TOPOLOGY PROCEEDINGS

Volume 40, 2012

Pages 215–226

http://topology.auburn.edu/tp/

Deformation Retracts and Hilbert Cubes in n-fold Hyperspaces

by

Sergio Macías

Electronically published on October 22, 2011

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS
Volume 40 (2012)
Pages 215-226

E-Published on October 22, 2011

DEFORMATION RETRACTS AND HILBERT CUBES IN n-FOLD HYPERSPACES

SERGIO MACÍAS

Abstract. We consider certain sets $\mathcal{OA}_n(B,A)$ of the n-fold hyperspaces of a continuum and prove that they are absolute retracts which implies, under the appropriate hypotheses, that they are deformation retracts of the n-fold hyperspace of a continuum. Given a locally connected continuum X, we also consider the sets $\mathcal{C}_n(B,X)$ which are absolute retracts too. We extend the results about intervals of continua that Carl Eberhart obtained for the hyperspace of subcontinua of a continuum to the n-fold hyperspace of a continuum to obtain Hilbert cubes.

1. Introduction

In [8, p. 1054] the notion of *property* (OA) was introduced to characterize the lightness of an induced map between n-fold hyperspaces [8, Theorem 5.7]. It turns out that this property is the right one to obtain that certain subsets, the sets $\mathcal{OA}_n(B,A)$, of the n-fold hyperspaces of a continuum are absolute retracts which implies, under the appropriate hypotheses, that they are deformation retracts of the n-fold hyperspace of a continuum.

The paper is divided into five sections. After the introduction and definitions, in §3 we present the preliminary results needed. In §4 we prove that the sets of the form $\mathcal{OA}_n(B,A)$ are absolute retracts (Theorem 4.3), and present some consequence of this result; we also prove that for locally connected continua X, the sets of the form $\mathcal{C}_n(B,X)$ are absolute retracts too (Theorem 4.1). In §5 we extend the results about intervals of

2010 *Mathematics Subject Classification.* Primary 54B20.

Key words and phrases. absolute retract, continuum, decomposable continuum, deformation retract, finite graph, Hilbert cube, indecomposable continuum, irreducible continuum, n-fold hyperspace, r-map, retract, retraction, Z-map, Z-set.

continua that Carl Eberhart obtained for the hyperspace of subcontinua of a continuum to the n-fold hyperspace of a continuum to obtain Hilbert cubes.

2. Definitions and Notation

Given a subset A of a metric space Z with metric d, the closure, the boundary, and the interior of A are denoted by $Cl_Z(A)$, $Bd_Z(A)$, and $Int_Z(A)$, respectively. Also, $\mathcal{V}_r^d(A)$ denotes the open ball of radius r about A.

A *map* is a continuous function. Let Z be a metric space and let A be a closed subset of Z. A *retraction from Z onto A* is a map $r\colon Z \to A$ such that $r(a) = a$ for all $a \in A$. The set A is called a *retract* of Z. Let $f\colon Z \to W$ be a map between metric spaces; f is said to be an *r-map* provided that there exists a map $g\colon W \to Z$ such that $f \circ g = 1_W$, where 1_W denotes the identity map on W.

Let A be a closed subset of a metric space Z; we say that A is a *Z-set* if for each $\varepsilon > 0$, there exists a map $f_\varepsilon\colon Z \to (Z \setminus A)$ such that $d(z, f_\varepsilon(z)) < \varepsilon$ for all $z \in Z$. A map $f\colon Z \to W$ between metric spaces is a *Z-map* if $f(Z)$ is Z-set in W.

Let Z be a metric space. By a *deformation* we mean a map $H\colon Z \times [0,1] \to Z$ such that $H((z,0)) = z$ for each $z \in Z$. Let $A = \{H((z,1)) \mid z \in Z\}$. If the map $r\colon Z \to A$, given by $r(z) = H((z,1))$, is a retraction from Z onto A, then H is a *deformation retraction from Z onto A*. If H is a deformation retraction from Z onto A and for each $a \in A$ and each $t \in [0,1]$, $H((a,t)) = a$, then H is a *strong deformation retraction from Z onto A*. The set A is called a *deformation retract of Z* (*strong deformation retract of Z*, respectively). A metric space Z is an *absolute retract* provided that for each embedding $e\colon Z \to X$ of Z into a metric space X such that $e(Z)$ is closed in X, $e(Z)$ is a retract of X.

