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by

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INDUCED MAPPINGS BETWEEN THE HYPERSPACES $\mathcal{C}(X)$ OF CONTINUA AND UNIVERSAL MAPPINGS

ANTONIO PELÁEZ

Abstract. In 1980, Professor Sam B. Nadler, Jr., proved that if $f : Y \to X$ is a surjective map from a continuum Y onto a metrizable chainable continuum X, then the induced map $\mathcal{C}(f) : \mathcal{C}(Y) \to \mathcal{C}(X)$ is universal. Later, in 2002, Jorge Bustamante, Raúl Escobedo, and Fernando Macías-Romero, proved that if $f : Y \to X$ is a surjective map between metrizable continua, where X has zero surjective semispan, then the induced map $\mathcal{C}(f) : \mathcal{C}(Y) \to \mathcal{C}(X)$ is universal. In this paper, we extend the first result to the non-metric case and the second one to the rim-metrizable case.

1. Preliminaries

Given the relations U and V on a set X, the inverse relation of U is the set

$$-U = \{(y, x) : (x, y) \in U\},$$

and the composition of U and V is the set

$$U + V = \{(x, z) : \text{there exists a } y \in X \text{ such that } (x, y) \in U \text{ and } (y, z) \in V\}.$$

We also write $1V = V$ and, for a positive integer n, $(n + 1)V = nV + V$.

The diagonal of X is the set $\Delta = \{(x, x) : x \in X\}$. An *entourage* of the diagonal is a set $V \subset X \times X$ containing Δ such that $V = -V$; the family of all entourages of the diagonal is denoted by $\mathcal{D}_X$. If we have $x, y \in X$ and $V \in \mathcal{D}_X$ such that $(x, y) \in V$, then we say that the *distance*

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between x and y is less than V and we write $\mid x-y \mid\ < V$; otherwise, we write $\mid x-y \mid\ \geq V$. If for every pair of points x, y of a set $A \subseteq X$ and $V \in \mathcal{D}_X$, we have that $\mid x-y \mid\ < V$, i.e., if $A \times A \subseteq V$, we say that the *diameter of A is less than V* and we write $\delta(A) < V$; otherwise, we write $\delta(A) \geq V$.

Given a point $x \in X$ and $V \in \mathcal{D}_X$, the *ball with center x and radius V* (briefly, the *V-ball about x*) is the set $B(x,V) = \{y \in X : \mid x-y \mid\ < V\}$. For a set $A \subseteq X$ and a $V \in \mathcal{D}_X$, the *V-ball about A* is the set $B(A,V) = \bigcup\{B(x,V) : x \in A\}$.

A *uniformity* on a set X is a subfamily $\mathcal{U}$ of $\mathcal{D}_X$ such that

U1. If $V \in \mathcal{U}$ and $V \subseteq W \in \mathcal{D}_X$, then $W \in \mathcal{U}$.
U2. If $V, W \in \mathcal{U}$, then $V \cap W \in \mathcal{U}$.
U3. For every $V \in \mathcal{U}$, there exists $W \in \mathcal{U}$ such that $2W \subseteq V$.
U4. $\bigcap \mathcal{U} = \Delta$.

Theorem 1.1 ([2, Theorem 8.1.1])**.** *For every uniformity $\mathcal{U}$ on a set X, the family*

$$\mathcal{O} = \{G \subseteq X : \textit{for every } x \in G \textit{ there exists a } V \in \mathcal{U} \\ \textit{such that } B(x,V) \subseteq G\}$$

is a topology on X and the topological space $(X, \mathcal{O})$ is a T_1-space.

The topology $\mathcal{O}$ is called the *topology induced by the uniformity $\mathcal{U}$*.

If X is a topological space and its topology is induced by a uniformity $\mathcal{U}$, we say that $\mathcal{U}$ is a *uniformity on the space X*.

The following lemma is a consequence of [2, Corollary 8.1.3].

