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by

Alexander Zupan

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Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
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A LOWER BOUND ON THE WIDTH OF SATELLITE KNOTS

ALEXANDER ZUPAN

Abstract. Thin position for knots in S^3 was introduced by David Gabai in *Foliations and the topology of 3-manifolds III* [J. Differential Geom. **26** (1987), 479–536] and has been used in a variety of contexts. We conjecture an analogue to a theorem of Horst Schubert and Jennifer Schultens concerning the bridge number of satellite knots. For a satellite knot K, we use the companion torus T to provide a lower bound for $w(K)$, proving the conjecture for K with a 2-bridge companion. As a corollary, we find thin position for any satellite knot with a braid pattern and 2-bridge companion.

1. Introduction

Thin position for knots in S^3 was introduced by David Gabai [2] and has since been studied extensively. Although thin position has been used in a variety of arguments, there are relatively few methods for putting specific knots into thin position. Thin position of a knot always provides a useful surface; either a level sphere is a bridge sphere for the knot or the thinnest thin sphere is incompressible in the complement of the knot, as shown by Ying-Qing Wu [9].

In some sense, width can be considered to be a refinement of bridge number, although recently Ryan Blair and Maggie Tomova [1] have shown that one cannot always recover the bridge number of a knot K from the thin position of K. On the other hand, if K is small, then $w(K) = 2{\cdot}b(K)^2$ and any thin position of K is a minimal bridge position [8]. In his seminal paper on the subject, Horst Schubert [6] proved that for any two knots K_1

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and K_2, $b(K_1\#K_2) = b(K_1) + b(K_2) - 1$. An updated proof by Jennifer Schultens appears in [7].

Unfortunately, we cannot hope for a similar statement to hold for width. In [5], Martin Scharlemann and Schultens establish $\max\{w(K_1), w(K_2)\}$ as a lower bound for $w(K_1\#K_2)$, and Blair and Tomova [1] prove that this bound is sharp in some cases, while Yo'av Rieck and Eric Sedgwick [4] demonstrate that the bound is never sharp for small knots. Both Schubert and Schultens also prove the following theorem.

Theorem 1.1. *Let K be a satellite knot with pattern $\hat{K}$ and companion J, and let n be the winding number of $\hat{K}$. Then*

$$b(K) \geq n \cdot b(J).$$

We make an analogous conjecture.

Conjecture 1.2. *Let K be a satellite knot with pattern $\hat{K}$ and companion J, and let n be the winding number of $\hat{K}$. Then*

$$w(K) \geq n^2 \cdot w(J).$$

In this paper, we provide a weaker lower bound for $w(K)$. Our main theorem follows.

Main Theorem. *Let K be a satellite knot with pattern $\hat{K}$, and let n be the winding number of $\hat{K}$. Then*

$$w(K) \geq 8n^2.$$

This proves the conjecture in the case that the companion J is a 2-bridge knot, since the width of such J is 8. As a corollary, if K is a satellite with a 2-bridge companion and its pattern $\hat{K}$ is a braid with index n, then any thin position is a minimal bridge position for K.

2. Preliminaries

Let K be a knot in S^3, and let $\mathcal{M}(K)$ denote the collection of Morse functions $h : S^3 \to \mathbb{R}$ with exactly two critical points, denoted $\pm\infty$, such that $h|_K$ is also Morse. (Equivalently, we could fix some Morse function h and look instead at the collection of embeddings of S^1 isotopic to K in S^3.) For every $h \in \mathcal{M}(K)$, let $c_0 < c_1 < \cdots < c_n$ denote the critical values of $h|_K$. Choose regular values $c_0 < r_1 < c_1 < \cdots < r_n < c_n$, and define

$$\begin{aligned} w(h) &= \sum_{i=1}^{n} |K \cap h^{-1}(r_i)|, \\ b(h) &= \frac{n+1}{2}, \\ \text{trunk}(h) &= \max |K \cap h^{-1}(r_i)|. \end{aligned}$$

Now, let

$$\begin{aligned} w(K) &= \min_{h\in\mathcal{M}(K)} w(h), \\ b(K) &= \min_{h\in\mathcal{M}(K)} b(h), \\ \operatorname{trunk}(K) &= \min_{h\in\mathcal{M}(K)} \operatorname{trunk}(h). \end{aligned}$$

These three knot invariants are called the *width*, the *bridge number*, and the *trunk* of K, respectively. Width was defined by Gabai [2], bridge number by Schubert [6], and trunk by Ozawa [3]. Observe that $b(K)$ is the least number of maxima (or minima) of any embedding of K. If $h \in \mathcal{M}(K)$ satisfies $w(K) = w(h)$, we say that h is a *thin position* for K. If all maxima of h occur above all minima, we say that h is a *bridge position* for K, and if $b(h) = b(K)$, we call h a *minimal bridge position* for K.

