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CHARACTERIZATIONS OF QUASI-PSEUDO-METRIZABLE SPACES

KEDIAN LI AND FUCAI LIN

Abstract. In this paper, we give some characterizations of quasi-pseudo-metrization spaces by means of pairwise weak base g-functions.

1. Introduction

J. C. Kelly [5] began the first systematized study of bitopological spaces, and then C. W. Patty [11] and S. Romaguera [12] obtained the necessary and sufficient conditions for characterizing quasi-pseudo-metrizability of bispaces. Recently, Josepha Marín [9] has systematically studied the quasi-pseudo-metrization theorem in the style of the Frínk metrization theorem by using weak bases. In this paper, we shall continue this approach and give a quasi-pseudo-metrization theorem of bispaces by means of pairwise weak base g-functions.

First, we list some concepts and notation used in this paper. Throughout, N denotes the set of positive integers. Suppose we are given a bitopological space (X, τ_i, τ_j), where τ_i and τ_j are topologies on $X (i, j = 1, 2$ and $i \neq j)$. For $A \subset X$, $cl_{\tau_i} A$ and $int_{\tau_i} A$ denote the closure and the interior of a set A in a topological space (X, τ_i), respectively, and "a sequence $\{y_n\}$ τ_i-converges to x" denotes "a sequence $\{y_n\}$ converges to x in a topological space (X, τ_i)." All spaces (X, τ_i) in this paper are assumed to be T_1.

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Definition 1.1. Let (X, τ) be a topological space. A family $\mathcal{B}$ of a subset of X is a *weak base* [1] for the topology τ if, for each $x \in X$, there is a subfamily $\mathcal{B}_x$ of $\mathcal{B}$, such that

(a) $x \in B$, for each $B \in \mathcal{B}_x$;
(b) if $A, B \in \mathcal{B}_x$, there is a $C \in \mathcal{B}_x$ such that $C \subset A \cap B$;
(c) a subset $U \subset X$ is τ-open if and only if, for each $x \in U$, there exists a subset $B \in \mathcal{B}_x$ such that $B \subset U$.

The family $\mathcal{B}_x$ is called a local weak base at x.

A topological space (X, τ) is said to have a *weak base g-function* [4], if there is a function $g : N \times X \to \mathcal{P}(X)$ such that

(a) $x \in g(n, x)$ for all $x \in X$ and $n \in N$;
(b) $g(n + 1, x) \subset g(n, x)$ for all $n \in N$;
(c) $\{g(n, x) : n \in N, x \in X\}$ is a weak base for X.

Let us recall that a function $d : X \times X \to R^+$ is a *quasi-pseudo-metric* on a set X if for all $x, y, z \in X$, it satisfies

(a) $d(x, x) = 0$;
(b) $d(x, z) \leq d(x, y) + d(y, z)$.

If d is a quasi-pseudo-metric on X, the function d^{-1}, defined by $d^{-1}(x, y) = d(y, x)$, is called the *conjugate quasi-pseudo-metric of* d on X. Each quasi-pseudo-metric d on a set X induces a topology $\tau(d)$ on X where for all $x \in X$ and all $r > 0$, $B_d(x, r) = \{y \in X : d(x, y) < r\}$ is an open d-ball and the family of open d-balls $\{B_d(x, r) : x \in X, r > 0\}$ is a base for the topology $\tau(d)$. A *bispace* (a bitopological space in [5]) is a triple (X, τ_i, τ_j) where X is a nonempty set and τ_i and τ_j are two topologies on X. A bispace (X, τ_i, τ_j) is quasi-pseudo-metrizable if there exists a quasi-pseudo-metric d on X such that $\tau(d) = \tau_i$ and $\tau(d^{-1}) = \tau_j$ (or $\tau(d) = \tau_j$ and $\tau(d^{-1}) = \tau_i$)$(i, j = 1, 2$ and $i \neq j)$.

Definition 1.2 ([9])**.** A *pair open cover* of a bispace $(X, \tau_i, \tau_j)(i, j = 1, 2$ and $i \neq j)$ is a family of pairs $(\mathcal{G}_i, \mathcal{G}_j) = \{(G_{i,\alpha}, G_{j,\alpha}) : \alpha \in I\}$ such that

(i) for each $\alpha \in I$, $G_{i,\alpha} \in \tau_i$ for $i = 1, 2$;
(ii) $\mathcal{G}_i = \{G_{i,\alpha} : \alpha \in I\}$ is a cover of X for $i = 1, 2$;
(iii) for each $x \in X$, there is an $\alpha \in I$ such that $x \in G_{1,\alpha} \cap G_{2,\alpha}$.