Lemma 1.2. *Let $\mathcal{U}$ be a uniformity on the space X. The net $\{x_\alpha\}_{\alpha\in\Lambda}$ in X converges to $x \in X$ if and only if, for every $U \in \mathcal{U}$, there exists $\alpha_0 \in \Lambda$ such that $\mid x_\alpha - x \mid\ < U$ for every $\alpha \geq \alpha_0$.*

Lemma 1.3. *Let $\mathcal{U}$ be a uniformity on the space X. Suppose that the nets $\{x_\alpha\}_{\alpha\in\Lambda}, \{y_\alpha\}_{\alpha\in\Lambda}$ in X converge to x and y, respectively. If, for every $U \in \mathcal{U}$, there exists $\alpha_0 \in \Lambda$ such that $\mid x_\alpha - y_\alpha \mid\ < U$ for every $\alpha \geq \alpha_0$, then $x = y$.*

Proof. Suppose that $x \neq y$. Let $W, U \in \mathcal{U}$ such that $3W \subseteq U$ and $\mid x-y \mid\ \geq U$. Let $\alpha_0 \in \Lambda$ such that $\mid x_\alpha - x \mid\ < W$, $\mid y_\alpha - y \mid\ < W$ and $\mid x_\alpha - y_\alpha \mid\ < W$ for every $\alpha \geq \alpha_0$. Then $\mid x-y \mid\ < 3W \subseteq U$, which is a contradiction. □

The space $(X, \mathcal{O})$, constructed in Theorem 1.1, is a Tychonoff space.

Theorem 1.4 ([2, Theorem 8.1.20])**.** *The topology of a space X can be induced by a uniformity on the set X if and only if X is a Tychonoff space.*

Theorem 1.5 ([2, Theorem 8.3.13])**.** *For every Hausdorff compact topological space X, there exists exactly one uniformity $\mathcal{U}$ on the set X that induces the original topology of X. All entourages of the diagonal $\Delta \subseteq X \times X$ which are open in the Cartesian product $X \times X$ form a base for the uniformity $\mathcal{U}$.*

Definition 1.6. A uniform space $(X, \mathcal{U})$ is *compact* if X with the topology induced by $\mathcal{U}$ is a compact space.

A mapping $f : X \to Y$ between the uniform spaces $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ is *uniformly continuous with respect to the uniformities $\mathcal{U}$ and $\mathcal{V}$* if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $\mid f(x) - f(y) \mid < V$ whenever $\mid x - y \mid < U$. In this case, we write $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$. It follows from the definition and Theorem 1.1 that f is a continuous mapping of the space X with the topology induced by $\mathcal{U}$ to the space Y with the topology induced by $\mathcal{V}$.

Theorem 1.7 ([9, Corollary 1.8])**.** *Let $\mathcal{U}$ and $\mathcal{V}$ be uniformities on the Hausdorff compact spaces X and Y, respectively. A mapping $f : X \to Y$ is continuous if and only if f is uniformly continuous with respect to the uniformities $\mathcal{U}$ and $\mathcal{V}$.*

A *hyperspace* of a topological space X is a family of subsets of X. The hyperspace 2^X is the family of all non-empty closed subsets of X. The hyperspace $\mathcal{Z}(X)$ is the subfamily of 2^X consisting of all non-empty compact closed subsets of X. The hyperspace $\mathcal{C}_n(X)$ is the subfamily of $\mathcal{Z}(X)$ consisting of all non-empty compact closed subsets of X with at most n components. We write $\mathcal{C}(X) = \mathcal{C}_1(X)$.

The *Vietoris topology* τ on 2^X is the topology generated by the family of all sets of the form

$$\begin{aligned}\langle U_1, \ldots, U_n \rangle = \{C \in 2^X : C \subseteq U_1 \cup \cdots \cup U_n \text{ and } C \cap U_i \neq \emptyset \\ \text{for each } 1 \leq i \leq n\},\end{aligned}$$

where $U_1, \ldots, U_n$ are open subsets of X (see [2, Problem 2.7.20(a)]). The hyperspaces $\mathcal{Z}(X)$ and $\mathcal{C}_n(X)$ are considered as subspaces of 2^X with the Vietoris topology.

The *weight* of a topological space X is denoted by $\omega(X)$.

From [2, Problem 3.12.27(a) and (b)], we have the following result.

