions, we can equivalently replace \mathcal{F} by \mathcal{G} .

Proof. The implication (a) \Rightarrow (b) is obvious, since λ is regular.

We now prove the implication (b) \Rightarrow (a). Suppose that (b) holds and that $(C_\alpha)_{\alpha \in \lambda}$ are closed sets of X such that, for every $Z \subseteq \lambda$ with $|Z| < \mu$, there exists $F \in \mathcal{F}$ such that $\bigcap_{\alpha \in Z} C_\alpha \supseteq F$.

For $\alpha \in \lambda$, define $D_\alpha = \bigcap_{\beta < \alpha} C_\beta$. The D_α 's are closed sets of X and satisfy the assumption in (b); hence, $\bigcap_{\alpha \in \lambda} D_\alpha \neq \emptyset$. But $\bigcap_{\alpha \in \lambda} C_\alpha = \bigcap_{\alpha \in \lambda} D_\alpha \neq \emptyset$; thus, (a) is proved.

(b) \Rightarrow (b₁) \Rightarrow (b₂) are trivial.

(b₂) \Rightarrow (c). Suppose that (b₂) holds and that $(F_\alpha)_{\alpha \in \lambda}$ are elements of \mathcal{F} . For $\alpha \in \lambda$, let C_α be the closure of $\bigcup_{\beta > \alpha} F_\beta$. The C_α 's satisfy the assumptions in (b₂); hence, $\bigcap_{\alpha \in \lambda} C_\alpha \neq \emptyset$. Let $x \in \bigcap_{\alpha \in \lambda} C_\alpha$. We want to show that x is a λ -complete accumulation point for $(F_\alpha)_{\alpha \in \lambda}$. Indeed, suppose by contradiction that $|\{\alpha \in \lambda \mid F_\alpha \cap U \neq \emptyset\}| < \lambda$ for some neighborhood U of x in X . If $\beta = \sup\{\alpha \in \lambda \mid F_\alpha \cap U \neq \emptyset\}$, then $\beta < \lambda$ since λ is regular and we are taking the supremum of a set of cardinality $< \lambda$. Thus, $F_\alpha \cap U = \emptyset$ for every $\alpha > \beta$; hence, $U \cap \bigcup_{\alpha > \beta} F_\alpha = \emptyset$ and $x \notin C_\beta$, a contradiction.

(c) \Rightarrow (b). Suppose that (c) holds, and that $(C_\alpha)_{\alpha \in \lambda}$ satisfies the premise of (b). For each $\alpha \in \lambda$, choose $F_\alpha \in \mathcal{F}$ with $F_\alpha \subseteq C_\alpha$. By (c), $(F_\alpha)_{\alpha \in \lambda}$ has a λ -complete accumulation point x . Hence, for every neighborhood U of x , there are arbitrarily large $\alpha < \lambda$ such that U intersects F_α , so there are arbitrarily large $\alpha < \lambda$ such that U intersects C_α ; hence, U intersects every C_α since the C_α 's form a decreasing sequence. In conclusion, for every $\alpha \in \lambda$, every neighborhood of x intersects C_α . That is, $x \in C_\alpha$, since C_α is closed.

(c) \Leftrightarrow (d) is immediate from Proposition 4.1.

(e) and (f) are obtained from (a) and (b), respectively, by taking complements.

It follows from various preceding remarks that we get equivalent conditions when we replace \mathcal{F} by $\overline{\mathcal{F}}$ or by \mathcal{G} , if $\mathcal{F} \triangleright \mathcal{G}$ and $\mathcal{G} \triangleright \mathcal{F}$. \square

When $\mathcal{F} = \mathcal{S}$, the equivalence of the conditions in Theorem 4.4 (except perhaps for (b₁) and (b₂)) is well known and, for the most part, dates back already to P. S. Alexandrov and P. S. Urysohn's classical survey [1]. See also [25] and [26] for further comments and references.

Remark 4.5. When $\lambda = \omega$, X is Tychonoff, and $\mathcal{F} = \mathcal{O}$, in Theorem 4.4 we get conditions equivalent to pseudocompactness, since, as we mentioned, a result by Glicksberg [10] implies that, for Tychonoff spaces, $\mathcal{O}\text{-CAP}_\omega^*$ is equivalent to pseudocompactness. Some of these equivalences are known: For example, condition (e) becomes condition (C₅) in [23]. Without assuming X to be Tychonoff, a space satisfying the conditions

in Theorem 4.4 (when $\mathcal{F} = \mathcal{O}$) is called *feebly compact* by many authors. See [23] for further references.