Let $(\mathcal{G}_i, \mathcal{G}_j)$ and $(\mathcal{G}'_i, \mathcal{G}'_j)$ be pair open covers of a bispace (X, τ_i, τ_j). We say that $(\mathcal{G}'_i, \mathcal{G}'_j)$ refines $(\mathcal{G}_i, \mathcal{G}_j)$ (that is $(\mathcal{G}'_i, \mathcal{G}'_j) < (\mathcal{G}_i, \mathcal{G}_j)$) if for each pair $(G'_{i,\alpha}, G'_{j,\alpha}) \in (\mathcal{G}'_i, \mathcal{G}'_j)$, there is a pair $(G_{i,\beta}, G_{j,\beta}) \in (\mathcal{G}_i, \mathcal{G}_j)$ such that $G'_{i,\alpha} \subset G_{i,\beta}$ and $G'_{j,\alpha} \subset G_{j,\beta}$ for $i, j = 1, 2$ and $i \neq j$.

Let $(\mathcal{G}_i, \mathcal{G}_j)$ be a pair open cover of a bispace (X, τ_i, τ_j). Let A be a nonempty subset of X. We define $St(A, \mathcal{G}_i, \mathcal{G}_j) = \bigcup\{G_{i,\alpha} \in \mathcal{G}_i : A \cap G_{j,\alpha} \neq \emptyset\}$ for $i, j = 1, 2$ and $i \neq j$. If $x \in X$, we define $St(x, \mathcal{G}_i, \mathcal{G}_j) = \bigcup\{G_{i,\alpha} \in \mathcal{G}_i : x \in G_{j,\alpha}\}$ and $St^2(x, \mathcal{G}_i, \mathcal{G}_j) = St(St(x, \mathcal{G}_i, \mathcal{G}_j), \mathcal{G}_i, \mathcal{G}_j)$.

Definition 1.3 ([9])**.** A *pair development* in a bispace (X, τ_i, τ_j) is a sequence $\{(\mathcal{G}_{i,n}, \mathcal{G}_{j,n}) : n \in N\}$ of pair open covers of X such that, for each $x \in X$, $\{St(x, \mathcal{G}_{i,n}, \mathcal{G}_{j,n}) : n \in N\}$ is a base of τ_i-neighborhoods of x, where $\mathcal{G}_{i,n} = \{G_i(n, \alpha) : \alpha \in I\}$ for $i = 1, 2$.

A bispace (X, τ_i, τ_j) is *pairwise developable* if it has a pair development $\{(\mathcal{G}_{i,n}, \mathcal{G}_{j,n}) : n \in N\}$ such that $(\mathcal{G}_{i,n+1}, \mathcal{G}_{j,n+1}) < (\mathcal{G}_{i,n}, \mathcal{G}_{j,n})$ for each $n \in N$.

Similarly, we have the following definition.

Definition 1.4. A *pair weak development* in a bispace (X, τ_i, τ_j) is a sequence $\{(\mathcal{G}_{i,n}, \mathcal{G}_{j,n}) : n \in N\}$ of pair covers of X such that, for each $x \in X$, $\{St(x, \mathcal{G}_{i,n}, \mathcal{G}_{j,n}) : n \in N\}$ is a weak base of τ_i-neighborhoods of x.

A bispace (X, τ_i, τ_j) is *pairwise weak developable* if it is a pair weak development $\{(\mathcal{G}_{i,n}, \mathcal{G}_{j,n}) : n \in N\}$ such that $(\mathcal{G}_{i,n+1}, \mathcal{G}_{j,n+1}) < (\mathcal{G}_{i,n}, \mathcal{G}_{j,n})$ for each $n \in N$.

A bitopological space (X, τ_i, τ_j) is said to have a *pairwise weak base g-function* if there is a function $g_i : N \times X \to \mathcal{P}(X)(i = 1, 2)$ such that

(a) $x \in g_i(n, x) \cap g_j(n, x)$ for all $x \in X$ and $n \in N(i, j = 1, 2$ and $i \neq j)$;
(b) $g_i(n + 1, x) \subset g_i(n, x)$ for all $n \in N(i = 1, 2)$;
(c) $\{g_i(n, x) : n \in N, x \in X\}$ is a weak base for space $(X, \tau_i)(i = 1, 2)$.

For $i = 1, 2$, let g_i be a pairwise weak base g-function for X and define $g_i^2(n, x) = \bigcup\{g_i(n, t) : t \in g_i(n, x)\}$.

Theorem 1.5 (Marín [9, Theorem 1])**.** *A bispace (X, τ_i, τ_j) is quasi-pseudo-metrizable if and only if for each $x \in X$, there exist two decreasing sequences $\mathcal{B}_{i,x} = \{V_i(n, x) : n \in N\}(i = 1, 2)$ of subsets of X such that $\mathcal{B}_i = \bigcup\{\mathcal{B}_{i,x} : x \in X\}$ is a weak base for a space (X, τ_i) and such that given $x \in X$ and $n \in N$, there exists $m = m(n, x) > n$ satisfying if $V_i(m, x) \cap V_j(m, y) \neq \emptyset$, then $V_i(m, y) \subset V_i(n, x)$ for $i, j = 1, 2$, and $i \neq j$.*

2. Main Results and Their Proofs

Theorem 2.1. *A bispace (X, τ_i, τ_j) is quasi-pseudo-metrizable if and only if it is pairwise weak developable and it has pair weak development $\{(\mathcal{G}_{i,n}, \mathcal{G}_{j,n}) : n \in N\}$ such that $\{St^2(x, \mathcal{G}_{i,n}, \mathcal{G}_{j,n}) : n \in N, x \in X\}$ is a weak base for a space $(X, \tau_i)(i, j = 1, 2$ and $i \neq j)$.*