Theorem 1.8. *If X is a Hausdorff compact space, then the hyperspace 2^X is a Hausdorff compact space and $\omega(2^X) = \omega(X)$.*

Definition 1.9. A *continuum* is a nonempty Hausdorff compact connected topological space. A *subcontinuum* is a continuum contained in a space.

By [3, Corollary 14.10], we obtain the following theorem.

Theorem 1.10. *If X is a metrizable continuum, then the hyperspaces 2^X and $\mathcal{C}(X)$ are metrizable continua.*

Given a continuous function $f : X \to Y$ between Hausdorff topological spaces, the *induced mapping between the hyperspaces $\mathcal{Z}(X)$ and $\mathcal{Z}(Y)$* is the mapping $\mathcal{Z}(f) : \mathcal{Z}(X) \to \mathcal{Z}(Y)$ defined by $\mathcal{Z}(f)(C) = f[C]$. The restriction $\mathcal{Z}(f)\mid_{\mathcal{C}_n(X)}$ is denoted by $\mathcal{C}_n(f)$. If X is compact, then $2^X = \mathcal{Z}(X)$, and we write $2^f = \mathcal{Z}(f)$.

Proposition 1.11 ([2, Problem 3.12.27(e)])**.** *If $f : X \to Y$ is a continuous function and $\mathcal{Z}(X)$ and $\mathcal{Z}(Y)$ have the Vietoris topology, then $\mathcal{Z}(f)$ is continuous.*

Let $\mathcal{U}$ be a uniformity on a Tychonoff space X. Given $U \in \mathcal{U}$, let

$$2^U = \left\{(C, D) \in 2^X \times 2^X : C \subseteq B(D, U) \text{ and } D \subseteq B(C, U)\right\}.$$

The family $\left\{2^U : U \in \mathcal{U}\right\}$ is a base for a uniformity $2^{\mathcal{U}}$ on the hyperspace 2^X (see [2, Problem 8.5.16(a)]).

Given a Tychonoff space X, we have two topologies on 2^X: the Vietoris topology τ and the induced topology $2^{\mathcal{O}}$ by the uniformity $2^{\mathcal{U}}$, where $\mathcal{U}$ is a uniformity on the space X. Those two topologies coincide on $\mathcal{Z}(X)$ (see [2, Problem 8.5.16(c)]). If $(X,\mathcal{U})$ is compact, then $(2^X, 2^{\mathcal{U}})$ is compact (see [2, Problem 8.5.16(f)]), and $\tau = 2^{\mathcal{O}}$.

Theorem 1.12. *Let $\mathcal{U}$ and $\mathcal{V}$ be uniformities on the Tychonoff spaces X and Y, respectively. If $f : X \to Y$ is uniformly continuous with respect to $\mathcal{U}$ and $\mathcal{V}$, then $\mathcal{Z}(f)$ is uniformly continuous with respect to ${2^{\mathcal{U}}}_{\mathcal{Z}(X)} = \left\{(\mathcal{Z}(X) \times \mathcal{Z}(X)) \cap 2^U : U \in \mathcal{U}\right\}$ and ${2^{\mathcal{V}}}_{\mathcal{Z}(Y)}$.*

Proof. Let $V \in \mathcal{V}$; then there exists $U \in \mathcal{U}$ such that $\mid f(x) - f(y) \mid < V$ whenever $\mid x - y \mid < U$. Let $C, D \in \mathcal{Z}(X)$ such that $\mid C - D \mid < 2^U$. Since $C \subseteq B(D, U)$ and $D \subseteq B(C, U)$, we have that $f[C] \subseteq B(f[D], V)$ and $f[D] \subseteq B(f[C], V)$. Then $\mid \mathcal{Z}(f)(C) - \mathcal{Z}(f)(D) \mid < 2^V$, which means that $\mathcal{Z}(f)$ is uniformly continuous with respect to ${2^{\mathcal{U}}}_{\mathcal{Z}(X)}$ and ${2^{\mathcal{V}}}_{\mathcal{Z}(Y)}$. $\square$

Definition 1.13. A mapping $f : (X,\mathcal{U}) \to (Y, \mathcal{V})$ is a *U-map*, where $U \in \mathcal{U}$, provided that $\delta(f^{-1}(y)) < U$ for each $y \in Y$.