Corollary 4.6. *Suppose that X is a topological space, \mathcal{F} is a family of subsets of X , and λ is a regular cardinal. If X is \mathcal{F} - D -compact for some ultrafilter D uniform over λ , then all the conditions in Theorem 4.4 hold.*

Proof. If X is \mathcal{F} - D -compact for some ultrafilter D uniform over λ , then Theorem 4.4(d) holds; hence, all the other equivalent conditions hold. \square

5. BEHAVIOR WITH RESPECT TO PRODUCTS

We now discuss the behavior of \mathcal{F} - D -compactness with respect to products.

Proposition 5.1. *Suppose that $(X_i)_{i \in I}$ is a sequence of topological spaces and let $X = \prod_{i \in I} X_i$ with the Tychonoff topology. Let D be an ultrafilter over λ .*

(a) *Suppose that, for each $i \in I$, $(Y_{i,\alpha})_{\alpha \in \lambda}$ is a sequence of subsets of X_i . Then some point $x = (x_i)_{i \in I}$ is a D -limit point of $(\prod_{i \in I} Y_{i,\alpha})_{\alpha \in \lambda}$ in X if and only if, for each $i \in I$, x_i is a D -limit point of $(Y_{i,\alpha})_{\alpha \in \lambda}$ in X_i .*

In particular, $(\prod_{i \in I} Y_{i,\alpha})_{\alpha \in \lambda}$ has a D -limit point in X if and only if, for each $i \in I$, $(Y_{i,\alpha})_{\alpha \in \lambda}$ has a D -limit point in X_i .

(b) *Suppose that, for each $i \in I$, \mathcal{F}_i is a family of subsets of X_i and suppose that, for some cardinal $\nu > 1$, \mathcal{F} is the family of all subsets of X of the form $\prod_{i \in I} F_i$, where, for some $I' \subseteq I$ with $|I'| < \nu$, we have that F_i belongs to \mathcal{F}_i for $i \in I'$, and that $F_i = X_i$ for $i \in I \setminus I'$.*

Then X is \mathcal{F} - D -compact if and only if X_i is \mathcal{F}_i - D -compact for every $i \in I$.

Theorem 5.2. *Suppose that X is a topological space and that \mathcal{F} is a family of subsets of X . For every cardinal δ , let X^δ be the δ^{th} power of X endowed with the Tychonoff topology, and let \mathcal{F}^δ be the family of all products of δ members of \mathcal{F} . Then, for every cardinal λ , the following are equivalent.*

- (1) *There exists some ultrafilter D uniform over λ such that X is \mathcal{F} - D -compact.*
- (2) *There exists some ultrafilter D uniform over λ such that, for every cardinal δ , the space X^δ is \mathcal{F}^δ - D -compact.*
- (3) *X^δ satisfies \mathcal{F}^δ -CAP $^*_\lambda$ for every cardinal δ .*
- (4) *X^δ satisfies \mathcal{F}^δ -CAP $^*_\lambda$ for $\delta = \min\{2^{2^\lambda}, |\mathcal{F}|^\lambda\}$.*

If λ is regular, then further conditions equivalent to (3) and (4) are obtained by applying Theorem 4.4 to X^δ and \mathcal{F}^δ .

Proof. (1) \Rightarrow (2) follows from Proposition 5.1(b), with $I = \delta$ and $\nu = \delta^+$.

(2) \Rightarrow (3) follows from Proposition 4.1.

(3) \Rightarrow (4) is trivial.

(4) \Rightarrow (1). We first consider the case $\delta = |\mathcal{F}|^\lambda$. Thus, there are δ -many λ -indexed sequences of elements of \mathcal{F} . Let us enumerate them as $(F_{\beta,\alpha})_{\alpha \in \lambda}$, β varying in δ .

In X^δ , consider the sequence $(\prod_{\beta \in \delta} F_{\beta,\alpha})_{\alpha \in \lambda}$ of elements of \mathcal{F}^δ . By (4), the sequence has a λ -complete accumulation point and, by Proposition 4.1, there exists a ultrafilter D uniform over λ such that $(\prod_{\beta \in \delta} F_{\beta,\alpha})_{\alpha \in \lambda}$ has a D -limit point x in X^δ . Say, $x = (x_\beta)_{\beta \in \delta}$. By Proposition 5.1(a), for every $\beta \in \delta$, x_β is a D -limit point of $(F_{\beta,\alpha})_{\alpha \in \lambda}$ in X .