In [5], the authors give an alternative formula for computing width, which involves thin and thick levels. Let $h \in \mathcal{M}(K)$ with critical and regular values as defined above. Then $h^{-1}(r_i)$ is a *thick level* if $|K \cap h^{-1}(r_i)| > |K \cap h^{-1}(r_{i-1})|, |K \cap h^{-1}(r_{i+1})|$ and $h^{-1}(r_i)$ is a *thin level* if $|K \cap h^{-1}(r_i)| < |K \cap h^{-1}(r_{i-1})|, |K \cap h^{-1}(r_{i+1})|$, where $1 < i < n$. Note that if h is a bridge position for K, then h has exactly one thick level and no thin levels. Letting $a_1, \dots, a_m$ denote the number of intersections of the thick levels with K and letting $b_1, \dots, b_{m-1}$ denote the number of intersection of the thin levels with K, the width of h is given by

$$w(h) = \frac{1}{2}\left(\sum_{i=1}^{m} a_i^2 - \sum_{i=1}^{m-1} b_i^2\right).$$

In particular, we see that for every $h \in \mathcal{M}(K)$, there exists $a_i \geq \operatorname{trunk}(K)$, which implies that

$$w(K) \geq \frac{\operatorname{trunk}(K)^2}{2}.$$

The knots we will be concerned with are satellite knots: Let $\hat{K}$ be a knot contained in a solid torus V with core C such that every meridian of V intersects $\hat{K}$, and let J be any nontrivial knot. Suppose that $\varphi : V \to S^3$ is an embedding such that $\varphi(C)$ is isotopic to J in S^3. Then $K = \varphi(\hat{K})$ is called a *satellite knot* with *companion* J and *pattern* $\hat{K}$. Essentially, to construct a satellite knot K, we start with a pattern in a solid torus and then "tie" the solid torus in the shape of the companion J.

We will need several more definitions to state the main result. Let $\hat{K}$ be a pattern contained in a solid torus V. The *winding number* of $\hat{K}$, $\#(\hat{K})$, is the absolute value of the algebraic intersection number of any meridian disk of V with $\hat{K}$. Equivalently, if $\alpha : S^1 \to V$ is an embedding such that $\alpha(S^1) = \hat{K}$ and $r : V \to S^1$ is a strong deformation retract of V onto its core, then $\#(\hat{K})$ agrees with the degree of the map $r \circ \alpha$. Let $\hat{K}$ be a pattern contained in a solid torus V. We say that $\hat{K}$ is a *braid* of index n if there is a foliation of V such that every leaf is a meridian disk intersecting $\hat{K}$ exactly n times.

In the case that $\hat{K}$ is a braid of index n, it is clear that $\#(\hat{K}) = n$. For an example, consider Figure 1. On the left, we see a braid pattern of index 3, $\hat{K}$, contained in a solid torus V. On the right, V is embedded in such a way that its core is a trefoil. Thus, the knot K on the right is a satellite knot with trefoil companion and pattern $\hat{K}$.

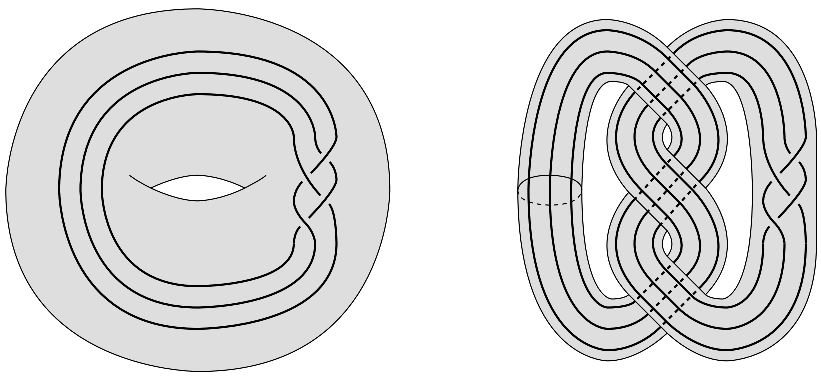

FIGURE 1. On the left, pattern $\hat{K}$ is shown contained in a solid torus. On the right, we see a satellite knot K with pattern $\hat{K}$ and trefoil companion.

