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MANUEL FERNÁNDEZ

Abstract. Taking into consideration a property of the basic open sets of the Sorgenfrey line, we define the concept of almost topological group, thus introducing a proper subclass of paratopological groups. In this paper we prove that any subgroup of a product Π of almost topological groups is saturated and its closure is a subgroup of Π.

1. Introduction

A *paratopological group* (G, τ) consists of an abstract group G together with a topology τ that makes the multiplication of G continuous. If, in addition, the inversion of G is continuous, then (G, τ) is a *topological group*. Not every paratopological group is a topological group; the standard counterexample is the Sorgenfrey line S. The underlying group of S is $\mathbb{R}$, the additive group of real numbers, and the canonical basic open sets in S are the half open intervals $[a, b)$ of $\mathbb{R}$. A natural question arises: Under what conditions does a paratopological group become a topological group? A *semitopological group* is a group endowed with a topology that makes continuous left and right translations. Paratopological groups are semitopological groups. In 1936, Deane Montgomery [8], proved that every separable semitopological group metrizable by a complete metric is a topological group. Another early result of this kind was obtained by Robert Ellis in [6]. Ellis proved that every locally compact Hausdorff semitopological group is a topological group. In 1960, W.

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Zelazko proved in [12] that every completely metrizable paratopological group is a topological group, and in 1982, Nikolaus Brand [5] showed that every Čech-complete paratopological group is a topological group. E. A. Reznichenko proved in [10] that every pseudocompact paratopological group is a topological group. A. V. Arhangel'skii and Reznichenko established in [1] some sufficient conditions for a paratopological group to be a topological group, generalizing the result of [10], and obtained some results on cardinal invariants of paratopological groups.

In this article we study paratopological groups in their own right. Following Guran (see [4]), we say that a paratopological group G is *saturated* if for every neighborhood U of the identity of G, the set $int(U^{-1})$ is not empty. We establish in Corollary 2.8 that any subgroup of an arbitrary power S^κ of the Sorgenfrey line is saturated. In [11], it is proved that the closure of any subgroup of S^κ is a subgroup of S^κ. The important property of S in the proof of these facts is the following: The usual topological group topology of $\mathbb{R}$ is weaker than the topology of S, and for every basic open set $U = [a, b)$ of S, the set $U \setminus \{a\}$ is open in the topological group $\mathbb{R}$. Bearing this property in mind, we define the class of almost topological groups and prove that any subgroup of a product Π of almost topological groups is saturated and its closure is a subgroup of Π. This result generalizes the main theorem in [11], where the authors prove that the closure of a subgroup of any power of the Sorgenfrey line is again a subgroup. In Example 3.10, we show that the closure of a subgroup of a saturated paratopological group G does not have to be a subgroup of G, even if the subgroup is discrete.

A paratopological group G that has a weaker Hausdorff topological group topology will be called *subtopological*. These groups have been studied by Taras Banakh and Olexandr Ravsky in [3] and [4], and they call them $\flat$-*separated*. It is easy to verify that almost topological groups and saturated Hausdorff paratopological groups are subtopological (see Definition 2.1 below and [4, Corollary 3]).

2. Subgroups of Products of Almost Topological Groups Are Saturated

With the Sorgenfrey line in mind, we define almost topological groups.

Definition 2.1. An *almost topological group* is a paratopological group (G, τ) that satisfies the following conditions:

(a) the group G admits a Hausdorff topological group topology γ, weaker than τ, and

(b) there exists a local base $\mathcal{B}$ at the identity e of the paratopological group (G, τ) such that the set $\widetilde{U} = U \setminus \{e\}$ is open in (G, γ), for every $U \in \mathcal{B}$.

If G, γ , τ, and $\mathcal{B}$ are as in Definition 2.1, we will say that G is an *almost topological group* with structure $(\tau, \gamma, \mathcal{B})$.

The following example is due to Constancio Hernández [7, Example 6]. It shows that an almost topological group need not be a topological group.

Example 2.2. Consider the additive group $\mathbb{R}^2$ with identity $e = (0, 0)$. For every $r > 0$, we define

$$B_r = \{e\} \cup \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < r^2, x > 0\right\}.$$

The family $\mathcal{B} = \{B_r : r > 0\}$ satisfies the conditions for a local base at e of a Hausdorff paratopological group (see [9, Proposition 1.1]). The group $G = (\mathbb{R}^2, \tau)$, where τ is the topology generated by the family $\mathcal{B}$, is a Hausdorff almost topological group which is not regular, so G is not a topological group.