Theorem 1.14. *Let $\mathcal{U}$ and $\mathcal{V}$ be uniformities on the Tychonoff spaces X and Y, respectively. If $f : (X,\mathcal{U}) \to (Y, \mathcal{V})$ is a U-map, then $\mathcal{Z}(f) : \left(\mathcal{Z}(X), {2^{\mathcal{U}}}_{\mathcal{Z}(X)}\right) \to \left(\mathcal{Z}(Y), {2^{\mathcal{V}}}_{\mathcal{Z}(Y)}\right)$ is a 2^U-map.*

Proof. By Theorem 1.12, the mapping $\mathcal{Z}(f)$ is uniformly continuous. Let $C, D \in (\mathcal{Z}(f))^{-1}(E)$, where $E \in \mathcal{Z}(Y)$. Let $c \in C$. Since $f[C] = f[D] =$

E, there exists $d \in D$ such that $f(c) = f(d)$, then $\mid c - d \mid < U$ which means that $C \subseteq B(D, U)$. Similarly, $D \subseteq B(C, U)$. So, $\mid C - D \mid < 2^U$; therefore, $\delta\left((\mathcal{Z}(f))^{-1}(E)\right) < 2^U$, which means that $\mathcal{Z}(f)$ is a 2^U-map. □

An *inverse system* is a family $S = \left\{X_\alpha, f_\alpha^\beta, \Lambda\right\}$, where $(\Lambda, \leq)$ is a directed set, X_α is a topological space for every $\alpha \in \Lambda$, and for any $\alpha, \beta \in \Lambda$ satisfying $\alpha \leq \beta$, $f_\alpha^\beta : X_\beta \to X_\alpha$ is a continuous mapping such that

i) f_α^α is the identity map on X_α for every $\alpha \in \Lambda$, and
ii) $f_\alpha^\gamma = f_\alpha^\beta \circ f_\beta^\gamma$ for any $\alpha, \beta, \gamma \in \Lambda$ satisfying $\alpha \leq \beta \leq \gamma$.

The maps f_α^β are called *bonding maps* and the spaces X_α are called *coordinate spaces*.

Given a point $\hat{x}$ in a product $\prod \{X_\alpha : \alpha \in \Lambda\}$, we write $\hat{x} = (x_\alpha)_{\alpha \in \Lambda}$.

Let $S = \left\{X_\alpha, f_\alpha^\beta, \Lambda\right\}$ be an inverse system. The subspace of the product $\prod \{X_\alpha : \alpha \in \Lambda\}$ consisting of all points $\hat{x}$ such that $x_\alpha = f_\alpha^\beta(x_\beta)$ for any $\alpha, \beta \in \Lambda$ satisfying $\alpha \leq \beta$ is called the *inverse limit* of the inverse system S, which is denoted by $\varprojlim S$ or by X_Λ. We define the projection map $f_\alpha^\Lambda : X_\Lambda \to X_\alpha$ by $f_\alpha^\Lambda(\hat{x}) = x_\alpha$.

A *morphism* from an inverse system $S = \left\{X_\alpha, f_\alpha^\beta, \Lambda\right\}$ into the inverse system $S' = \left\{Y_\alpha, g_\alpha^\beta, \Lambda\right\}$ is a family $\mathfrak{h} = \{h_\alpha : \alpha \in \Lambda\}$ of continuous mappings $h_\alpha : X_\alpha \to Y_\alpha$ such that $g_\alpha^\beta \circ h_\beta = h_\alpha \circ f_\alpha^\beta$ for any $\alpha, \beta \in \Lambda$ satisfying $\alpha \leq \beta$. Every morphism $\mathfrak{h}$ between the inverse systems S and S' induces a continuous mapping $h_\Lambda : X_\Lambda \to Y_\Lambda$ between their inverse limits such that $g_\alpha^\Lambda \circ h_\Lambda = h_\alpha \circ f_\alpha^\Lambda$ (see [2, p. 101]).