3. Reducing the Saddle Points on the Companion Torus

From this point on, we set the convention that K is a satellite knot with companion J and pattern $\hat{K}$ contained in a solid torus $\hat{V}$, φ is an embedding of $\hat{V}$ into S^3 that takes a core of $\hat{V}$ to J, $V = \varphi(\hat{V})$, and $T = \partial V$. Further, we will let $h \in \mathcal{M}(K)$ and perturb V slightly so that $h|_T$ is Morse. We wish to restrict our investigation to tori T with only certain types of saddle points.

In this vein, we follow [7], from which the next definition is taken. Consider the singular foliation, F_T, of T induced by $h|_T$. Let σ be a leaf corresponding to a saddle point. Then one component of σ is the wedge of two circles s_1 and s_2. If either is inessential in T, we say that σ is an *inessential saddle*. Otherwise, σ is an *essential saddle*.

The next lemma is the Pop Over Lemma [7, Lemma 1].

Lemma 3.1. *If F_T contains inessential saddles, then after a small isotopy of T, there is an inessential saddle σ in T such that*

(1) *s_1 bounds a disk $D_1 \subset T$ such that F_T restricted to D_1 contains only one maximum or minimum,*
(2) *for L, the level surface of h containing σ, D_1 co-bounds a 3-ball B with a disk $\tilde{D}_1 \subset L$ such that B does not contain $\pm\infty$ and such that s_2 lies outside of $\tilde{D}_1$.*

In the following lemma, we mimic Lemma 2 of [7] with a slight modification to preserve the height function h on K.

Lemma 3.2. *There exists an isotopy $f_t : S^3 \to S^3$ such that $f_0 = Id$, $h = h \circ f_1$ on K, and the foliation of T induced by $h \circ f_1$ contains no inessential saddles.*

Proof. Suppose that T has an inessential saddle, σ, lying in the level 2-sphere L. By the previous lemma, we may suppose that σ is as described above, and suppose without loss of generality that D_1 contains only one maximum. By slightly pushing D_1 into $\text{int}(B)$, we can create another 3-ball B' such that $B' \cap D_1 = \emptyset$ and $(K \cup T) \cap \text{int}(B) \subset B'$. First, we construct an isotopy which pushes B' below L into a small neighborhood of $\tilde{D}_1$ and then cancels the maximum of D_1 with the saddle point σ. Now, there exists a monotone increasing arc beginning at the highest point of B', passing through the disk $\tilde{D}_2 \subset L$ bounded by s_2, intersecting only maxima of T, and disjoint from K. Thus, we may construct another isotopy which pushes B' upward through a regular neighborhood of α, increasing the heights of maxima of T if necessary, until the minima and maxima of $K \cap \text{int}(B')$ are restored to their original heights. We see that after performing both isotopies, T has one fewer inessential saddle and no new critical points have been created. See Figure 2. Repeating this process, we eliminate all inessential saddles via isotopy. □

Thus, from this point forward, we may replace any $h \in \mathcal{M}(K)$ with $h \circ f_1$ from the lemma without changing the information carried by $h|_K$; thus, we may suppose that the foliation F_T contains no inessential saddles. It follows that if γ is a loop contained in a level 2-sphere that bounds a disk $D \subset T$, then $h|_D$ has exactly one critical point, a minimum or a maximum. For if not, D would contain a saddle point, which would necessarily be inessential.