A *continuum* is a nonempty compact connected metric space. A continuum X is *decomposable* provided that there exist two proper subcontinua A and B of X such that $X = A \cup B$. A continuum is *indecomposable* if it is not decomposable. A *finite graph* is a continuum which can be written as a finite union of arcs, any two of which either are disjoint or intersect at one or both of their end points. A continuum X is *irreducible* if there exist two points x_1 and x_2 of X such that if A is a subcontinuum of X and $\{x_1, x_2\} \subset A$, then $A = X$. Given a continuum X and a point $p \in X$, the *composant of p in X* is the union of all proper subcontinua of X containing p.

Given a continuum X, we consider the following hyperspaces of X:

$$2^X = \{A \subset X \mid A \text{ is nonempty and closed}\}$$

and

$$\mathcal{C}_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ components}\},$$

where n is a positive integer. $\mathcal{C}_n(X)$ is called the *n-fold hyperspace* of X. These spaces are topologized with the Hausdorff metric defined as

$$\mathcal{H}(A, B) = \inf\{\varepsilon > 0 \mid A \subset \mathcal{V}^d_\varepsilon(B) \text{ and } B \subset \mathcal{V}^d_\varepsilon(A)\};$$

$\mathcal{H}$ always denotes the Hausdorff metric on 2^X. When $n = 1$, we write $\mathcal{C}(X)$ instead of $\mathcal{C}_1(X)$. Given an element B of $\mathcal{C}_n(X)$, the *mesh of* B, denoted by $\text{mesh}(B)$, is

$$\text{mesh}(B) = \max\{\text{diam}(K) \mid K \text{ is a component of } B\}.$$

The symbol $\mathcal{F}_n(X)$ denotes the *n-fold symmetric product* of X; that is,

$$\mathcal{F}_n(X) = \{A \in \mathcal{C}_n(X) \mid A \text{ has at most } n \text{ points}\}.$$

Note that, by definition, $\mathcal{F}_n(X) \subset \mathcal{C}_n(X)$. It is known that if X is a continuum, then 2^X and $\mathcal{C}_n(X)$ are arcwise connected continua (for 2^X and $\mathcal{C}(X)$, see [10, (1.13)]; for $\mathcal{C}_n(X)$ and $n \geq 2$, see [9, 1.8.12]). Also, $\mathcal{F}_n(X)$ is a continuum for all positive integers n [2, p. 877].

Let X be a continuum and let n be a positive integer. An *order arc* in $\mathcal{C}_n(X)$ is an arc $\alpha\colon [0, 1] \to \mathcal{C}_n(X)$ such that if $0 \leq s < t \leq 1$, then $\alpha(s) \subset \alpha(t)$ and $\alpha(s) \neq \alpha(t)$.

Let B and A be two elements of $\mathcal{C}_n(X)$. We say that the pair (B, A) satisfies *property* (OA) provided that $B \subset A$ and each component of A intersects B. Let us note that this condition guarantees the existence of an order arc in $\mathcal{C}_n(X)$ from B to A when $B \subsetneq A$ [10, (1.8)].

Let X be a continuum and let n be a positive integer. If $B \in \mathcal{C}_n(X)$, define

$$\mathcal{C}_n(B, X) = \{A \in \mathcal{C}_n(X) \mid B \subset A\};$$

$$\mathcal{OA}_n(B, X) = \{A \in \mathcal{C}_n(X) \mid (B, A) \text{ satisfies property } (OA)\}.$$

If $A \in \mathcal{OA}_n(B, X)$, then

$$\mathcal{OA}_n(B, A) = \{D \in \mathcal{OA}_n(B, X) \mid D \subset A\}.$$

Note that, in general, if $n \geq 2$, then $\mathcal{OA}_n(B, X) \subsetneq \mathcal{C}_n(B, X)$. In the next two propositions, we present two instances in which these sets are equal.

Proposition 2.1. *If X is an irreducible continuum between x_1 and x_2, $B \in \mathcal{C}_2(X) \setminus \mathcal{C}_1(X)$, and $\{x_1, x_2\} \subset B$, then $\mathcal{OA}_2(B, X) = \mathcal{C}_2(B, X)$.*

Using the fact that indecomposable continua have uncountably many pairwise disjoint composants [6, Theorem 3–46 and Theorem 3–47], we obtain the following proposition.