The following is a well-known result (see [8, Theorem 2.5]).

Theorem 1.15. *Let $S = \{X_\alpha, f_\alpha^\beta, \Lambda\}$ be an inverse system of Hausdorff compact spaces. Then the families $2^S = \left\{2^{X_\alpha}, 2^{f_\alpha^\beta}, \Lambda\right\}$ and $\mathcal{C}(S) = \left\{\mathcal{C}(X_\alpha), \mathcal{C}(f_\alpha^\beta), \Lambda\right\}$ are inverse systems, and the continuous mapping $h : 2^{\varprojlim S} \to \varprojlim 2^S$ defined by $h(E) = \left(f_\alpha^\Lambda[E]\right)_{\alpha \in \Lambda}$ is a homeomorphism and $h\left[\mathcal{C}\left(\varprojlim S\right)\right] = \varprojlim \mathcal{C}(S)$.*

A subset Σ of a directed set Λ is *cofinal* provided that, for every $\alpha \in \Lambda$, there exists $\beta \in \Sigma$ such that $\alpha \leq \beta$. A subset Σ of a directed set Λ is a *chain* provided that, for any $\alpha, \beta \in \Sigma$, we have that $\alpha \leq \beta$ or $\beta \leq \alpha$. Let $\tau \geq \aleph_0$ be a cardinal number. A directed set Λ is called *τ-complete* provided that, for each chain $\Sigma \subseteq \Lambda$ with $\mid \Sigma \mid \leq \tau$, there exists $\sup\Sigma \in \Lambda$.

Let $S = \{X_\alpha, f_\alpha^\beta, \Lambda\}$ be an inverse system and let $\Sigma \subseteq \Lambda$ be a chain with $\gamma = \sup\Sigma \in \Lambda$. By [2, Exercise 2.5.F], the morphism $\{f_\alpha^\gamma : \alpha \in \Sigma\}$ induces a continuous mapping $h_\gamma : X_\gamma \to \varprojlim\{X_\alpha, f_\alpha^\beta, \Sigma\}$ such that $f_\alpha^\gamma = f_\alpha^\Sigma \circ h_\gamma$ for every $\alpha \in \Sigma$. Note that h_γ is defined by $h_\gamma(x_\gamma) = (f_\alpha^\gamma(x_\gamma))_{\alpha\in\Sigma}$.

Definition 1.16. An inverse system $S = \{X_\alpha, f_\alpha^\beta, \Lambda\}$ is *τ-continuous* provided that, for each chain $\Sigma \subseteq \Lambda$ with $|\ \Sigma\ | < \tau$ and $\gamma = \sup\Sigma$, the induced map $h_\gamma : X_\gamma \to \varprojlim\{X_\alpha, f_\alpha^\beta, \Sigma\}$, by the morphism $\{f_\alpha^\gamma : \alpha \in \Sigma\}$, is a homeomorphism.

From the proof of [8, Theorem 3.4], we obtain the following theorem.

Theorem 1.17. *If $S = \{X_\alpha, f_\alpha^\beta, \Lambda\}$ is a τ-continuous inverse system of Hausdorff compact spaces, then the inverse systems 2^S and $\mathcal{C}(S)$ are τ-continuous.*

Definition 1.18. An inverse system $S = \{X_\alpha, f_\alpha^\beta, \Lambda\}$ is *τ-complete* if S is τ-continuous and Λ is τ-complete.

From Theorem 1.17 and the definition of a τ-complete inverse system, we obtain the following result.

Theorem 1.19. *If $S = \{X_\alpha, f_\alpha^\beta, \Lambda\}$ is a τ-complete inverse system of Hausdorff compact spaces, then the inverse systems 2^S and $\mathcal{C}(S)$ are τ-complete.*

Definition 1.20. An inverse system $S = \{X_\alpha, f_\alpha^\beta, \Lambda\}$ is an *inverse τ-system* if S is τ-complete and $\omega(X_\alpha) \leq \tau$ for each $\alpha \in \Lambda$. If $\tau = \aleph_0$, then inverse τ-system is called an inverse σ-system.