Our first lemma follows easily from Definition 2.1.

Lemma 2.3. *Let G be an almost topological group with structure $(\tau, \gamma, \mathcal{B})$ and identity e. Then*

(a) *If $U \in \mathcal{B}$, then $\widetilde{U}^{-1}$ is open in G, where $\widetilde{U} = U \setminus \{e\}$.*
(b) *For every subset $M \subseteq G$, the closure $\overline{M}$ of M in G is contained in the closure $\overline{M}^{\gamma}$ of M in (G, γ).*

Proposition 2.4. *Every almost topological group is saturated.*

Proof. Let G be an almost topological group with structure $(\tau, \gamma, \mathcal{B})$. If G is discrete, then clearly it is saturated. We suppose that G is not discrete. Let $U \in \mathcal{B}$ be a basic open neighborhood of the identity e of G. Clearly, U is infinite. By Lemma 2.3(a), $\widetilde{U}^{-1}$ is open in G and non-empty. Since $\widetilde{U} \subseteq U$, $int(U^{-1})$ is non-empty. □

The following proposition is intuitively evident.

Proposition 2.5. *Let G be a non-discrete almost topological group with structure $(\tau, \gamma, \mathcal{B})$. Then γ is the strongest Hausdorff topological group topology of G weaker than τ.*

Proof. Let σ be the strongest Hausdorff topological group topology of G weaker than τ. We have to prove that $\sigma = \gamma$. Since γ is a Hausdorff topological group topology, we have that $\gamma \subseteq \sigma$. By Proposition 2.4, G is saturated; then, by [3, Proposition 3], a local base at e in (G, σ) is the family $\mathcal{L}$ of all the sets UU^{-1}, with U an open neighborhood of e in G.

We will show that $\mathcal{L} \subseteq \gamma$. Let U be an open neighborhood of e in G. Since $UU^{-1} = \widetilde{U}\widetilde{U}^{-1} \cup \widetilde{U} \cup \widetilde{U}^{-1}$, then UU^{-1} is open in (G, γ). We have then that $\mathcal{L} \subseteq \gamma$, and from here, $\sigma \subseteq \gamma$. We conclude that $\sigma = \gamma$. □

Now we will show that subgroups of arbitrary products of almost topological groups are saturated. An important step towards the proof of Proposition 2.7 is the case of finite products.

Proposition 2.6. *Any subgroup of a finite product of almost topological groups is saturated.*

Proof. Let H be a non-discrete subgroup of a finite product $G = \prod_{i=1}^{n} A_i$ of almost topological groups with structures $(\tau_i, \gamma_i, \mathcal{B}_i)$, for $i = 1, \ldots, n$, and $V = H \cap \prod_{i=1}^{n} U_i$ a basic open neighborhood of the identity $e = (e_1, \ldots, e_n)$ in H, with $U_i \in \mathcal{B}_i$. We will prove that $int_H V^{-1}$ is not empty.

For every $x = (x_1, \ldots, x_n) \in V$, we define N_x as the number of non-identity coordinates of x, i.e., $N_x = |\{i \in \{1, \ldots, n\} : x_i \neq e_i\}|$, and let $k = max\,\{N_x : x \in V\}$. If $k = n$, choose any $x \in V$ with $N_x = n$. It follows that $x^{-1} \in H \cap \prod_{i=1}^{n} \widetilde{U_i}^{-1} \subseteq V^{-1}$; therefore, $x^{-1} \in int_H V^{-1}$. Suppose now that $k < n$. Pick an element $x = (x_1, \ldots, x_n) \in V$ such that $N_x = k$. Without loss of generality, we may assume that $x_i \neq e_i$ if and only if $i \leq k$. For every $i = 1, \ldots, k$ there exists an open symmetric neighborhood W_i of e_i in (G, γ_i) such that $x_i W_i \subseteq \widetilde{U}_i$ and $W_i x_i \subseteq \widetilde{U}_i$. The set $O = H \cap ((\prod_{i=1}^{k} x_i^{-1} W_i) \times (\prod_{j=k+1}^{n} U_j))$ is an open neighborhood of x^{-1} in H.

Claim. $O \subseteq V^{-1}$.