Since every λ -indexed sequence of elements of \mathcal{F} has the form $(F_{\beta,\alpha})_{\alpha \in \lambda}$ for some $\beta \in \delta$, we have that every λ -indexed sequence of elements of \mathcal{F} has some D -limit point in X , that is, X is \mathcal{F} - D -compact.

Now we consider the case $\delta = 2^{2^\lambda}$. We shall prove that if $\delta = 2^{2^\lambda}$ and (1) fails, then (4) fails. If (1) fails, then, for every ultrafilter D uniform over λ , there is a sequence $(F_\alpha)_{\alpha \in \lambda}$ of elements in \mathcal{F} which has no D -limit point. Since there are δ -many ultrafilters over λ , we can enumerate the above sequences as $(F_{\beta,\alpha})_{\alpha \in \lambda}$, β varying in δ .

Now the sequence $(\prod_{\beta \in \delta} F_{\beta,\alpha})_{\alpha \in \lambda}$ in X^δ has no λ -complete accumulation point in X^δ since, otherwise, by Proposition 4.1, for some ultrafilter D uniform over λ , it would have some D -limit point in X^δ . However, this contradicts Proposition 5.1(a) since, by assumption, there is a β such that $(F_{\beta,\alpha})_{\alpha \in \lambda}$ has no D -limit point. \square

Remark 5.3. Suppose that $\mathcal{F} = \mathcal{O}$ in Theorem 5.2. Then in (3) and (4), we cannot replace \mathcal{O}^δ by the family $\mathcal{O}(X^\delta)$ of all nonempty open subsets of X^δ . Indeed, if X is a Tychonoff space and we take $\lambda = \omega$, then $\mathcal{O}(X^\delta)$ -CAP $^*_\omega$ for X^δ is equivalent to the pseudocompactness of X^δ . However, Ginsburg and Saks [9, Example 4.4] constructed a Tychonoff space X such that all powers of X are pseudocompact, but there exists no uniform ultrafilter D over ω such that X is D -pseudocompact. Thus, (3) \Rightarrow (1) becomes false, in general, if we choose $\mathcal{O}(X^\delta)$ instead of \mathcal{O}^δ .

Remark 5.4. When $\lambda = \omega$ and $\mathcal{F} = \mathcal{S}$, the equivalence of (1), (3), and (4) in Theorem 5.2 is due to Ginsburg and Saks [9, Theorem 2.6], here in equivalent form via Theorem 4.4. See also [22, Theorem 5.6] for a related result.

More generally, when $\mathcal{F} = \mathcal{S}$, the equivalence of (1) and (3) in Theorem 5.2 is due to [20, Theorem 6.2]. See also [7, Corollary 2.15], [2], and [3, Theorem 3.4].

Let us mention the special case of Theorem 5.2 dealing with D -pseudocompactness.

Corollary 5.5. *Let X be a topological space, let λ be an infinite cardinal, and, for every cardinal δ , let X^δ be endowed with the Tychonoff topology. Then the following are equivalent.*

- (1) *There exists some ultrafilter D uniform over λ such that X is D -pseudocompact.*
- (2) *There exists some ultrafilter D uniform over λ such that, for every cardinal δ , every λ -indexed sequence of members of \mathcal{O}^δ has some D -limit point in X^δ .*
- (3) *For every cardinal δ , in X^δ every λ -indexed sequence of members of \mathcal{O}^δ has a λ -complete accumulation point.*
- (4) *Let $\delta = \min\{2^{2^\lambda}, \kappa^\lambda\}$, where κ is the weight of X . In X^δ every λ -indexed sequence of members of \mathcal{O}^δ has a λ -complete accumulation point.*

If λ is regular, then the above conditions are also equivalent to

- (5) *For every cardinal δ , X^δ is \mathcal{O}^δ - $[\lambda, \lambda]$ -compact.*
- (6) *Suppose that δ is a cardinal, $(C_\alpha)_{\alpha \in \lambda}$ is a sequence of closed sets of X^δ , and $C_\alpha \supseteq C_\beta$, whenever $\alpha \leq \beta < \lambda$. If, for every $\alpha \in \lambda$, there exists $F \in \mathcal{O}^\delta$ such that $C_\alpha \supseteq F$, then $\bigcap_{\alpha \in \lambda} C_\alpha \neq \emptyset$.*

In all of the above statements, we can equivalently replace \mathcal{O}^δ by the family of the nonempty open sets of X^δ in the box topology.

Proof. In order to prove the equivalence of conditions (1)–(3), just take $\mathcal{F} = \mathcal{O}$ in Theorem 5.2.