Proposition 2.2. *Let X be an indecomposable continuum, let n be an integer greater than or equal to 2, and let $B \in \mathcal{C}_n(X) \setminus \mathcal{C}_{n-1}(X)$. If the components of B are contained in n different composants of X, then $\mathcal{OA}_n(B, X) = \mathcal{C}_n(B, X)$.*

3. Preliminary Results

We begin by stating a special case of a theorem of H. Toruńczyk [12, Theorem 1].

Theorem 3.1. *If X is a compact absolute retract such that the identity map of X onto X is a uniform limit of Z-maps, then X is homeomorphic to the Hilbert cube.*

The following lemma is due to Hiroshi Hosokawa, and a proof may be found in [7, Lemma 3.1].

Lemma 3.2. *If X is a continuum, $\mathcal{A}$ is a subcontinuum of 2^X, and $A \in \mathcal{A}$, then $\cup\mathcal{A}$ intersects each component of A.*

We extend [10, (0.74.1)] to n-fold hyperspaces.

Lemma 3.3. *Let X be a continuum and let n be a positive integer. If Λ is a locally connected continuum of $\mathcal{C}_n(X)$ such that $\cup\Gamma \in \Lambda$ for each subcontinuum Γ of Λ, then Λ is an absolute retract.*

Proof. Since Λ is a locally connected continuum, $\mathcal{C}(\Lambda)$ is an absolute retract [10, (1.96)]. By hypothesis, the map $\sigma|_{\mathcal{C}(\Lambda)}\colon \mathcal{C}(\Lambda) \to \Lambda$ given by $\sigma(\Gamma) = \cup\Gamma$ is well defined. Note that σ is continuous [10, (1.48)]. Let $\xi\colon \Lambda \to \mathcal{C}(\Lambda)$ be given by $\xi(A) = \{A\}$. Then ξ is continuous and $\sigma|_{\mathcal{C}(\Lambda)} \circ \xi = 1_\Lambda$. Hence, $\sigma|_{\mathcal{C}(\Lambda)}$ is an r-map. Therefore, Λ is an absolute retract by [1, Corollary (2.2), p. 86]. □

In the next two lemmas, we extend Lemma 3 and Lemma 5 of Eberhart [5] to n-fold hyperspaces, and the proofs are similar to those given by Eberhart; we present the details for the convenience of the reader.

Let X be a continuum, let n be a positive integer, and let $\{a_1, \dots, a_\ell\} \in \mathcal{F}_n(X)$. An *annular pair* is a pair (C, B) of elements of $\mathcal{OA}_n(\{a_1, \dots, a_\ell\}, X)$ such that $C \subset B$ and $B \setminus C$ is a nonempty open subset of X.

Lemma 3.4. *Let X be a continuum, let n be a positive integer, and let $\{a_1, \dots, a_\ell\} \in \mathcal{F}_n(X)$. If $A \in \mathcal{OA}_n(\{a_1, \dots, a_\ell\}, X)$ and, for each $\delta > 0$, there exists an annular pair (C, B) such that* $\operatorname{mesh}(B) < \delta$ *and $(B \setminus C) \cap A \neq \emptyset$, then $\mathcal{OA}_n(A, X)$ is a Z-set in $\mathcal{OA}_n(\{a_1, \dots, a_\ell\}, X)$.*

Proof. Let $A \in \mathcal{OA}_n(\{a_1, \ldots, a_\ell\}, X)$ and let $\delta > 0$. Then there exists an annular pair (C, B) such that $\text{mesh}(B) < \frac{\delta}{2}$ and $(B \setminus C) \cap A \neq \emptyset$. Let $\zeta\colon \mathcal{OA}_n(\{a_1, \ldots, a_\ell\}, X) \to \mathcal{OA}_n(B, X)$ be given by $\zeta(D) = B \cup D$. Since $B, D \in \mathcal{OA}_n(\{a_1, \ldots, a_\ell\}, X)$, we have that ζ is well defined. Note that ζ is continuous [10, (1.49)]. Since $\text{mesh}(B) < \frac{\delta}{2}$, we have that $\mathcal{H}(D, \zeta(D)) < \frac{\delta}{2}$.