Proof. Necessity. Let (X, τ_i, τ_j) be a quasi-pseudo-metrizable bispace satisfying the conditions in Theorem 1.5. For each $x \in X$ and each $n \in N$, put $\mathcal{G}_{i,n} = \{V_i(n, x) : x \in X\}$. Then $\mathcal{G}_{i,n}$ is a cover of X for each $n \in N$ and $i = 1, 2$, and hence $\{(\mathcal{G}_{i,n}, \mathcal{G}_{j,n}) : n \in N\}$ is a sequence of pair covers of X. We need only to show that $\{St^2(x,\mathcal{G}_{i,n},\mathcal{G}_{j,n}) : n \in N, x \in X\}$ is a weak base of space $(X,\tau_i)(i,j=1,2$ and $i \neq j)$.

For each $x \in X$ and $i = 1,2$, let $\mathcal{B}_{i,x} = \{St^2(x,\mathcal{G}_{i,n},\mathcal{G}_{j,n}) : n \in N\}$. If $A, B \in \mathcal{B}_{i,x}$, then $A = St^2(x,\mathcal{G}_{i,k},\mathcal{G}_{j,k})$ and $B = St^2(x,\mathcal{G}_{i,m},\mathcal{G}_{j,m})$. Let $n = max\{k,m\}$, then $C = St^2(x,\mathcal{G}_{i,n},\mathcal{G}_{j,n}) \in \mathcal{B}_{i,x}$. Since $\{V_i(n,x) : n \in N\}$ is a decreasing sequence, we have $C \subset A \cap B$.

Let $U \in \tau_i$ and $x \in U$. Since $\{V_i(n,x) : n \in N\}$ is a weak base of x in space (X,τ_i), there exists a $k \in N$ such that $V_i(k,x) \subset U$. By Theorem 1.5, there exists an $l \in N$ with $l > k$ satisfying if $V_i(l,x) \cap V_j(l,y) \neq \emptyset$, then $V_i(l,y) \subset V_i(k,x)$ for $i,j = 1,2$ and $i \neq j$. Similarly, for $x \in X$ and $l \in N$, there exists a $p \in N$ with $p > l$ satisfying if $V_i(p,x) \cap V_j(p,y) \neq \emptyset$, then $V_i(p,y) \subset V_i(l,x)$ for $i,j = 1,2$. For $x \in V_j(p,y)$, we have $V_i(p,x) \cap V_j(p,y) \neq \emptyset$, and therefore $V_i(p,y) \subset V_i(l,x)$ by Theorem 1.5. Hence, $St(x,G_{i,p},G_{j,p}) \subset V_i(l,x)$. If $St(x,G_{i,p},G_{j,p}) \cap V_j(p,y) \neq \emptyset$, then $V_i(l,x) \cap V_j(p,y) \neq \emptyset$, and hence $V_i(l,x) \cap V_j(l,y) \neq \emptyset$, which implies that $V_i(p,y) \subset V_i(l,y) \subset V_i(k,x)$ by Theorem 1.5. Therefore, $St^2(x,\mathcal{G}_{i,p},\mathcal{G}_{j,p}) \subset V_i(k,x) \subset U$. On the other hand, if $U \subset X$ and $x \in U$, there exists an $s \in N$ such that $St^2(x,\mathcal{G}_{i,s},\mathcal{G}_{j,s}) \subset U$. For $t \in N$ and $t > s$, we have $V_i(t,x) \subset St^2(x,\mathcal{G}_{i,s},\mathcal{G}_{j,s})$. Then $V_i(t,x) \subset U$, and hence U is a τ_i-open set.

Sufficiency. Let $\{(\mathcal{G}_{i,n},\mathcal{G}_{j,n}) : n \in N\}$ be a pair weak development of X such that

$$\{St^2(x,\mathcal{G}_{i,n},\mathcal{G}_{j,n}) : n \in N, x \in X\}$$

is a weak base of a space $(X,\tau_i)(i,j=1,2$ and $i \neq j)$.

CLAIM 1. If a sequence $\{x_n\}$ τ_i-converges to x, then $A = \{x\} \cup \{x_n : n \in N\}$ is τ_i-closed.