Consider any element $y = (x_1^{-1} w_1, \ldots, x_k^{-1} w_k, y_{k+1}, \ldots, y_n) \in O$, where $w_i \in W_i$ for each $i \leq k$. Since H is a group and $x_i W_i \subseteq \widetilde{U_i}$ for $i \leq k$, the element $x^2 y = (x_1 w_1, \ldots, x_k w_k, y_{k+1}, \ldots, y_n)$ belongs to V and $N_{x^2 y} = k$. Thus, $y_j = e_j$ for $j = k+1, \ldots, n$. This fact, together with the symmetry of the sets W_i, implies that $O^{-1} \subseteq V$ or, equivalently, $O \subseteq V^{-1}$. This proves the claim.

Since O is open and not empty, we conclude that the interior of V^{-1} in H is not empty. □

Proposition 2.7. *Any subgroup of an arbitrary product of almost topological groups is saturated.*

Proof. Let H be a non-discrete subgroup of a product $G = \prod_{\alpha \in \Lambda} A_\alpha$ of almost topological groups and V an open neighborhood of e in H. There exists a canonical basic open neighborhood $U = \prod_{\alpha \in \Lambda} U_\alpha$ of e in G such that $U \cap H \subseteq V$. Since U is an open set in G, there is a finite subset F of Λ

such that $U_\alpha \neq A_\alpha$ if and only if $\alpha \in F$. If $F = \emptyset$, then clearly $V = H$ and $V^{-1} = H$; so we suppose that $F \neq \emptyset$. The projection $p : G \to \prod_{\alpha\in F} A_\alpha$ is a continuous open epimorphism. Thus, $p(H)$ is a subgroup of $\prod_{\alpha\in F} A_\alpha$. By Proposition 2.6, $p(H)$ is saturated. Since $p(U) \cap p(H)$ is an open neighborhood of the identity of $p(H)$,

$$int_{p(H)}(p(U) \cap p(H))^{-1} = int_{p(H)}(p(U^{-1}) \cap p(H)) \neq \emptyset.$$

Take an open set W in $\prod_{\alpha\in F} A_\alpha$ such that $\emptyset \neq W \cap p(H) \subseteq p(U^{-1}) \cap p(H)$. Since $p^{-1}p(U^{-1}) = U^{-1}$, we have that

$$\emptyset \neq p^{-1}(W \cap p(H)) \cap H \subseteq U^{-1} \cap H \subseteq V^{-1}.$$

From here it follows that $int_H(V^{-1}) \neq \emptyset$. We conclude that H is saturated. □

Corollary 2.8. *Any subgroup of an arbitrary power of the Sorgenfrey line is saturated.*

3. Closures of Subgroups of Products of Almost Topological Groups

It is known that in paratopological groups, closures of subgroups need not be subgroups, even if they are discrete; see [2, Example 1.4.17] or Example 3.10 in this section. Things change in almost topological groups—we will prove in Theorem 3.7 that in arbitrary products of almost topological groups, closures of subgroups are subgroups. Our starting point is the following fact proved in [11].

Proposition 3.1. *The closure of any subgroup of an arbitrary power S^κ of the Sorgenfrey line is a subgroup of S^κ.*

Given a finite family $\{A_i\}_{i=1}^n$ of almost topological groups with structures $(\tau_i, \gamma_i, \mathcal{B}_i)$, we will denote by $\mathcal{R}$ the family of all the topologies σ on $A = \prod_{i=1}^n A_i$ of the form $\sigma = \prod_{i=1}^n \sigma_i$, with $\sigma_i \in \{\tau_i, \gamma_i\}$, for each $i \leq n$.

Lemma 3.2. *Let H be a subgroup of a finite product $A = \prod_{i=1}^n A_i$ of almost topological groups. For every $i \leq n$, let $(\tau_i, \gamma_i, \mathcal{B}_i)$ be a structure for A_i. Suppose that for every topology $\sigma \in \mathcal{R}$ strictly weaker than the topology of A, the closure $\overline{H}^\sigma$ of H in (A, σ) is a subgroup of A. Then the closure $\overline{H}$ of H in A is a subgroup of A.*