Now, let $\xi\colon \mathcal{OA}_n(B, X) \to \mathcal{OA}_n(\{a_1, \ldots, a_\ell\}, X)$ be given by $\xi(D) = D \setminus (B \setminus C)$. We show that ξ is well defined. To this end, first note that $\xi(D)$ is a closed subset of X since $B \setminus C$ is an open subset of X. Let $x \in D \setminus B$, let K_x be the closure of the component of $D \setminus B$ containing x, and let D_1 be the component of D containing x. Then $K_x \subset D_1$. Observe that $K_x \subset \xi(D)$ since $\xi(D)$ is closed in X and $D \setminus B \subset \xi(D)$. By [11, 5.6], $K_x \cap B \neq \emptyset$. Since $B \setminus C$ is open in X, we have that $K_x \cap C \neq \emptyset$. Thus, $\xi(D) = C \cup (\cup_{x \in D \setminus B} K_x)$ and $\xi(D)$ has at most as many components as C. Hence, $\xi(D) \in \mathcal{C}_n(X)$. Note that $\{a_1, \ldots, a_\ell\} \subset C \cap D$. Therefore, $\xi(D) \in \mathcal{OA}_n(\{a_1, \ldots, a_\ell\}, X)$ and ξ is well defined. Observe that since $A \cap (B \setminus C) \neq \emptyset$, $\xi(D) \notin \mathcal{OA}_n(A, X)$.

It is not difficult to see that if $D, E \in \mathcal{OA}_n(B, X)$, then $\mathcal{H}(\xi(D), \xi(E)) = \mathcal{H}(D, E)$. Thus, ξ is continuous.

Next, we prove that if $D \in \mathcal{OA}_n(B.X)$, then $\mathcal{H}(D, \xi(D)) < \frac{\delta}{2}$. Clearly, $\xi(D) \subset \mathcal{V}^d_{\frac{\delta}{2}}(D)$. Let $x \in D$, and assume that $x \notin D \setminus (B \setminus C)$. Then $x \in B \setminus C$. Let B_1 be the component of B containing x. Let C_1 be a component of C such that $C_1 \subset B_1$. Let $x_0 \in C_1$. Then $d(x, x_0) \leq \text{diam}(B_1) \leq \text{mesh}(B) < \frac{\delta}{2}$. Hence, $D \subset \mathcal{V}^d_{\frac{\delta}{2}}(\xi(D))$. Therefore, $\mathcal{H}(D, \xi(D)) < \frac{\delta}{2}$.

Let $\varphi = \xi \circ \zeta$. Then φ is a map from $\mathcal{OA}_n(\{a_1, \ldots, a_\ell\}, X)$ into itself and $\varphi(\mathcal{OA}_n(\{a_1, \ldots, a_\ell\}, X)) \cap \mathcal{OA}_n(A, X) = \emptyset$. Also, if $D \in \mathcal{OA}_n(\{a_1, \ldots, a_\ell\}, X)$, then $\mathcal{H}(D, \varphi(D)) < \delta$. Therefore, we have that $\mathcal{OA}_n(A, X)$ is a Z-set in $\mathcal{OA}_n(\{a_1, \ldots, a_\ell\}, X)$. □

Let X be a continuum, let n be a positive integer, and let $\{a_1, \ldots, a_\ell\} \in \mathcal{F}_n(X)$. If $A \in \mathcal{OA}_n(\{a_1, \ldots, a_\ell\}, X)$, then A is said to *cover* $\{a_1, \ldots, a_\ell\}$ provided that

$$\{a_1, \ldots, a_\ell\} \in Int_{\mathcal{OA}_n(\{a_1, \ldots, a_\ell\}, X)}(\mathcal{OA}_n(\{a_1, \ldots, a_\ell\}, A)).$$