Suppose A is not τ_i-closed. There exists a point $y \in cl_{\tau_i}A - A$ such that $St(y,G_{i,n},G_{j,n}) \cap A \neq \emptyset$ for all $n \in N$; that is, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \in St(y,G_{i,k},G_{j,k})$ for each $k \in N$. Let $B = \{y\} \cup \{x_{n_k} : k \in N\}$ and suppose that B is not τ_i-closed. Then there exists a point $z \in cl_{\tau_i}B - B$ such that $St(z,G_{i,n},G_{j,n}) \cap B \neq \emptyset$ for all $n \in N$; that is, there exists a subsequence $\{x_{n_{k_s}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_s}} \in St(z,G_{i,s},G_{j,s})$ for each $s \in N$. We also have $x_{n_{k_s}} \in St(y,G_{i,k_s},G_{j,k_s})$ for each $s \in N$. Since $k_s > s$, it follows that $St(y,G_{i,k_s},G_{j,k_s}) \subset St(y,G_{i,s},G_{j,s})$, and therefore $St(y,G_{i,s},G_{j,s}) \cap St(z,G_{i,s},G_{j,s}) \neq \emptyset$ for each $s \in N$. Since $z \neq y$, this contradicts that $\{(\mathcal{G}_{i,n},\mathcal{G}_{j,n}) : n \in N\}$ is a pair weak development of X; hence, B is τ_i-closed. Since the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ τ_i-converges to x and $x \notin B$, B is not τ_i-closed, which is a contradiction. Thus, A must be τ_i-closed.

CLAIM 2. If a sequence $\{x_n\}$ τ_i-converges to x in (X,τ_i), then $\{x_n\}$ is eventually in $St(x,\mathcal{G}_{i,m},\mathcal{G}_{j,m})$ for each $m \in N$.

Suppose that the sequence $\{x_n\}$ is not eventually in $St(x, \mathcal{G}_{i,m}, \mathcal{G}_{j,m})$ for some $m \in N$; then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \notin St(x, \mathcal{G}_{i,m}, \mathcal{G}_{j,m})$ for each $k \in N$. Since $\{x_{n_k}\}$ τ_i-converges to x, by Claim 1, $C = \{x\} \cup \{x_{n_k} : k \in N\}$ is τ_i-closed. However, $x_{n_k} \notin St(x, \mathcal{G}_{i,m}, \mathcal{G}_{j,m})$ for each $k \in N$, which implies that the set $C - \{x\}$ is also τ_i-closed, which contradicts the fact that the sequence $\{x_{n_k}\}$ converges to x. It follows that the sequence $\{x_n\}$ is eventually in the set $St(x, \mathcal{G}_{i,n}, \mathcal{G}_{j,n})$ for each $n \in N$; that is, the sequence $\{x_n\}$ τ_i-converges to x if and only if the sequence $\{x_n\}$ is eventually in $St(x, \mathcal{G}_{i,m}, \mathcal{G}_{j,m})$ for each $m \in N$.

CLAIM 3. For each $x \in X$, $\{\text{int}_{\tau_i} St^2(x, \mathcal{G}_{i,n}, \mathcal{G}_{j,n}) : n \in N\}$ is a base of τ_i-neighborhoods of x in the space $(X, \tau_i)(i = 1, 2)$.

For each $n \in N$, let $U = X - St^2(x, \mathcal{G}_{i,n}, \mathcal{G}_{j,n})$ and $F = \{y \in X : St^2(y, \mathcal{G}_{i,m}, \mathcal{G}_{j,m}) \cap U \neq \emptyset$ for each $m \in N\}$. Assume that F is not τ_i-closed; then there exists a point $t \notin F$ such that $St(t, \mathcal{G}_{i,k}, \mathcal{G}_{j,k}) \cap F \neq \emptyset$ for each $k \in N$, and hence there exists a sequence $\{x_k\} \subset F$ with $x_k \in St(t, \mathcal{G}_{i,k}, \mathcal{G}_{j,k})$ for each $k \in N$. For each $k \in N$, by the definition of F, we have that $St^2(x_k, \mathcal{G}_{i,m}, \mathcal{G}_{j,m}) \cap U \neq \emptyset$ for each $m \in N$, and thus there exists a sequence $\{y_m\} \subset U$ such that $y_m \in St^2(x_k, \mathcal{G}_{i,m}, \mathcal{G}_{j,m})$ for each $m \in N$. Then the sequence $\{y_m\}$ τ_i-converges to x_k, and therefore $\{y_m\}$ is eventually in the set $St(x_k, \mathcal{G}_{i,t}, \mathcal{G}_{j,t})$ for each $t \in N$ by Claim 2. Thus, there exists a subsequence $\{y_{m_s}\}$ of $\{y_m\}$ such that $y_{m_s} \in St(x_k, \mathcal{G}_{i,s}, \mathcal{G}_{j,s})$ for each $s \in N$. In particular, we have $\{y_{m_k}\} \in St(x_k, \mathcal{G}_{i,k}, \mathcal{G}_{j,k})$ for each $k \in N$. Since $x_k \in St(t, \mathcal{G}_{i,k}, \mathcal{G}_{j,k})$ for each $k \in N$, we have $y_{m_k} \in St^2(t, \mathcal{G}_{i,k}, \mathcal{G}_{j,k})$ for each $k \in N$. By the definition of F, we have $t \in F$, a contradiction which shows that the set F is τ_i-closed. Hence, $x \in X - F \subset St^2(x, \mathcal{G}_{i,n}, \mathcal{G}_{j,n})$; that is, $x \in \text{int}_{\tau_i} St^2(x, \mathcal{G}_{i,n}, \mathcal{G}_{j,n})$ for each $n \in N$. Therefore, $\{\text{int}_{\tau_i} St^2(x, \mathcal{G}_{i,n}, \mathcal{G}_{j,n}) : n \in N\}$ is a base of τ_i-neighborhoods of x in (X, τ_i), which implies that (X, τ_i) is first countable $(i = 1, 2)$.