In order to get the right bound in condition (4), recall that if \mathcal{B} is a base (consisting of nonempty sets) of X , then, by Remark 2.2, $\mathcal{O} \triangleright \mathcal{B}$ and $\mathcal{B} \triangleright \mathcal{O}$. Notice also that $\mathcal{O}^\delta \triangleright \mathcal{B}^\delta$ and $\mathcal{B}^\delta \triangleright \mathcal{O}^\delta$, as well. Thus, we can apply Theorem 5.2 with \mathcal{B} in place of \mathcal{O} , getting the right bound in which $|\mathcal{B}| = \kappa$ is the weight of X .

If λ is regular, then conditions (5) and (6) are equivalent to (3), by Theorem 4.4.

The last statement follows from Remark 2.2, since, for every δ , \mathcal{O}^δ is a base for the box topology on X^δ . \square

When λ is regular, we can use Theorem 4.4 in order to get still more conditions equivalent to (3) and (4) above.

6. TWO CARDINALS TRANSFER RESULTS

We are now going to show that there are very nontrivial cardinal transfer properties for the conditions dealt with in Theorem 5.2.

Let D be an ultrafilter over λ and let $f : \lambda \rightarrow \mu$. The ultrafilter $f(D)$ over μ is defined by $Y \in f(D)$ if and only if $f^{-1}(Y) \in D$.

Fact 6.1. *Suppose that X is a topological space, \mathcal{F} is a family of subsets of X , D is an ultrafilter over λ , and $f : \lambda \rightarrow \mu$.*

If X is \mathcal{F} - D -compact, then X is \mathcal{F} - $f(D)$ -compact.

If D is an ultrafilter over some set Z and μ is a cardinal, D is said to be μ -decomposable if and only if there exists a function $f : Z \rightarrow \mu$ such that $f(D)$ is uniform over μ .

The next corollary implies that if every ultrafilter uniform over λ is μ -decomposable and the conditions in Theorem 5.2 hold for the cardinal λ , then they hold for the cardinal μ , too.

Corollary 6.2. *Suppose that λ is an infinite cardinal and K is a set of infinite cardinals, and suppose that every uniform ultrafilter over λ is μ -decomposable for some $\mu \in K$.*

If X is a topological space, \mathcal{F} is a family of subsets of X and one (and hence all) of the conditions in Theorem 5.2 hold for λ , then there is $\mu \in K$ such that the conditions in Theorem 5.2 hold when λ is everywhere replaced by μ .

The same applies with respect to Corollary 5.5.

Proof. Suppose that the conditions in Theorem 5.2 hold for λ . By Theorem 5.2(1), there exists some ultrafilter D uniform over λ such that X is \mathcal{F} - D -compact. By assumption, there exist $\mu \in K$ and $f : \lambda \rightarrow \mu$ such that $D' = f(D)$ is uniform over μ . By Fact 6.1, X is \mathcal{F} - D' -compact; hence, Theorem 5.2(1) holds for μ . \square

There are many results asserting that, for some cardinal λ and some set K , the assumption in Corollary 6.2 holds. In order to state some of these results in a more concise way, let us denote by $\lambda \overset{\infty}{\cong} K$, for K a set of infinite cardinals, the statement that the assumption in Corollary 6.2 holds. That is, $\lambda \overset{\infty}{\cong} K$ means that every uniform ultrafilter over λ is μ -decomposable for some $\mu \in K$. When $K = \{\mu\}$, we simply write $\lambda \overset{\infty}{\cong} \mu$ in place of $\lambda \overset{\infty}{\cong} K$. The only reason for the superscript ∞ is to keep the notation consistent with the notation used in former papers. Notice that many conditions equivalent to $\lambda \overset{\infty}{\cong} K$ can be obtained from [14, Theorem 8 and Theorem 10], by letting $\kappa = 2^\lambda$ (equivalently, letting κ be arbitrarily large) there.

The following are trivial facts about the relation $\lambda \overset{\infty}{\cong} K$. If $\lambda \in K$, then $\lambda \overset{\infty}{\cong} K$ holds. In particular, $\lambda \overset{\infty}{\cong} \lambda$ holds. If $\lambda \overset{\infty}{\cong} K$ holds and $K' \supseteq K$, then $\lambda \overset{\infty}{\cong} K'$ holds, too.