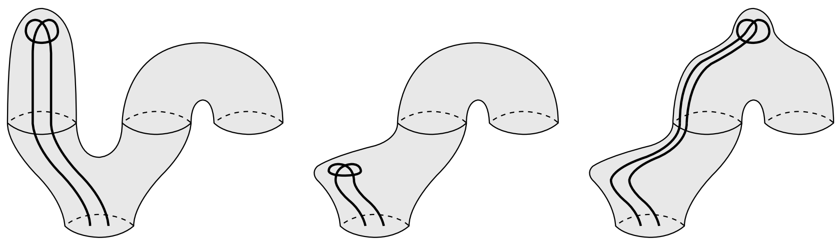

FIGURE 2. An illustration of the process of eliminating an inessential saddle described in the proof of Lemma 3.2

4. The Connectivity Graph

For each regular value r of $h|_{T,K}$, we have that $h^{-1}(r)$ is a level 2-sphere and $h^{-1}(r)\cap T$ is a collection of simple closed curves. Let $\gamma_1, \ldots, \gamma_n$ denote these curves.

A *bipartite* graph is a graph together with a partition of its vertices into two sets $\mathcal{A}$ and $\mathcal{B}$ such that no two vertices from the same set share an edge. We will create a bipartite graph Γ_r from $h^{-1}(r)$ as follows: Cut the 2-sphere $h^{-1}(r)$ along $\gamma_1, \ldots, \gamma_n$, splitting $h^{-1}(r)$ into a collection of planar regions $R_1, \ldots, R_m$. The vertex set $\{v_1, \ldots, v_m\}$ of Γ_r corresponds to the regions $R_1, \ldots, R_m$, and the edges correspond to the curves $\gamma_1, \ldots, \gamma_n$ that do not bound disks in T. For each such γ_i, make an edge between v_j and v_k if $\gamma_i = R_j \cap R_k$ in $h^{-1}(r)$. To see that Γ_r is bipartite, we create two vertex sets $\mathcal{A}_r$ and $\mathcal{B}_r$, letting $v_i \in \mathcal{A}_r$ if $R_i \subset V$, and $v_i \in \mathcal{B}_r$ otherwise. We call Γ_r the *essential connectivity graph* with respect to the regular value r of h, where the word "essential" emphasizes the fact that edges correspond to only those γ_i that are essential in T. Note that since each γ_i separates $h^{-1}(r)$, the graph Γ_r must be a tree. An *endpoint* of Γ_r is a vertex that is incident to exactly one edge.

We remark that the term "connectivity graph" also appears in [5], but the two notions are not related. In the above definition, the essential connectivity graph represents adjacencies between components of intersections of V and $\overline{S^3 \setminus V}$ with a single level 2-sphere. In [5], the connectivity graph is more global, representing adjacencies between components of a 3-manifold cut along certain level surfaces.

For instance, Figure 3 depicts a possible level 2-sphere and corresponding essential connectivity graph. Observe that since V is a knotted solid torus, T is only compressible on one side, and every compressing disk for T is a meridian of V. This leads to Lemma 4.1.

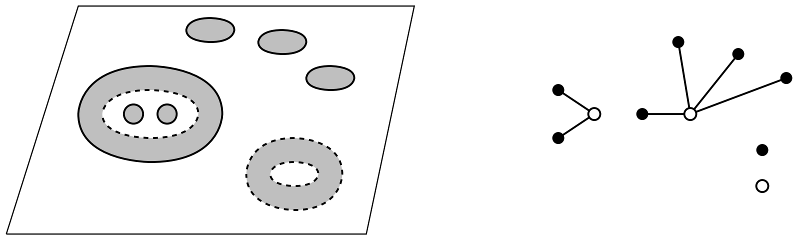

FIGURE 3. A level 2-sphere at left and its corresponding essential connectivity graph at right. Note that dotted curves on the left correspond to curves bounding disks in T.

Lemma 4.1. *If $v_i \in \Gamma_r$ is an endpoint, then $v_i \in \mathcal{A}_r$.*

Proof. Suppose R_i is the region in $h^{-1}(r)$ corresponding to v_i. Then ∂R_i contains exactly one essential curve in T, call it γ, and some (possibly empty) set of curves that bound disks in T. Since each of these disks contains only one maximum or minimum by the discussion above, any two such disks must be pairwise disjoint. Thus, we can glue each disk to R_i to create an embedded disk D such that $\partial D = \gamma$. Now, push each glued disk into a collar of T in V, so that $T \cap \text{int}(D) = \emptyset$, and thus D is a compressing disk for T. We conclude $D \subset V$ and $\text{int}(R_i) \cap \text{int}(D) \neq \emptyset$, implying $R_i \subset V$ and $v_i \in \mathcal{A}_r$. □

Using similar arguments, we prove the next lemma.