CLAIM 4. For each $x \in X$, $x \in \text{int}_{\tau_i} St(x, \mathcal{G}_{i,n}, \mathcal{G}_{j,n})$ for each $n \in N$.

Suppose $x \notin \text{int}_{\tau_i} St(x, \mathcal{G}_{i,n}, \mathcal{G}_{j,n})$ for some $n \in N$. By Claim 3, there exists a sequence $\{z_m\} \subset X - \text{int}_{\tau_i} St(x, \mathcal{G}_{i,n}, \mathcal{G}_{j,n})$ such that $\{z_m\}$ τ_i-converges to x. By Claim 2, there exists a subsequence $\{z_{m_k}\}$ of $\{z_m\}$ such that $z_{m_k} \in St(x, \mathcal{G}_{i,k}, \mathcal{G}_{j,k})$ for each $k \in N$, which is a contradiction.

For each $x \in X$ and each $n \in N$, let $V_i(n, x) = St(x, \mathcal{G}_{i,n}, \mathcal{G}_{j,n})$. By claims 3 and 4, there exists $m = m(n, x) > n$ such that $St^2(x, \mathcal{G}_{i,m}, \mathcal{G}_{j,m}) \subset St(x, \mathcal{G}_{i,n}, \mathcal{G}_{j,n}) = V_i(n, x)$. Similarly, for $x \in X$ and $m \in N$, there exists $k = k(x, m) > m$ such that $St^2(x, \mathcal{G}_{i,k}, \mathcal{G}_{j,k}) \subset St(x, \mathcal{G}_{i,m}, \mathcal{G}_{j,m}) = V_i(m, x)$.

If $V_i(k, x) \cap V_j(k, y) \neq \emptyset$, we shall show that $V_i(k, y) \subset V_i(n, x)$.

Let $V_i(k, x) \cap V_j(k, y) \neq \emptyset$. Then there exists $t \in V_i(k, x) \cap V_j(k, y)$. For $t \in V_j(k, y) = St(y, \mathcal{G}_{j,k}, \mathcal{G}_{i,k})$, there exists $G_j(k, \beta) \in \mathcal{G}_{j,k}$ such that $t \in G_j(k,\beta)$ and $y \in G_i(k,\beta)$, and thus $y \in St(t,\mathcal{G}_{i,k},\mathcal{G}_{j,k})$, which implies that $St(y,\mathcal{G}_{i,k},\mathcal{G}_{j,k}) \subset St^2(t,\mathcal{G}_{i,k},\mathcal{G}_{j,k})$. Similarly, for $t \in V_i(k,x)$, we have $St(t,\mathcal{G}_{i,k},\mathcal{G}_{j,k}) \subset St^2(x,\mathcal{G}_{i,k},\mathcal{G}_{j,k})$. Because $St^2(x,\mathcal{G}_{i,k},\mathcal{G}_{j,k}) \subset St(x,\mathcal{G}_{i,m},\mathcal{G}_{j,m})$, we have $St^2(t,\mathcal{G}_{i,k},\mathcal{G}_{j,m}) \subset St^2(x,\mathcal{G}_{i,m},\mathcal{G}_{j,m})$. Therefore,

$$V_i(k,y) = St(y,\mathcal{G}_{i,k},\mathcal{G}_{j,k}) \subset St^2(x,\mathcal{G}_{i,m},\mathcal{G}_{j,m}) \subset V_i(n,x).$$

Let $\mathcal{B}_i = \{V_i(n,x) : x \in X \text{ and } n \in N\}$ be a collection of subsets of $(X,\tau_i)(i=1,2)$. Then the family $\mathcal{B}_i(i=1,2)$ satisfies the conditions of Theorem 1.5, and hence a bispace (X,τ_i,τ_j) is quasi-pseudo-metrizable. □

Theorem 2.2. *A bispace (X,τ_i,τ_j) is quasi-pseudo-metrizable if and only if it has a pairwise weak base g-function $g_i : N \times X \to \mathcal{P}(X)(i=1,2)$ satisfying that*

1. *for each $x \in X$ and τ_i-neighborhood U of x, there exists an $m \in N$ such that $x \notin cl_{\tau_i}\bigcup\{g_j(m,y) : y \in X-U\}$;*
2. *for any $Y \subset X$ and each $n \in N$, $cl_{\tau_i}Y \subset \bigcup\{cl_{\tau_i}g_j^2(n,y) : y \in Y\}$.*

Proof. Necessity. Let (X,τ_i,τ_j) be a quasi-pseudo-metrizable bispace which satisfies the conditions in Theorem 1.5. For each $x \in X$ and each $n \in N$, let

$$g_i(n,x) = V_i(n,x) \quad \text{and} \quad \mathcal{G}_{i,n} = \{g_i(n,x) : x \in X\};$$

then $\mathcal{G}_{i,n}$ is a cover of X for each $n \in N$ and $i = 1,2$.