In the next theorem, we reformulate, according to the present terminology, some of the results on decomposability of ultrafilters collected in [16]. In order to state the theorem, we need to introduce some notational conventions. By λ^{+n} , we denote the n^{th} successor of λ . By $\beth_n(\lambda)$, we denote the n^{th} iteration of the power set of λ ; that is, $\beth_0(\lambda) = \lambda$ and $\beth_{n+1}(\lambda) = 2^{\beth_n(\lambda)}$. As usual, $[\mu, \lambda]$ denotes the interval $\{\nu \mid \mu \leq \nu \leq \lambda\}$.

Theorem 6.3. *The following hold.*

- (1) *If λ is a regular cardinal, then $\lambda^+ \cong \lambda$.*
- (2) *More generally, if λ is a regular cardinal, then $\lambda^{+n} \cong \lambda$.*
- (3) *If λ is a singular cardinal, then $\lambda \cong \text{cf } \lambda$.*
- (4) *If λ is a singular cardinal, then $\lambda^+ \cong \{\text{cf } \lambda\} \cup K$ for every set K of regular cardinals $< \lambda$ such that K is cofinal in λ .*
- (5) *$\nu^{\kappa^{+n}} \cong [\kappa, \nu^\kappa]$.*
- (6) *If $m \geq 1$, then $\beth_m(\kappa^{+n}) \cong [\kappa, 2^\kappa]$.*
- (7) *If κ is a strong limit cardinal, then $\beth_m(\kappa^{+n}) \cong \{\text{cf } \kappa\} \cup [\kappa', \kappa]$ for every $\kappa' < \kappa$.*
- (8) *If λ is smaller than the first measurable cardinal (or no measurable cardinal exists), then $\lambda \cong \omega$.*
- (9) *More generally, for every cardinal λ , we have that $\lambda \cong \{\omega\} \cup M$, M being the set of all measurable cardinals $\leq \lambda$.*
- (10) *If there is no inner model with a measurable cardinal and $\lambda \geq \mu$ are infinite cardinals, then $\lambda \cong \mu$.*

In particular, Corollary 6.2 applies in each of the above cases.

Remark 6.4. Notice that, by [16, Properties 1.1(iii),(x)], and arguing as in [16, Consequence 1.2], the relation $\lambda \cong \mu$ is equivalent to “every λ -decomposable ultrafilter is μ -decomposable.”

Similarly, $\lambda \cong K$ is equivalent to “every λ -decomposable ultrafilter is μ -decomposable, for some $\mu \in K$.”

Proof of Theorem 6.3. (1–4), (8), and (9) are immediate from classical results about ultrafilters; see the comments after Problem 6.8 in [16].

(5)–(7) follow from [16, Theorem 4.3 and Properties 1.1(vii)].

(10) is immediate from [5, Theorem 4.5] by using [16, Properties 1.1 and Remarks 1.5(b)]. \square

By Remark 6.4, we get the following transitivity properties of the relation $\lambda \cong K$.

Proposition 6.5. *The following hold.*

- (1) *If $\lambda \cong \mu$ and $\mu \cong K$, then $\lambda \cong K$.*

- (2) More generally, suppose that $\lambda \overset{\infty}{\cong} K$ and, for every $\mu \in K$, it happens that $\mu \overset{\infty}{\cong} H_\mu$ for some set H_μ depending on μ . Then $\lambda \overset{\infty}{\cong} \bigcup_{\mu \in K} H_\mu$.
- (3) Suppose that $\lambda \overset{\infty}{\cong} K$, $\mu \in K$, and $\mu \overset{\infty}{\cong} K'$ for some set $K' \subseteq K$ such that $\mu \notin K'$. Then $\lambda \overset{\infty}{\cong} K \setminus \{\mu\}$.
- (4) More generally, suppose that $\lambda \overset{\infty}{\cong} K$, $H \subseteq K$, and, for every $\mu \in H$, it happens that $\mu \overset{\infty}{\cong} K \setminus H$. Then $\lambda \overset{\infty}{\cong} K \setminus H$.

Proof. (1) and (2) follow from Remark 6.4.

(4) is immediate from (2) by taking $H_\mu = K \setminus H$ if $\mu \in H$, and by taking $H_\mu = \{\mu\}$ if $\mu \in K \setminus H$, since, trivially, $\mu \overset{\infty}{\cong} \mu$.