Lemma 4.2. *Suppose that $v_1, \ldots, v_n \subset \Gamma_r$ are endpoints corresponding to regions $R_1, \ldots, R_n \subset h^{-1}(r)$. Then $\gamma_1, \ldots, \gamma_n$ bound meridian disks $D_1, \ldots, D_n \subset V$ such that $K \cap D_i \subset R_i$ for all i.*

Proof. The existence of the disks $D_1, \ldots, D_n$ is given in the proof of Lemma 4.1. Thus, suppose that Δ is a disk glued to R_i to construct D_i. When we push Δ into a collar of T, we can choose this collar to be small enough so that it does not intersect K. Thus, we may suppose that $\Delta \cap K = \emptyset$ for every such Δ, which implies that all intersections of K with D_i must be contained in R_i. □

We note that Lemma 4.1 and Lemma 4.2 are inspired by the proof of Theorem 1.9 of [3]. Essentially, Lemma 4.2 demonstrates that even though the set of meridian disks $D_1, \ldots, D_n$ may not be level, we may assume they are level for the purpose of bounding below the number of intersections of K with $h^{-1}(r)$, since any intersection of K with one of these disks occurs in one of the level regions R_i. Let r be a regular value of

$h|_{T,K}$. We define the *trunk* of the level 2-sphere $h^{-1}(r)$, denoted trunk(r), to be the number of endpoints of Γ_r.

For example, if r is the regular value whose essential connectivity graph is pictured in Figure 3, then trunk$(r) = 6$. We are now in a position to use the winding number of the pattern $\hat{K}$.

Lemma 4.3. *Let r be a regular value of $h|_{T,K}$.*

- *If trunk(r) is even, then $|K \cap h^{-1}(r)| \geq \#(\hat{K}) \cdot$ trunk(r).*
- *If trunk(r) is odd, then $|K \cap h^{-1}(r)| \geq \#(\hat{K}) \cdot [$trunk$(r) + 1]$.*

Proof. First, suppose that $m = $ trunk(r) is even and let $n = \#(\hat{K})$. Since each meridian of V has algebraic intersection $\pm n$ with K, we know that each meridian must intersect K in at least n points. Let $v_1, \ldots, v_m$ be endpoints of Γ_r corresponding to regions $R_1, \ldots, R_m$. By Lemma 4.2, $|K \cap R_i| = |K \cap D_i| \geq n$ for each i. Further, since these regions are pairwise disjoint, it follows that $|K \cap h^{-1}(r)| \geq n \cdot m$, completing the first part of the proof.

Now, suppose that m is odd. If N_1 is the algebraic intersection number of K with $R = \cup R_i$, we have that

$$N_1 = \sum_{i=1}^{m} \pm n.$$

In particular, as m is odd, it follows that $|N_1| \geq n$. Let $R' = \overline{h^{-1}(r) - R}$. Then $R' \cap R \subset T$, so K does not intersect $R' \cap R$. Let N_2 denote the algebraic intersection number of K with R'. Since $h^{-1}(r)$ is a 2-sphere which bounds a ball in S^3, $h^{-1}(r)$ is homologically trivial, implying that the algebraic intersection of K with $h^{-1}(r)$ is zero. This means $N_1 + N_2 = 0$, so $|N_2| \geq n$, and thus $|K \cap R'| \geq n$. Lastly,

$$|K \cap h^{-1}(r)| = |K \cap R| + |K \cap R'| = \sum_{i=1}^{m} |K \cap R_i| + |K \cap R'| \geq n \cdot (m+1).$$

□

5. Bounding the Width of Satellite Knots

We will use the trunk of the level surfaces to impose a lower bound on the trunk of a knot K, which in turn forces a lower bound on the width of K. We need the following lemma.

Lemma 5.1 ([3, Claim 2.4])**.** *Let S be a torus embedded in S^3, and let $h : S^3 \to \mathbb{R}$ be a Morse function with two critical points on S^3 such that $h|_S$ is also Morse. Suppose that for every regular value r of $h|_S$, all curves in $h^{-1}(r) \cap S$ that are essential in S are mutually parallel in $h^{-1}(r)$. Then S bounds solid tori V_1 and V_2 in S^3 such that $V_1 \cap V_2 = T$.*

As a result of this lemma, we have the following corollary.