(1) For each $x \in X$ and τ_i-neighborhood U of x, there exists a $k \in N$ such that $g_i(k,x) = V_i(k,x) \subset U$. For, given x and $k \in N$, by Theorem 1.5, there is an $l > k$ such that if $V_i(l,x) \cap V_j(l,y) \neq \emptyset$, then $V_i(l,y) \subset V_i(k,x)$ for $i,j = 1,2$, and $i \neq j$. Assume that, for any $n \in N$, we have $x \in cl_{\tau_i}\bigcup\{g_j(n,y) : y \in X-U\}$. Since X is first countable, there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n \in g_j(n,y_n)$ and $y_n \in X-U$, and the sequence $\{x_n\}$ τ_i-converges to x. By [7, Corollary 1.6.19], $V_i(l,x)$ is a τ_i-sequence neighborhood of x in (X,τ_i), and thus the sequence $\{x_n\}$ is eventually in the set $V_i(l,x)$. Then there exists an $n_l > l$ such that $x_{n_l} \in V_i(l,x)$. Since $x_{n_l} \in g_j(n_l,y_{n_l}) = V_j(n_l,y_{n_l})$, $V_i(l,x) \cap V_j(n_l,y_{n_l}) \neq \emptyset$, which implies that $V_i(l,x) \cap V_j(l,y_{n_l}) \neq \emptyset$, and therefore $V_i(l,y_{n_l}) \subset V_i(k,x)$. Then $y_{n_l} \in V_i(l,y_{n_l}) \subset V_i(k,x) \subset U$, which contradicts $y_{n_l} \in X-U$. Thus, (1) is satisfied.

(2) For any $Y \subset X$ and each $n \in N$, since $\bigcup\{cl_{\tau_i}g_j(n,y) : y \in Y\} \subset \bigcup\{cl_{\tau_i}g_j^2(n,y) : y \in Y\}$, it is sufficient to show that $cl_{\tau_i}Y \subset \bigcup \{cl_{\tau_i}g_j(n,y) : y \in Y\}$.

Assume that there are a subset $Y \subset X$ and an $m \in N$ such that $cl_{\tau_i}Y \not\subset \bigcup\{cl_{\tau_i}g_j(m,y) : y \in Y\}$; then there exists a point $x \in cl_{\tau_i}Y - \bigcup$

$\{cl_{\tau_i} g_j(m,y) : y \in Y\}$. Since (X,τ_i) is first countable, there is a sequence $\{y_n\} \subset Y$ such that $\{y_n\}$ τ_i-converges to x. Because $x \notin \bigcup\{cl_{\tau_i} g_j(m,y) : y \in Y\}$, we have $g_j(m,y_k) \cap \{y_n : n \in N\}$ is at most finite for each $k \in N$. Hence, there exists a subsequence $\{y_{n_s}\}$ of $\{y_n\}$ such that $y_{n_t} \notin g_j(m,y_{n_s}) = V_j(m,y_{n_s})$ for each $t > s$. By the definition of the $V_j(n,x)(j=1,2)$ in the proof of Theorem 1 in [9], we have $d(y_{n_s},y_{n_t}) > 1/2^m$ or $d(y_{n_t},y_{n_s}) > 1/2^m$ for each $t > s$, where d is a quasi-pseudo-metric on X. Hence, the subsequence $\{y_{n_s} : s \in N\}$ is τ_i-closed discrete, which is a contradiction to the sequence $\{y_{n_s}\}$ τ_i-converging to x. Therefore, (2) is satisfied.

Sufficiency. Assume that a bispace (X,τ_i,τ_j) has a pairwise weak base g-function $g_i : N \times X \to \mathcal{P}(X)$ satisfying conditions (1) and (2). For each $(n,x) \in N \times X$, let

$$h_i(n,x) = \{y \in X : x \in cl_{\tau_i} g_j^2(n,y)\}, \quad k_i(n,x) = g_i(n,x) \cap h_i(n,x),$$
$$\text{and } \mathcal{G}_{i,n} = \{k_i(n,x) : x \in X\}(i=1,2).$$

By virtue of Theorem 2.1, we need to show that $\{(\mathcal{G}_{i,n},\mathcal{G}_{j,n}) : n \in N\}$ is a pairwise weak development of bispace $(X,\tau_i,\tau_j)(i,j=1,2$ and $i \neq j)$ such that $\{St^2(x,\mathcal{G}_{i,n},\mathcal{G}_{j,n}) : x \in X, n \in N\}$ is a weak base for $(X,\tau_i)(i=1,2)$.

First, we shall show $h_i(n,x)$ is a τ_i neighborhood of x in (X,τ_i). To do this, we need to prove $x \in U(n,x) = X - cl_{\tau_i}\{y \in X : x \notin cl_{\tau_i} g_j^2(n,y)\} \subset h_i(n,x)$.