(3) is a particular case of (4), since $K' \subseteq K \setminus \{\mu\}$. \square

Corollary 6.6. *Suppose that $\kappa < \nu$ are infinite cardinals and that either $K = [\kappa, \nu]$ or $K = [\kappa, \nu)$.*

(a) *If $\lambda \overset{\infty}{\cong} K$, then $\lambda \overset{\infty}{\cong} S$, where S is the set containing κ , containing all limit cardinals of K , and containing all cardinals of K which are successors of singular cardinals.*

(b) *More generally, if $\lambda \overset{\infty}{\cong} K$, then $\lambda \overset{\infty}{\cong} L$, where L is the set of all $\mu \in K$ such that either*

- (1) $\mu = \kappa$, or
- (2) μ is singular and $\text{cf } \mu < \kappa$, or
- (3) $\mu = \varepsilon^+$ for some singular ε such that $\text{cf } \varepsilon < \kappa$, or
- (4) μ is weakly inaccessible.

The above statements can be used to refine Theorem 6.3(5) and (6).

Proof. Clearly, (a) follows from (b).

In order to prove (b), let $H = K \setminus L$; thus, $L = K \setminus H$.

By Proposition 6.5(4), it is enough to show that if $\mu \in H$, then $\mu \not\overset{\infty}{\cong} L$. This is trivial if $H = \emptyset$. Otherwise, suppose by contradiction that there is some $\mu \in H$ such that $\mu \overset{\infty}{\cong} L$ fails. Let μ_0 be the least such μ .

We now show that there is some $\mu' < \mu_0$ such that $\mu' \geq \kappa$ and $\mu_0 \overset{\infty}{\cong} \mu'$. This follows from Theorem 6.3(1) if μ_0 is the successor of some regular cardinal, since $\mu_0 > \kappa \notin H$ by (1). The existence of μ' follows from Theorem 6.3(4) if $\mu_0 = \varepsilon^+$ with ε singular such that $\text{cf } \varepsilon \geq \kappa$. Finally, the existence of μ' follows from Theorem 6.3(3) if μ_0 is singular and $\text{cf } \mu_0 \geq \kappa$. By (2)–(4), no other possibility can occur for μ_0 since $\mu_0 \in H$; that is, $\mu_0 \notin L$.

Since $\kappa \leq \mu' < \mu_0$, then $\mu' \overset{\infty}{\cong} L$. This is trivial if $\mu' \in L$ and follows from the minimality of μ_0 if $\mu' \notin L$, which means $\mu' \in H = K \setminus L$.

From $\mu_0 \overset{\infty}{\cong} \mu'$ and $\mu' \overset{\infty}{\cong} L$, we infer $\mu_0 \overset{\infty}{\cong} L$ by applying Proposition 6.5(1). We have reached the desired contradiction. \square

Some more results about the relation $\lambda \overset{\infty}{\cong} K$ follow from results in [16]. See [18] and, in particular, the comments after [16, Problem 6.8] for some open problems concerning transfer of decomposability for ultrafilters.

When $\mathcal{F} = \mathcal{S}$, many versions of Corollary 6.2 are known and are usually stated by means of conditions involving $[\lambda, \lambda]$ -compactness (for regular cardinals, the conditions are equivalent by Theorem 4.4). In [2] and [3, Corollary 1.8(ii)], Caicedo proved, among other things, that every productively $[\lambda^+, \lambda^+]$ -compact family of topological spaces is productively $[\lambda, \lambda]$ -compact. More generally, among other things, we proved in [12, Theorem 16] that if λ is regular and a product of topological spaces is $[\lambda^+, \lambda^+]$ -compact, then all but at most λ factors are $[\lambda, \lambda]$ -compact. Results related to Corollary 6.2 appear in [2], [3], [11], and [16, Corollary 4.6]. Generally, they deal with (λ, μ) -regularity of ultrafilters, which is a notion tightly connected to decomposability since, for λ a regular cardinal, an ultrafilter is λ -decomposable if and only if it is (λ, λ) -regular. Stronger related results appear in [13] and [14], dealing also with equivalent notions from model theory and set theory; in particular, see [14, Theorem 8]. Even in the case when $\mathcal{F} = \mathcal{S}$, some consequences of Theorem 6.3 and corollaries 6.6 and 6.2 appear to be new, particularly in the case of singular cardinals.

Already the special case $\mu = \omega$ for pseudocompactness of Corollary 6.2 appears to have some interest.

Corollary 6.7. *Suppose that λ is an infinite cardinal and suppose that every uniform ultrafilter over λ is ω -decomposable (for example, this happens when either $\text{cf } \lambda = \omega$ or when λ is less than the first measurable cardinal or if there exists no inner model with a measurable cardinal).*

Suppose that X is a topological space satisfying one of the conditions in Corollary 5.5. Then X is D -pseudocompact for some ultrafilter D uniform over ω . In particular, if X is Tychonoff, then X is pseudocompact, and, furthermore, all powers of X are pseudocompact.