Lemma 3.5. *Let X be a continuum, let n be a positive integer, and let $\{a_1, \ldots, a_\ell\} \in \mathcal{F}_n(X)$. If $A \in \mathcal{OA}_n(\{a_1, \ldots, a_\ell\}, X)$ does not cover $\{a_1, \ldots, a_\ell\}$, then $\mathcal{OA}_n(\{a_1, \ldots, a_\ell\}, A))$ is a Z-set in*

$$\mathcal{OA}_n(\{a_1, \ldots, a_\ell\}, X).$$

Proof. Let $\varepsilon > 0$ and let $A \in \mathcal{OA}_n(\{a_1, \ldots, a_\ell\}, X)$ be such that it does not cover $\{a_1, \ldots, a_\ell\}$. Then there exists an element $B \in \mathcal{OA}_n(\{a_1, \ldots, a_\ell\}, X)$ such that $\text{mesh}(B) < \varepsilon$ and $B \cap (X \setminus A) \neq \emptyset$. Let

$\xi\colon \mathcal{OA}_n(\{a_1,\dots,a_\ell\},X) \to \mathcal{OA}_n(B,X)$ be given by $\xi(D) = B \cup D$. Then ξ is well defined and it is continuous [10, (1.49)]. Since $\mathrm{mesh}(B) < \varepsilon$, $\mathcal{H}(D,\xi(D)) < \varepsilon$. The condition $B \cap (X \setminus A) \neq \emptyset$ implies that

$$\xi(\mathcal{OA}_n(\{a_1,\dots,a_\ell\},X)) \cap \mathcal{OA}_n(\{a_1,\dots,a_\ell\},A) = \emptyset.$$

Therefore, $\mathcal{OA}_n(\{a_1,\dots,a_\ell\},A)$ is a Z-set in $\mathcal{OA}_n(\{a_1,\dots,a_\ell\},X)$. $\square$

4. Deformation Retracts

Let X be a continuum and let n be a positive integer. One might try to prove that if $B \in \mathcal{C}_n(X)$, then $\mathcal{C}_n(B,X)$ is an absolute retract. It turns out that even though this set is an arcwise connected continuum, in general, it is not locally connected as can be seen when X is an indecomposable continuum and B has fewer than n components (compare with Corollary 4.5) or when $X = Cl_{\mathbb{R}^2}\left(\left\{(x,\sin(\frac{1}{x})) \mid x \in (0,1]\right\}\right)$. However, when X is a locally connected continuum, $\mathcal{C}_n(B,X)$ is an absolute retract (Theorem 4.1). On the other hand, the set $\mathcal{OA}_n(B,X)$ is always an absolute retract (Theorem 4.3).

Theorem 4.1. *Let X be a locally connected continuum and let n be a positive integer. If $B \in \mathcal{C}_n(X)$, then $\mathcal{C}_n(B,X)$ is an absolute retract.*

Proof. Let $\varepsilon > 0$. Then, by [6, Lemma 3–29], there exists $\delta > 0$ such that if $x, x' \in X$ and $d(x,x') < \delta$, then there exists an arc $Z(x,x')$ from x to x' such that $\mathrm{diam}(Z(x,x')) < \varepsilon$. Without loss of generality, we assume that $\delta < \frac{\varepsilon}{4}$.

Let $B \in \mathcal{C}_n(X)$, let $A \in \mathcal{C}_n(B,X)$, and let $A_1,\dots,A_k$ be the components of A. Let $E \in \mathcal{V}^{\mathcal{H}}_\delta(A) \cap \mathcal{C}_n(B,X)$ and let $E_1,\dots,E_\ell$ be the components of E. Fix $j \in \{1,\dots,k\}$ and let $a_j \in A_j$. Since $\mathcal{H}(A,E) < \delta$, there exists $e_j \in E$ such that $d(a_j,e_j) < \delta$. Then there exists an arc $Z(a_j,e_j)$ in X from a_j to e_j such that $\mathrm{diam}(Z(a_j,e_j)) < \frac{\varepsilon}{4}$. Fix $s \in \{1,\dots,\ell\}$ and let $e'_s \in E_s$. Since $\mathcal{H}(A,E) < \delta$, there exists $a'_s \in A$ such that $d(e'_s,a'_s) < \delta$. Then there exists an arc $Z(e'_s,a'_s)$ in X from e'_s to a'_s such that $\mathrm{diam}(Z(e'_s,a'_s)) < \frac{\varepsilon}{4}$.