From Theorem 1.8 and Theorem 1.19, we have the following.

Theorem 1.21. *If $S = \{X_\alpha, f_\alpha^\beta, \Lambda\}$ is an inverse τ-system of Hausdorff compact spaces, then the inverse systems 2^S and $\mathcal{C}(S)$ are inverse τ-systems.*

Theorem 1.22 ([6, Theorem 1.5])**.** *Let X be a Hausdorff compact space with $\omega(X) \geq \aleph_1$. Then for each cardinal number $\tau < \omega(X)$, there exists an inverse τ-system $\{X_\alpha, f_\alpha^\beta, \Lambda\}$ of Hausdorff compact spaces with surjective bonding maps such that X is homeomorphic to X_Λ.*

Remark 1.23. In the previous theorem, the directed set Λ is constructed using the weight of X, and this set can be used in the inverse τ-system satisfying Theorem 1.22 for another space Y with the same weight.

Definition 1.24. A topological space X has the *fixed point property* provided that, for every continuous mapping $f : X \to X$, there exists $x \in X$ such that $f(x) = x$.

Definition 1.25. Let X and Y be two topological spaces. A continuous mapping $f : X \to Y$ is *universal* if, for every continuous mapping $g : X \to Y$, there exists $x \in X$ such that $f(x) = g(x)$.

It is not difficult to see that if $f : X \to Y$ is universal, then f is surjective and Y has the fixed point property.

Theorem 1.26 ([6, Theorem 2.1]). *Let $S = \left\{X_\alpha, f_\alpha^\beta, \Lambda\right\}$ and $S' = \left\{Y_\alpha, g_\alpha^\beta, \Lambda\right\}$ be two inverse τ-systems of Hausdorff compact spaces with onto bonding maps. If $\mathfrak{h} = \{h_\alpha : \alpha \in \Lambda\} : S \to S'$ is a morphism of universal mappings, then the induced map $h_\Lambda : X_\Lambda \to Y_\Lambda$ is universal.*

Definition 1.27. A continuous mapping $f : X \to Y$ between topological spaces is *monotone* provided that all fibers $f^{-1}(y)$ are connected.

From the proof of [5, Theorem 3.7(1) and (3)], we have the following.

Theorem 1.28. *Let $S = \{X_\alpha, f_\alpha^\beta, \Lambda\}$ be a τ-complete inverse system of Hausdorff compact spaces with onto bonding maps. Then there exists a τ-complete inverse system $M(S) = \{M_\alpha, m_\alpha^\beta, \Lambda\}$ of Hausdorff compact spaces with monotone surjective bonding maps such that the space $\varprojlim S$ is homeomorphic to the space $\varprojlim M(S)$.*

Definition 1.29. A topological space X is *rim-metrizable* if it has a basis $\mathcal{B}$ such that every $U \in \mathcal{B}$ has metrizable boundary.

In the previous theorem, each space M_α is obtained by the monotone-light factorization of f_α^Λ. So, if X_Λ is rim-metrizable, by [12, Theorem 1.2 and Theorem 3.2], we have that $\omega(X_\alpha) = \omega(M_\alpha)$. This implies the following result.

Theorem 1.30. *Let $S = \{X_\alpha, f_\alpha^\beta, \Lambda\}$ be an inverse τ-system of Hausdorff compact spaces with onto bonding maps. If X_Λ is rim-metrizable, then there exists an inverse τ-system $M(S) = \{M_\alpha, m_\alpha^\beta, \Lambda\}$ of Hausdorff compact spaces with monotone surjective bonding maps such that the space X_Λ is homeomorphic to the space M_Λ.*

From [11, Theorem 15], we have the following result.