Proof. Immediate from Remark 4.5. □

Garcia-Ferreira [8] provides results related to Corollary 6.7. In particular, he analyzes the relationship between D -pseudocompactness and D' -pseudocompactness for various ultrafilters D and D' . Indeed, by using results from [8], we can show that $\lambda \overset{\infty}{\cong} K$ is actually equivalent to the statement that, for every topological space X , if Corollary 5.5 holds for λ , then it holds for some $\mu \in K$.

Corollary 6.8. *Suppose that λ is an infinite cardinal and K is a set of infinite cardinals $\leq \lambda$. Then the following are equivalent.*

- (a) $\lambda \overset{\infty}{\cong} K$ holds; that is, every uniform ultrafilter over λ is μ -decomposable for some $\mu \in K$.
- (b) For every topological space X , if one (and hence, all) of the conditions in Corollary 5.5 hold for λ , then there is $\mu \in K$ such that the conditions in Corollary 5.5 hold when λ is everywhere replaced by μ .
- (c) Same as (b), restricted to Tychonoff spaces.

Proof. (a) \Rightarrow (b) is from Corollary 6.2.

(b) \Rightarrow (c) is trivial.

(c) \Rightarrow (a). Garcia-Ferreira [8, Lemma 1.4] constructs, for every ultrafilter D uniform over λ , a Tychonoff space $P_{RK}(D)$ such that, for every ultrafilter E , the space $P_{RK}(D)$ is E -pseudocompact if and only if $E = f(D)$ for some function f (this is usually expressed by saying that E is $\leq D$ in the *Rudin-Keisler order*).

Let D be an ultrafilter uniform over λ . By the above, $X = P_{RK}(D)$ is D -pseudocompact; hence, X satisfies condition (1) in Corollary 5.5. By (c), condition (1) in Corollary 5.5 holds for some $\mu \in K$; hence, X is E -pseudocompact for some ultrafilter E uniform over some $\mu \in K$. By the above-mentioned result in [8], $E = f(D)$ for some function $f : \lambda \rightarrow \mu$; that is, $\lambda \overset{\infty}{\cong} K$ holds. \square

7. $[\mu, \lambda]$ -COMPACTNESS RELATIVE TO A FAMILY \mathcal{F}

We can generalize the notion of $[\mu, \lambda]$ -compactness in another direction.

Definition 7.1. If X is a topological space and \mathcal{G} is a family of subsets of X , we say that X is $[\mu, \lambda]$ -compact relative to \mathcal{G} if and only if the following holds.

For every sequence $(G_\alpha)_{\alpha \in \lambda}$ of elements of \mathcal{G} , if, for every $Z \subseteq \lambda$ with $|Z| < \mu$, $\bigcap_{\alpha \in Z} G_\alpha \neq \emptyset$, then $\bigcap_{\alpha \in \lambda} G_\alpha \neq \emptyset$.

The usual notion of $[\mu, \lambda]$ -compactness can be obtained from the above definition when \mathcal{G} is the family of all closed sets of X .

If \mathcal{G} is the family \mathcal{Z} of all zero sets of some space X , then $[\omega, \lambda]$ -compactness relative to \mathcal{Z} is called λ -quasicompactness in [6]. The property of being $[\mu, \lambda]$ -compact relative to \mathcal{Z} for every cardinal λ is called μ -quasi-Lindelöfness in [6].

If X is Tychonoff, then X is $[\omega, \lambda]$ -compact relative to \mathcal{Z} if and only if X is λ -pseudocompact. See, e. g., [6], [8], [19], and [23] for results about λ -pseudocompactness, equivalent formulations, and further references. Notice that [8, Example 1.9] shows that it is possible, under some

set-theoretical assumptions, to construct a space which is not ω_1 -pseudo-compact, but which is D -pseudocompact for some ultrafilter D uniform over ω_1 .

Proposition 7.2. *Suppose that X is a topological space and \mathcal{G} is a family of subsets of X . Then the following are equivalent.*

- (a) X is $[\mu, \lambda]$ -compact relative to \mathcal{G} .
- (b) X is $[\kappa, \kappa]$ -compact relative to \mathcal{G} for every κ with $\mu \leq \kappa \leq \lambda$.

Proof. Similar to the proof of the classical result for $[\mu, \lambda]$ -compactness, see, e. g., [12, Proposition 8]. \square

There is some connection between the compactness properties introduced in Definition 4.2 and Definition 7.1. In order to deal with the relationship between the two properties, it is convenient to introduce a common generalization.