Let $R = A \cup E \cup \left(\cup_{j=1}^k Z(a_j,e_j)\right) \cup \left(\cup_{s=1}^\ell Z(e'_s,a'_s)\right)$. Then R is compact; also, it has at most $\min\{k,\ell\}$ components and $B \subset R$. Hence, $R \in \mathcal{C}_n(B,X)$. Observe that $\mathcal{H}(A,R) < \frac{\varepsilon}{2}$ and $\mathcal{H}(E,R) < \frac{\varepsilon}{2}$. Let $\alpha_A\colon [0,1] \to \mathcal{C}_n(X)$ be an order arc such that $\alpha_A(0) = A$ and $\alpha_A(1) = R$ and let $\alpha_E\colon [0,1] \to \mathcal{C}_n(X)$ be an order arc such that $\alpha_E(0) = A$ and $\alpha_E(1) = R$ [10, (1.8)]. Then $\alpha_A([0,1]) \cup \alpha_E([0,1])$ is a connected set such that $\{A,E\} \subset \alpha_A([0,1]) \cup \alpha_E([0,1])$. We show that $\alpha_A([0,1]) \cup \alpha_E([0,1]) \subset \mathcal{V}^{\mathcal{H}}_\delta(A) \cap \mathcal{C}_n(B,X)$. Clearly, $\alpha_A([0,1]) \cup \alpha_E([0,1]) \subset \mathcal{C}_n(B,X)$. Let $F \in \alpha_A([0,1]) \cup \alpha_E([0,1])$. If $F \in \alpha_A([0,1])$, then $\mathcal{H}(A,F) \leq \mathcal{H}(A,R) < \frac{\varepsilon}{2}$. Now, if $F \in \alpha_E([0,1])$, then $\mathcal{H}(A,F) \leq \mathcal{H}(A,E) + \mathcal{H}(E,F) < \frac{\varepsilon}{4} +$

$\mathcal{H}(E,R) < \frac{\varepsilon}{4} + \frac{\varepsilon}{2} < \varepsilon$. Thus, $\alpha_A([0,1]) \cup \alpha_E([0,1]) \subset \mathcal{V}^{\mathcal{H}}_{\delta}(A) \cap \mathcal{C}_n(B,X)$. Hence, $\mathcal{C}_n(B,X)$ is connected im kleinen at A. Therefore, $\mathcal{C}_n(B,X)$ is locally connected.

In order to conclude that $\mathcal{C}_n(B,X)$ is an absolute retract, we use Lemma 3.3. To this end, let Γ be a subcontinuum of $\mathcal{C}_n(B,X)$. Then, by Lemma 3.2, $\cup\Gamma \in \mathcal{C}_n(X)$. Clearly, $B \subset \cup\Gamma$. Thus, $\cup\Gamma \in \mathcal{C}_n(B,X)$. Therefore, by Lemma 3.3, $\mathcal{C}_n(B,X)$ is an absolute retract. □

Corollary 4.2. *Let X be a locally connected continuum and let n be a positive integer. If $B \in \mathcal{C}_n(X)$, then $\mathcal{C}_n(B,X)$ is a deformation retract of $\mathcal{C}_n(X)$.*

Proof. Since X is locally connected, X has the property of Kelley [10, (16.11)]. Hence, $\mathcal{C}_n(X)$ is contractible [9, 6.1.16]. Thus, since $\mathcal{C}_n(B,X)$ is a retract of $\mathcal{C}_n(X)$ by Theorem 4.1, we have that $\mathcal{C}_n(B,X)$ is a deformation retract of $\mathcal{C}_n(X)$ [13, 32E.4]. □

Now we show that the sets of the form $\mathcal{OA}_n(B,A)$ are absolute retracts.

Theorem 4.3. *Let X be a continuum, let n be a positive integer, and let $B \in \mathcal{C}_n(X)$. If $A \in \mathcal{OA}_n(B,X)$, then $\mathcal{OA}_n(B,A)$ is an absolute retract.*

Proof. Clearly, $\mathcal{OA}_n(B,A)$ is closed in $\mathcal{C}_n(X)$. By [10, (1.48)], for each $D \in \mathcal{OA}_n(B,A)$, there exists an order arc $\alpha\colon [0,1] \to \mathcal{C}_n(X)$ such that $\alpha(0) = B$ and $\alpha(1) = D$. Note that for each $t \in [0,1]$, $\alpha(t) \subset A$. Hence, $\alpha([0,1]) \subset \mathcal{OA}_n(B,A)$. Therefore, $\mathcal{OA}_n(B,A)$ is an arcwise connected continuum.