In fact, if $x \notin U(n,x)$, then $x \in cl_{\tau_i}\{y \in X : x \notin g_j^2(n,y)\}$. Let $A = \{y \in X : x \notin cl_i g_j^2(n,y)\}$. By (2), we have $x \in \bigcup\{cl_{\tau_i} g_j^2(n,y) : y \in A\}$. Then there exists a point $y_0 \in A$ such that $x \in cl_{\tau_i} g_j^2(n,y_0)$. So $y_0 \notin A$. This is a contradiction. Further, it is easy to check that $U(n,x) \subset h_i(n,x)$. Thus, we have verified that $h_i(n,x)$ is a τ_i-neighborhood of x in (X,τ_i).

Next, we shall show that $\mathcal{B}_i = \{k_i(n,x) : x \in X, n \in N\}$ is a weak base for $(X,\tau_i)(i=1,2)$. Let $U \subset X$ be nonempty. If U is τ_i-open and $x \in U$, then there is $(n,x) \in N \times X$ such that $g_i(n,x) \subset U$ since g_i is a pairwise weak base g-function. Hence, $k_i(n,x) \subset U$. Conversely, suppose for any $x \in U$, we have some $n \in N$ with $k_i(n,x) \subset U$. Since $h_i(n,x)$ is a τ_i-neighborhood of x in (X,τ_i), there exists an $m \in N$ such that $g_i(m,x) \subset h_i(n,x)$. Let $p = max\{m,n\}$. Then $g_i(p,x) \subset g_i(n,x) \cap h_i(n,x) = k_i(n,x) \subset U$. Since $\{g_i(n,x) : n \in N, x \in X\}$ is a weak base for space (X,τ_i), U must be τ_i-open.

Now, we shall show that $\{St^2(x,\mathcal{G}_{i,n},\mathcal{G}_{j,n}) : x \in X, n \in N\}$ is a weak base for space (X,τ_i). To do this, let $U \subset X$ be nonempty. Suppose for any $x \in U$, there is some $n \in N$ such that $St^2(x,\mathcal{G}_{i,n},\mathcal{G}_{j,n}) \subset U$. Then $k_i(n,x) \subset U$. Since $\{k_i(n,x) : x \in X, n \in N\}$ is a weak base for space (X,τ_i), U must be τ_i-open. On the other hand, suppose U is

τ_i-open and $x \in U$. We want to verify $St^2(x, \mathcal{G}_{i,n}, \mathcal{G}_{j,n}) \subset U$ for some $n \in N$. Suppose not, then we can take $y_n \in St^2(x, \mathcal{G}_{i,n}, \mathcal{G}_{j,n}) - U$ for each $n \in N$. Also, for each $n \in N$, we can get $t_n \in X$ such that $y_n \in k_i(n, t_n)$ with $k_j(n, t_n) \cap st(x, \mathcal{G}_{i,n}, \mathcal{G}_{j,n}) \neq \emptyset$, and thus there exist $z_n, s_n \in X$ with $s_n \in k_i(n, z_n) \cap k_j(n, t_n)$ and $x \in k_j(n, z_n)$ for each $n \in N$.

Claim 1. The sequence $\{z_n\}$ τ_i-converges to x.

In fact, if not, then there is a subsequence $\{z_{n_m}\}$ of $\{z_n\}$ with $x \notin cl_{\tau_i}\{z_{n_m} : m \in N\} = B$. By (1), there exists an $n_0 \in N$ such that $x \notin cl_{\tau_i} \bigcup\{g_j(n_0, t) : t \in B\}$. In particular, $x \notin cl_{\tau_i} \bigcup\{g_j(n_0, z_{n_m}) : m \in N\}$; then $x \notin g_j(n_0, z_{n_m})$ for each $n_m > n_0$. On the other hand, for each $n \in N$, since $x \in k_j(n, z_n) \subset g_j(n, z_n)$, $x \in g_j(n_m, z_{n_m}) \subset g_j(n_0, z_{n_m})$ for each $n_m > n_0$. This contradiction implies that the sequence $\{z_n\}$ τ_i-converges to x.

Claim 2. The sequence $\{s_n\}$ τ_i-converges to x.

Assume that $\{s_n\}$ does not τ_i-converge to x; then there is a subsequence $\{s_{n_m}\}$ of $\{s_n\}$ with $x \notin cl_{\tau_i}\{s_{n_m} : m \in N\} = F$. By (1), there exists an $n_1 \in N$ such that $x \notin cl_{\tau_i} \bigcup\{g_j(n_1, p) : p \in F\} = F_1$. Note that $F \subset F_1$; by (1), we have $x \notin cl_{\tau_i} \bigcup\{g_j(n_2, p) : p \in F_1\}$ for some $n_2 \in N$ with $n_2 > n_1$. Since $g_j(n+1, x) \subset g_j(n, x)$ for each $n \in N$, $x \notin cl_{\tau_i} \bigcup\{g_j^2(n_2, p) : p \in F\}$. In particular, $x \notin cl_{\tau_i} \bigcup\{g_j^2(n_2, s_{n_m}) : m \in N\}$. On the other hand, for each $n \in N$, since $s_n \in k_i(n, z_n) \subset h_i(n, z_n) = \{y \in X : z_n \in cl_{\tau_i} g_j^2(n, y)\}$, $z_n \in cl_{\tau_i} g_j^2(n, s_n)$. By Claim 1, we have that $x \in cl_{\tau_i}\{z_{n_m} : m > n_2\} \subset cl_{\tau_i} \bigcup\{g_j^2(n_m, s_{n_m}) : m > n_2\} \subset cl_{\tau_i} \bigcup\{g_j^2(n_2, s_{n_m}) : m > n_2\} \subset cl_{\tau_i} \bigcup\{g_j^2(n_2, s_{n_m}) : m \in N\}$. This contradiction implies that the sequence $\{s_n\}$ τ_i-converges to x.