Theorem 1.31. *Let $\{X_\alpha, f_\alpha^\beta, \Lambda\}$ and $\{Y_\alpha, g_\alpha^\beta, \Lambda\}$ be two inverse τ-systems of Hausdorff compact spaces with onto bonding maps. If $h : X_\Lambda \to Y_\Lambda$ is a continuous mapping, then there exist a cofinal subset Σ of Λ and a morphism $\mathfrak{h} = \{h_\alpha : \alpha \in \Sigma\} : \{X_\alpha, f_\alpha^\beta, \Sigma\} \to \{Y_\alpha, g_\alpha^\beta, \Sigma\}$ such that $h_\Sigma = g' \circ h \circ g^{-1}$, where $g : X_\Lambda \to X_\Sigma$ and $g' : Y_\Lambda \to Y_\Sigma$ are the homeomorphisms defined by $g((x_\alpha)_{\alpha\in\Lambda}) = (x_\alpha)_{\alpha\in\Sigma}$ and $g'((y_\alpha)_{\alpha\in\Lambda}) = (y_\alpha)_{\alpha\in\Sigma}$ (see* [2, Corollary 2.5.11]*).*

2. Universal Mappings Theorems

In this section, we extend the results in [7, Theorem 2.11] and [1] to the non-metric case. In particular, we give an easier proof of [6, Theorem 4.2] in Theorem 2.3.

Theorem 2.1. *Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be two compact uniform spaces and let $f : X \to Y$ be a continuous mapping. If, for every $V \in \mathcal{V}$, there exists a V-map f_V from Y into a space Z_V such that $f_V \circ f$ is universal, then f is universal.*

Proof. Let $g : X \to Y$ be a continuous mapping and let $V \in \mathcal{V}$. Since $f_V \circ f : X \to Z_V$ is universal, there exists $x_V \in X$ such that $f_V(f(x_V)) = f_V(g(x_V))$. Then $\mid f(x_V) - g(x_V) \mid \, < V$. By the compactness of X, we can assume that the net $\{x_V\}_{V\in\mathcal{V}}$ converges to a point $x \in X$. Since the net $\{f(x_V)\}_{V\in\mathcal{V}}$ converges to $f(x)$ and the net $\{g(x_V)\}_{V\in\mathcal{V}}$ converges to $g(x)$, by Lemma 1.3, we have that $f(x) = g(x)$. Then f is universal. $\square$

Definition 2.2. Let $\mathcal{U}$ be a uniformity on a continuum X. The continuum X is *chainable* provided that for every $U \in \mathcal{U}$ there exists a surjective U-map $f_U : X \to [0, 1]$ (see [9, Theorem 2.10]).

Theorem 2.3. *Let $\mathcal{U}$ be a uniformity on a chainable continuum X and let $\mathcal{V}$ be a uniformity on a continuum Y. For any continuous onto mapping $f : Y \to X$, the induced mapping $\mathcal{C}(f) : \mathcal{C}(Y) \to \mathcal{C}(X)$ is universal.*

Proof. Let $U \in \mathcal{U}$ and let $f_U : X \to [0, 1]$ be a surjective U-map. By [7, Theorem 2.11], the induced mapping $\mathcal{C}(f_U \circ f) : \mathcal{C}(Y) \to \mathcal{C}([0, 1])$ is universal. It is not difficult to see that $\mathcal{C}(f_U \circ f) = \mathcal{C}(f_U) \circ \mathcal{C}(f)$. By Theorem 1.14, $\mathcal{C}(f_U)$ is a 2^U-map. Then, by Theorem 2.1, $\mathcal{C}(f)$ is universal. $\square$

We couldn't translate the proof of [1, Theorem 4.1] using only uniformities, so we give a generalization using inverse limits in Theorem 2.7.

Let π_1 denote the projection map from $X \times Y$ onto X.

Definition 2.4. Let $\mathcal{U}$ be a uniformity on a continuum X. The *surjective semispan* of X is the set

$$\begin{aligned}\sigma_0^*(X) = \{V \in \mathcal{U} : \ &\text{there exists a continuum } Z_V \subseteq X \times X \\ &\text{such that } \pi_1[Z_V] = X \text{ and } Z_V \cap V = \emptyset\}.\end{aligned}$$

In the realm of metric spaces, the emptiness of the surjective semispan is characterized with the condition of having zero surjective semispan as A. Lelek defined in [4] (see [9, Theorem 3.2]).

From [10, Theorem 3.7], we have the following theorem.