Proof. Clearly the identity $e = (e_1, \ldots, e_n)$ of A belongs to $\overline{H}$. By continuity of the multiplication, $\overline{H} \cdot \overline{H} = \overline{H}$. In order to show that $\overline{H}$ is a subgroup of A it remains to prove that $\overline{H} \subseteq \overline{H}^{-1}$. Let $x = (x_1, \ldots, x_n) \in \overline{H}$ and $U = \prod_{i=1}^n U_i$ be an open neighborhood of e in A such that $U_i \in \mathcal{B}_i$ for each $i \leq n$. We define $V = \prod_{i=1}^n V_i$ as follows: $V_i = U_i$ if A_i is not a topological group; otherwise, V_i is an open symmetric neighborhood of e_i

such that $V_i \subseteq U_i$ and $x_i V_i x_i^{-1} \subseteq U_i$. Clearly, V is an open neighborhood of e contained in U. Since $x \in \overline{H}$ and xV is a neighborhood of x in A, the intersection $xV \cap H$ is not empty, so there exist $v = (v_1, \ldots, v_n) \in V$ and $h = (h_1, \ldots, h_n) \in H$ such that $xv = h$. If $v = e$, we have that $x \in H$, and since H is a subgroup, $x^{-1} \in H \subseteq \overline{H}$ or $x \in \overline{H}^{-1}$.

If $v \neq e$, let $J = \{i \leq n : v_i \neq e_i\}$. We consider the topology $\sigma = \prod_{i=1}^n \sigma_i$ on A, defined by $\sigma_i = \gamma_i$ if $i \in J$ and $\sigma_i = \tau_i$ otherwise. By the definition, $\sigma \in \mathcal{R}$. Suppose first that the topology σ is strictly weaker than the topology of A. We define $O = \prod_{i=1}^n O_i$, with $O_i = V_i \setminus \{e_i\}$ if $i \in J$ and $O_i = V_i$ otherwise. Then $v \in O$ and the set Oh^{-1} is an open neighborhood of x^{-1} in (A, σ). Since $x \in \overline{H} \subseteq \overline{H}^{\sigma}$ and, by our hypothesis, $\overline{H}^{\sigma}$ is a subgroup of A, we have that $x^{-1} \in \overline{H}^{\sigma}$. Thus, $Oh^{-1} \cap H \neq \emptyset$, and then $O \cap H \neq \emptyset$. So $e \in OH$ and $x^{-1}OH$ is an open neighborhood of x^{-1} in (A, σ). Since $x^{-1} \in \overline{H}^{\sigma}$, we have that $x^{-1}OH \cap H \neq \emptyset$ or, equivalently, $x^{-1}O \cap H \neq \emptyset$. Hence,

$$\emptyset \neq x^{-1}V \cap H \subseteq x^{-1}U \cap H.$$

Therefore, $x^{-1} \in \overline{H}$.

Now we suppose that the topology σ coincides with the topology of A. This means that A_i is a topological group for every $i \in J$. We consider the element $h^{-1} = (v_1^{-1}x_1^{-1}, \ldots, v_n^{-1}x_n^{-1})$. If $i \notin J$, then $h_i^{-1} = x_i^{-1} \in x_i^{-1}U_i$. If $i \in J$, using the symmetry of V_i and the fact that $x_i V_i x_i^{-1} \subseteq U_i$, we have

$$h_i^{-1} = v_i^{-1}x_i^{-1} = x_i^{-1}(x_i v_i^{-1} x_i^{-1}) \in x_i^{-1}U_i.$$

Therefore, $h^{-1} \in x^{-1}U$. Since $x^{-1}U \cap H \neq \emptyset$, we have that $x^{-1} \in \overline{H}$. We conclude that $\overline{H}$ is a subgroup of A. □

Proposition 3.3. *Let H be a subgroup of an almost topological group G. Then $\overline{H}$ is a subgroup of G.*

Proof. This is the single factor version of Lemma 3.2. □

Lemmas 3.4 and 3.5 below will be used to prove that closures of subgroups of finite products of almost topological groups are subgroups.

Lemma 3.4. *Let H be a subgroup of $G = G_1 \times G_2$, where G_1 is an almost topological group and G_2 is a topological group. Then $\overline{H}$ is a subgroup of G.*

Proof. For $i = 1, 2$, let $(\tau_i, \gamma_i, \mathcal{B}_i)$ be a structure for G_i, with $\tau_2 = \gamma_2$. Let H be a subgroup of G. If there is a topology $\sigma \in \mathcal{R}$ strictly weaker than the topology of G, then $\sigma = \gamma_1 \times \tau_2$ is a topological group topology. Since in topological groups, closures of subgroups are subgroups, we conclude, by Lemma 3.2, that $\overline{H}$ is a subgroup of G. □

Proposition 3.5. *Let* $\{A_i\}_{i=1}^n$ *be a family of almost topological groups and* A_{n+1} *any topological group. Then the closure of any subgroup of* $G = \prod_{i=1}^{n+1} A_i$ *is a subgroup of* G*.*

Proof. We proceed by induction on n. For $n = 1$, the statement follows from Lemma 3.4.