Definition 7.3. If X is a topological space and \mathcal{F} and \mathcal{G} are families of subsets of X , we say that X is \mathcal{F} - $[\mu, \lambda]$ -compact relative to \mathcal{G} if and only if the following holds.

For every sequence $(G_\alpha)_{\alpha \in \lambda}$ of elements of \mathcal{G} , if, for every $Z \subseteq \lambda$ with $|Z| < \mu$, there exists $F \in \mathcal{F}$ such that $\bigcap_{\alpha \in Z} G_\alpha \supseteq F$, then $\bigcap_{\alpha \in \lambda} G_\alpha \neq \emptyset$.

Thus, \mathcal{F} - $[\mu, \lambda]$ -compactness is \mathcal{F} - $[\mu, \lambda]$ -compactness relative to \mathcal{G} when \mathcal{G} is the family of all closed subsets of X .

On the other hand, $[\mu, \lambda]$ -compactness relative to \mathcal{G} is \mathcal{F} - $[\mu, \lambda]$ -compactness relative to \mathcal{G} when $\mathcal{F} = \mathcal{S}$.

Frolík [6] introduced a notion equivalent to \mathcal{Z} - $[\omega, \lambda]$ -compactness relative to \mathcal{O} under the name *almost λ -quasicompactness*. Additionally, he called the property of being \mathcal{Z} - $[\mu, \lambda]$ -compact relative to \mathcal{O} for every cardinal λ *almost μ -quasi-Lindelöfness*.

Proposition 7.4. *Suppose that λ and μ are infinite cardinals, and let $\kappa = \sup\{\lambda^{\mu'} \mid \mu' < \mu\}$. Suppose that X is a topological space and \mathcal{F} is a family of subsets of X . Let \mathcal{F}^* ($\mathcal{F}_{\leq \kappa}^*$, respectively) be the family of all subsets of X which are the closure of the union of some family of ($\leq \kappa$, respectively) sets in \mathcal{F} . Then*

- (1) *the following conditions are equivalent:*
 - (a) X is \mathcal{F} - $[\mu, \lambda]$ -compact.
 - (b) X is \mathcal{F} - $[\mu, \lambda]$ -compact relative to \mathcal{F}^* .
 - (c) X is \mathcal{F} - $[\mu, \lambda]$ -compact relative to $\mathcal{F}_{\leq \kappa}^*$.
- (2) *Suppose, in addition, that all members of \mathcal{F} are nonempty. If X is $[\mu, \lambda]$ -compact relative to $\mathcal{F}_{\leq \kappa}^*$, then X is \mathcal{F} - $[\mu, \lambda]$ -compact.*

Proof. In (1), the implications (a) \Rightarrow (b) \Rightarrow (c) are trivial.

In order to show that (c) \Rightarrow (a) holds, let $(C_\alpha)_{\alpha \in \lambda}$ be a sequence of closed sets of X such that, for every $Z \subseteq \lambda$ with $|Z| < \mu$, there exists $F_Z \in \mathcal{F}$ such that $\bigcap_{\alpha \in Z} C_\alpha \supseteq F_Z$.

For $\alpha \in \lambda$, let C'_α be the closure of $\bigcup_{\alpha \in Z} F_Z$. Clearly, for every $\alpha \in \lambda$, we have $C_\alpha \supseteq C'_\alpha$. Since there are κ subsets of λ of cardinality $< \mu$, that is, we can choose Z in κ -many ways, we have that each C'_α is the closure of the union of $\leq \kappa$ elements from \mathcal{F} . Thus, we can apply (c) in order to get $\bigcap_{\alpha \in \lambda} C'_\alpha \neq \emptyset$; hence, $\bigcap_{\alpha \in \lambda} C_\alpha \supseteq \bigcap_{\alpha \in \lambda} C'_\alpha \neq \emptyset$.

(2) is immediate from (1)(c) \Rightarrow (a), since if \mathcal{F} is a family of *nonempty* subsets of X , then $[\mu, \lambda]$ -compactness relative to some family \mathcal{G} implies \mathcal{F} - $[\mu, \lambda]$ -compactness relative to \mathcal{G} . \square

Remark 7.5. The value $\kappa = \sup\{\lambda^{\mu'} \mid \mu' < \mu\}$ in Proposition 7.4 can be improved to $\kappa =$ the cofinality of the partial order $S_\mu(\lambda)$ (see [16]).

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