Theorem 2.5. *Let $f : X \to Y$ be a monotone surjective map. If $\sigma_0^*(X) = \emptyset$ then $\sigma_0^*(Y) = \emptyset$.*

Theorem 2.6 ([1, Theorem 4.1])**.** *Let X be a metrizable continuum with zero surjective semispan. If $f : Y \to X$ is a continuous map from a metrizable continuum Y onto X, then the induced map $\mathcal{C}(f) : \mathcal{C}(Y) \to \mathcal{C}(X)$ is universal.*

The following theorem generalizes Theorem 2.6.

Theorem 2.7. *If $h : Y \to X$ is a surjective continuous mapping between rim-metrizable continua, where X has empty surjective semispan, then the induced map $\mathcal{C}(h) : \mathcal{C}(Y) \to \mathcal{C}(X)$ is universal.*

Proof. By [2, Theorem 3.1.22], we consider only the following two cases.

Case 1: $\omega(X) = \omega(Y) \geq \aleph_1$. By Theorem 1.22, there exist two inverse σ-systems $S = \left\{X_\alpha, f_\alpha^\beta, \Lambda\right\}$ and $S' = \left\{Y_\alpha, g_\alpha^\beta, \Lambda\right\}$ of Hausdorff compact spaces with surjective bonding maps such that X is homeomorphic to X_Λ and Y is homeomorphic to Y_Λ. So, we can assume that $h : Y_\Lambda \to X_\Lambda$. By Theorem 1.30, we can assume that each g_α^β and each f_α^β are monotone; then $\mathcal{C}(S')$ and $\mathcal{C}(S)$ have surjective bonding maps and, by Theorem 1.21, are inverse σ-systems. By [2, Problem 6.3.16(a)], each projection map f_α^Λ is monotone. So, by Theorem 2.5, each X_α has empty surjective semispan. By Theorem 1.31, we can assume that h is the induced map by a morphism $\mathfrak{h} = \{h_\alpha : \alpha \in \Lambda\} : S' \to S$. Then $\mathcal{C}(h)$ is the induced mapping by the morphism $\{\mathcal{C}(h_\alpha) : \alpha \in \Lambda\} : \mathcal{C}(S') \to \mathcal{C}(S)$. By Theorem 2.6, each $\mathcal{C}(h_\alpha) : \mathcal{C}(Y_\alpha) \to \mathcal{C}(X_\alpha)$ is universal. Thus, by Theorem 1.26, $\mathcal{C}(h)$ is universal.

Case 2: $\omega(Y) > \omega(X) \geq \aleph_1$. Let $\tau = \omega(X)$. By Theorem 1.22, there exists an inverse τ-system $S' = \left\{Y_\alpha, g_\alpha^\beta, \Lambda\right\}$ of Hausdorff compact spaces with surjective bonding maps such that Y is homeomorphic to Y_Λ. By Theorem 1.30, we can assume that each g_α^β is monotone; then $\mathcal{C}(S')$ has surjective bonding maps and, by Theorem 1.21, is an inverse τ-system**s**. By [2, Problem 6.3.16(a)], each projection map g_α^Λ is monotone. So, by [12, Theorem 3.2], each Y_α is rim-metrizable. Consider the inverse system $S = \left\{X_\alpha, f_\alpha^\beta, \Lambda\right\}$, where each $X_\alpha = X$ and each f_α^β is the identity map on X. So, we can assume that $h : Y_\Lambda \to X_\Lambda$. By Theorem 1.31, we can assume that h is the induced map by a morphism $\mathfrak{h} = \{h_\alpha : \alpha \in \Lambda\} : S' \to S$. Then $\mathcal{C}(h)$ is the induced mapping by the morphism $\{\mathcal{C}(h_\alpha) : \alpha \in \Lambda\}$. Since $\omega(X_\alpha) = \omega(Y_\alpha)$ for each $\alpha \in \Lambda$, by Case 1, each $\mathcal{C}(h_\alpha) : \mathcal{C}(Y_\alpha) \to \mathcal{C}(X_\alpha)$ is universal. Thus, by Theorem 1.26, $\mathcal{C}(h)$ is universal. □

Question 2.8. Is Theorem 2.7 valid if we remove rim-metrizable?

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