Now suppose that the proposition holds for every positive integer $k < n$. Let $G = \prod_{i=1}^{n+1} A_i$, where (A_{n+1}, τ_{n+1}) is a topological group and the other factors are almost topological groups with structures $(\tau_i, \gamma_i, \mathcal{B}_i)$, for $i = 1, \ldots, n+1$, with $\tau_{n+1} = \gamma_{n+1}$. Let $\tau = \prod_{i=1}^{n+1} \tau_i$. Any topology $\sigma \in \mathcal{R}$ strictly weaker than τ is the topology of a product of less than n almost topological groups times a topological group. By the hypothesis of induction, closures of subgroups of (G, σ) are subgroups of G, and the required conclusion follows from Lemma 3.2. □

Again, the case of finite products is an important step towards the proof of Theorem 3.7.

Proposition 3.6. *Let* $\{A_i\}_{i=1}^n$ *be a finite family of almost topological groups. Then the closure of any subgroup of* $G = \prod_{i=1}^n A_i$ *is a subgroup of* G*.*

Proof. G is topologically isomorphic to $G \times \{e\}$, where $\{e\}$ is a one point topological group. The conclusion now follows from Proposition 3.5. □

The following theorem, our main result, is a consequence of Proposition 3.6.

Theorem 3.7. *Let* $\{A_i\}_{i \in I}$ *be a family of almost topological groups and* $G = \prod_{i \in I} A_i$*. Then the closure of any subgroup of* G *is a subgroup of* G*.*

Proof. Let H be any subgroup of G. Let $x \in (cl_G H)^{-1}$. We have to prove that $x \in cl_G H$. Let $U = \prod_{i \in I} U_i$ be a basic open neighborhood of x in G. Suppose that $U \neq G$. There exists a finite subset F of I such that $U_i = A_i$ for every $i \in I \setminus F$. Let $K = \prod_{i \in F} A_i$ and consider the projection $p : G \to K$. Since p is a homomorphism, $p(H)$ is a subgroup of K, and since K is a finite product of almost topological groups, $cl_K(p(H))$ is a subgroup of K by Proposition 3.6. The element $p(x)$ belongs to $cl_K(p(H))$, for

$$p(x) \in p((cl_G H)^{-1}) = p(cl_G H)^{-1} \subseteq cl_K(p(H))^{-1} = cl_K(p(H)).$$

Since p is open, $p(U) \in \mathcal{N}_K(p(x))$; thus, $p(U) \cap p(H) \neq \emptyset$. This implies that $U \cap H \neq \emptyset$. Therefore, $x \in cl_G(H)$. We conclude that $cl_G(H)$ is a subgroup of G. □

Proposition 3.8. *Any discrete subgroup of a product of almost paratopological groups is closed.*

Proof. Let G be a product of almost topological groups and H a discrete subgroup of G. Suppose, contrary to the conclusion of the corollary, that H is not closed. Pick any $x \in \overline{H} \setminus H$. Choose open neighborhoods U and V of the identity e of G such that $U \cap H = \{e\}$ and $V^2 \subseteq U$. Since $x \in \overline{H}$, the open neighborhood xV of x contains an element $xv \in H$, for some $v \in V$. By Theorem 3.7, the closure $\overline{H}$ of H in G is a subgroup of G; thus, $x^{-1} \in \overline{H}$. Choose $v' \in V$ such that $v'x^{-1} \in (Vx^{-1} \setminus \{v^{-1}x^{-1}\}) \cap H$. Then $v'v \in V^2 \cap H \subseteq U \cap H$, and $v'v \neq e$. This contradiction completes the proof. □

From Theorem 3.7, and adapting the proof of Corollary 3.4 in [11], we obtain the following.