Corollary 5.2. *There exists a regular value r of $h|_{T,K}$ such that trunk(r) ≥ 3.*

Proof. Suppose not, and let r be any regular value of $h|_{T,K}$ such that $h^{-1}(r)$ contains essential curves in T. Such a regular value must exist; otherwise, T could not contain a saddle point. By assumption, trunk$(r) \leq 2$, so Γ_r has exactly two endpoints, v_1 and v_2. But this implies that Γ_r is a path, and thus all essential curves in $h^{-1}(r)$ are mutually parallel. As this is true for every such regular value r, we conclude by Lemma 5.1 that V is an unknotted solid torus, contradicting the fact that K is a satellite knot with nontrivial companion J. □

This brings us to our main theorem.

Theorem 5.3 (Main Theorem)**.** *Suppose K is a satellite knot with pattern $\hat{K}$, where $n = \#(\hat{K})$. Then*

$$w(K) \geq 8n^2.$$

Proof. Choose a height function $h \in \mathcal{M}(K)$ such that trunk(h) $=$ trunk(K). Since K is a satellite knot, K is contained in a knotted solid torus V. Let $T = \partial V$, and if necessary perturb T slightly so that $h|_T$ is also Morse. By Corollary 5.2, there exists a regular value r of h such that trunk$(r) \geq 3$. From Lemma 4.3, it follows that $|K \cap h^{-1}(r)| \geq 4n$. Since trunk$(K) =$ trunk(h), and trunk(h) corresponds to the level of h with the greatest number of intersections with K, we have trunk$(h) \geq 4n$. Finally, using the lower bound for width based on trunk,

$$w(K) \geq \frac{\text{trunk}(K)^2}{2} \geq 8n^2,$$

as desired. □

Corollary 5.4. *Suppose K is a satellite knot, with pattern $\hat{K}$ and companion J. If $\hat{K}$ is a braid of index n and J is a 2-bridge knot, then $w(K) = 8n^2$ and any thin position for K is a minimal bridge position.*

Proof. For such K we can exhibit a Morse function $h \in \mathcal{M}(K)$ such that $w(h) = 8n^2$, $b(h) = 2n$, and trunk$(h) = 4n$. By [7], $b(K) = b(h)$, so h is both a bridge and thin position for h, and further every minimal bridge position h' for K satisfies $w(h') = 8n^2$ and is also thin. It follows from the proof of the above theorem that trunk$(K) = 4n$, so any $h \in \mathcal{M}(K)$ that is not a minimal bridge position satisfies $w(h) > 8n^2$. □

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References

[1] Ryan Blair and Maggie Tomova, *Width is not additive*. Preprint. Available at arxiv:1005.1359v1[math.GT].

[2] David Gabai, *Foliations and the topology of* 3*-manifolds. III*, J. Differential Geom. **26** (1987), no. 3, 479–536.

[3] Makoto Ozawa, *Waist and trunk of knots*, Geom. Dedicata **149** (2010), 85–94.

[4] Yo'av Rieck and Eric Sedgwick, *Thin position for a connected sum of small knots*, Algebr. Geom. Topol. **2** (2002), 297–309.

[5] Martin Scharlemann and Jennifer Schultens, *3-manifolds with planar presentations and the width of satellite knots*, Trans. Amer. Math. Soc. **358** (2006), no. 9, 3781–3805.

[6] Horst Schubert, *Über eine numerische knoteninvariante*, Math. Z. **61** (1954), 245–288.

[7] Jennifer Schultens, *Additivity of bridge numbers of knots*, Math. Proc. Cambridge Philos. Soc. **135** (2003), no. 3, 539–544.

[8] Abigail Thompson, *Thin position and bridge number for knots in the 3-sphere*, Topology **36** (1997), no. 2, 505–507.

[9] Ying-Qing Wu, *Thin position and essential planar surfaces*, Proc. Amer. Math. Soc. **132** (2004), no. 11, 3417–3421.

Department of Mathematics; University of Iowa; Iowa City, Iowa 52242
E-mail address: alexander-zupan@uiowa.edu