Claim 3. The sequence $\{t_n\}$ τ_i-converges to x.

In fact, if not, then there is a subsequence $\{t_{n_m}\}$ of $\{t_n\}$ with $x \notin cl_{\tau_i}\{t_{n_m} : m \in N\} = E$. By (1), there exists an $n_3 \in N$ such that $x \notin cl_{\tau_i} \bigcup\{g_j(n_3, t) : t \in E\}$. In particular, $x \notin cl_{\tau_i} \bigcup\{g_j(n_3, t_{n_m}) : m \in N\}$. On the other hand, since $s_n \in k_j(n, t_n) \subset g_j(n, t_n)$ and $\{s_n\}$ τ_i-converges to x by Claim 2, $x \in cl_{\tau_i}\{s_{n_m} : n_m > n_3\} \subset \bigcup\{g_j(n_m, t_{n_m}) : n_m > n_3\} \subset cl_{\tau_i} \bigcup\{g_j(n_3, t_{n_m}) : n_m > n_3\}$. This contradiction implies that the sequence $\{t_n\}$ τ_i-converges to x.

Claim 4. The sequence $\{y_n\}$ τ_i-converges to x.

Since $\{y_n\} \in k_i(n, t_n) \subset h_i(n, t_n)$, for each $n \in N$, and the sequence $\{t_n\}$ τ_i-converges to x by Claim 3, using the method in Claim 2, we can easily show that the sequence $\{y_n\}$ τ_i-converges to x.

By definition of the sequence $\{y_n\}$, we have $y_n \notin U$ for each $n \in N$. Since $U \in \tau_i$, the sequence $\{y_n\}$ does not τ_i-converge to x. This contradicts Claim 4. Hence, $\{St^2(x, \mathcal{G}_{i,n}, \mathcal{G}_{j,n}) : x \in X, n \in N\}$ is a weak base for space $(X, \tau_i)(i, j = 1, 2$ and $i \neq j)$. □

It easily seen that we can use "$cl_{\tau_i} g_j$" in place of "$cl_{\tau_i} g_j^2$" in the sufficiency of the proof of Theorem 2.2. Therefore, we have the following corollary.

Corollary 2.3. *A bispace (X, τ_i, τ_j) is quasi-pseudo-metrizable if and only if it has a pairwise weak base g-function $g_i : N \times X \to \mathcal{P}(X)(i = 1, 2)$ satisfying* (1) *of Theorem 2.2 and*

(2′) *for any $Y \subset X$ and each $n \in N$, $cl_{\tau_i} Y \subset \bigcup\{cl_{\tau_i} g_j(n, y) : y \in Y\}$.*

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References

[1] A. V. Arhangel'skiĭ, *Mappings and spaces*, Russian Math. Surveys **21** (1966), no. 4, 115–162.

[2] Ryszard Engelking, *General Topology*. Translated from the Polish by the author. 2nd ed. Sigma Series in Pure Mathematics, 6. Berlin: Heldermann Verlag, 1989.

[3] Paul Gartside, *Generalized metric spaces, Part I*, in Encyclopedia of General Topology. Ed. Klaas Pieter Hart, Jun-iti Nagata, and Jerry E. Vaughan. Amsterdam: Elsevier, 2004. 273–275.

[4] Zhi Min Gao, *Metrizability of spaces and weak base g-functions*, Topology Appl. **146/147** (2005), 279–288.

[5] J. C. Kelly, *Bitopological spaces*, Proc. London Math. Soc. (3) **13** (1963), 71–89.

[6] K. Li and S. Lin, *Notes on symmetric g-functions*, Acta Math. Hungar. **116** (2007), no. 1-2, 73–77.

[7] Shou Lin, *Generalized Metric Spaces and Maps* (Chinese). 2nd ed. Beijing: Kexue Chubanshe (Science Press), 2007.

[8] Harold W. Martin, *Weak bases and metrization*, Trans. Amer. Math. Soc. **222** (1976), 337–344.

[9] Josefa Marín, *Weak bases and quasi-pseudo-metrization of bispaces*, Topology Appl. **156** (2009), no. 18, 3070–3076.

[10] Jun-iti Nagata, *Characterizations of metrizable and Lašnev spaces in terms of g-function*, Questions Answers Gen. Topology **4** (1986/87), no. 2, 129–139.

[11] C. W. Patty, *Bitopological spaces*, Duke Math. J. **34** (1967), 387–391.

[12] S. Romaguera, *On bitopological quasipseudometrization*, J. Austral. Math. Soc. Ser. A **36** (1984), no. 1, 126–129.

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