Corollary 3.9. *Let $f : G \to H$ be an open continuous epimorphism, where G is an arbitrary product of almost topological groups and H is a semitopological group. Then H is a paratopological group and the closure of any subgroup of H is again a subgroup of H.*

Proof. By Theorem 3.7, the closure of any subgroup of G is again a subgroup of G. From [11, Proposition 3.3], it follows that the closure of any subgroup of H is a subgroup of H. We shall prove now that H is a paratopological group. Let U be an open neighborhood of the identity e_H of H. Then $W = f^{-1}(U)$ is an open neighborhood of the identity e of the paratopological group G. Then there exists an open neighborhood O of e in G such that $O^2 \subseteq W$. Clearly, $V = f(O)$ is an open neighborhood of the identity in H, and we have that $V^2 = f(O^2) \subseteq f(W) = U$. We have proved that $V^2 \subseteq U$, whence it follows that H is a paratopological group. □

Proposition 3.3 and Theorem 3.7 cannot be extended to the wider class of saturated paratopological groups, as the following example shows.

Example 3.10. *There exist a saturated paratopological group G_1 and a subgroup H of G_1 such that $cl(H)$ is not a subgroup of G_1.*

Proof. Let $\mathbb{N}$ be the set of positive integers. The family

$$\mathcal{V} = \left\{ \{0\}^k \times (\mathbb{N} \cup \{0\})^{\omega \setminus k} \right\}_{k=1}^{\infty}$$

is a local base at the identity e_G for a Hausdorff topology that makes G into a paratopological group. The subset $A = \{a_0 + a_n : n \in \mathbb{N}\}$ of G, where a_i is the point with the ith coordinate equal to 1 and the other coordinates equal to 0, for $i \in \mathbb{N} \cup \{0\}$, generates a discrete subgroup H

of G. It is shown in [3, Example 1.4.17] that $cl_G H$ is not a subgroup of G. The identity map $G \to \mathbb{Z}^\omega$ is a continuous group isomorphism, where $\mathbb{Z}$ is discrete and $\mathbb{Z}^\omega$ carries the usual product topology. Hence, G is subtopological. By [4, Corollary 3], G is a closed subgroup of a saturated paratopological group G_1. The set $cl_{G_1}(H)$ is not a subgroup of G_1. Otherwise, since G is closed in G_1, $cl_G H = cl_{G_1} H$ would be a subgroup of G. □

Following [3], we will say that a paratopological group is 2-*oscillating* (3-*oscillating*) provided that for every open neighborhood U of the identity e there is an open neighborhood V of e such that $V^{-1}V \subseteq UU^{-1}$ ($V^{-1}VV^{-1} \subseteq UU^{-1}U$). Clearly, 2-oscillating paratopological groups are 3-oscillating. It is shown in [3, Proposition 3] that saturated Hausdorff paratopological groups are 2-oscillating, and in [3, Theorem 3], the authors prove that every Hausdorff 3-oscillating paratopological group is subtopological. Example 3.10 shows that Theorem 3.7 is no longer valid in the class of Hausdorff saturated paratopological groups or in the class of subtopological groups.

We finish with another proposition and a question.

Proposition 3.11. *Any Hausdorff first countable 3-oscillating paratopological group admits a weaker Hausdorff first countable topological group topology.*

Proof. Let (G, τ) be a Hausdorff first countable 3-oscillating paratopological group. By [3, Theorem 3], G is subtopological. Let γ be the strongest Hausdorff topological group topology for the algebraic group G contained in τ, and $G' = (G, \gamma)$. Let $\mathcal{L}$ be a countable base at the identity e in G. It is shown in [3] that a local base at e in G' is the family of the sets $UU^{-1}U$, with U an open neighborhood of e in G. Then the family $\mathcal{L}' = \{UU^{-1}U : U \in \mathcal{L}\}$ is a countable base for e in G'. □

Corollary 3.12. *Let G be a Hausdorff first countable subtopological group with a Hausdorff topological group topology γ weaker than the topology of G. Let $G' = (G, \gamma)$. Suppose that for every open neighborhood U of the identity e in G, the set $int_{G'}U$ is not empty. Then G admits a first countable topological group topology.*

Proof. For every open neighborhood U of the identity e in G, the set $int_{G'}U$ is not empty; from here, it follows that $int_G U^{-1}$ is not empty. Thus, G is saturated. Since saturated paratopological groups are 3-oscillating, by Proposition 3.11, G admits a first countable topological group topology. □

Question 3.13. Does every Hausdorff first countable subtopological group admit a weaker Hausdorff first countable topological group topology?

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