Let $D \in \mathcal{OA}_n(B,A)$ and let $\varepsilon > 0$. If $E \in \mathcal{V}^{\mathcal{H}}_{\frac{\varepsilon}{2}}(D) \cap \mathcal{OA}_n(B,A)$, then it is clear that $D \cup E \in \mathcal{OA}_n(B,A)$, $\mathcal{H}(D, D \cup E) < \frac{\varepsilon}{2}$, and $\mathcal{H}(E, D \cup E) < \frac{\varepsilon}{2}$. Since it is easy to see that each component of $D \cup E$ intersects both D and E, there exist order arcs $\alpha_D, \alpha_E\colon [0,1] \to \mathcal{C}_n(X)$ such that $\alpha_D(0) = D$, $\alpha_D(1) = D \cup E$, $\alpha_E(0) = E$, and $\alpha_E(1) = D \cup E$ [10, (1.48)]. Note that $\alpha_D([0,1]) \subset \mathcal{OA}_n(B,A)$ and $\alpha_E([0,1]) \subset \mathcal{OA}_n(B,A)$. In fact, $\alpha_D([0,1]) \subset \mathcal{V}^{\mathcal{H}}_{\varepsilon}(D) \cap \mathcal{OA}_n(B,A)$ and $\alpha_E([0,1]) \subset \mathcal{V}^{\mathcal{H}}_{\varepsilon}(D) \cap \mathcal{OA}_n(B,A)$. Hence, $\alpha_D([0,1]) \cup \alpha_E([0,1])$ is a connected subset of $\mathcal{V}^{\mathcal{H}}_{\varepsilon}(A) \cap \mathcal{OA}_n(B,A)$ such that $\{D,E\} \subset \alpha_D([0,1]) \cup \alpha_E([0,1])$. This implies that $\mathcal{OA}_n(B,A)$ is connected im kleinen at D. Therefore, $\mathcal{OA}_n(B,A)$ is locally connected [9, 1.7.12].

Let Γ be a subcontinuum of $\mathcal{OA}_n(B,A)$. Then $\cup\Gamma \in \mathcal{C}_n(X)$ [9, 6.1.2]. In fact, $\cup\Gamma \subset A$. By Lemma 3.2, each component of $\cup\Gamma$ intersects B. Hence, $\cup\Gamma \in \mathcal{OA}_n(B,A)$. Therefore, by Lemma 3.3, $\mathcal{OA}_n(B,A)$ is an absolute retract. □

From Proposition 2.1 and Theorem 4.3, we obtain the following.

Corollary 4.4. *If X is an irreducible continuum between x_1 and x_2, $B \in \mathcal{C}_2(X) \setminus \mathcal{C}_1(X)$, and $\{x_1, x_2\} \subset B$, then $\mathcal{C}_2(B, X)$ is an absolute retract.*

As a consequence of Proposition 2.2 and Theorem 4.3, we have the following corollary.

Corollary 4.5. *Let X be an indecomposable continuum, let n be an integer greater than or equal to 2, and let $B \in \mathcal{C}_n(X) \setminus \mathcal{C}_{n-1}(X)$. If the components of B are contained in n different composants of X, then $\mathcal{C}_n(B, X)$ is an absolute retract.*

Corollary 4.6. *Let X be a continuum, let n be a positive integer, and let $B \in \mathcal{C}_n(X)$. If $\mathcal{C}_n(X)$ is contractible and $A \in \mathcal{OA}_n(B, X)$, then $\mathcal{OA}_n(B, A)$ is a deformation retract of $\mathcal{C}_n(X)$.*

Proof. Since $\mathcal{OA}_n(B, A)$ is a retract of $\mathcal{C}_n(X)$ by Theorem 4.3, we have that $\mathcal{OA}_n(B, A)$ is a deformation retract of $\mathcal{C}_n(X)$ [13, 32E.4]. □

A continuum X has the *property of Kelley* provided that for each $\varepsilon > 0$, there exists $\delta > 0$ such that for every pair of points x and y of X such that $d(x, y) < \delta$ and each subcontinuum A of X such that $x \in A$, there exists a subcontinuum B of X such that $y \in B$ and $\mathcal{H}(A, B) < \varepsilon$.

Corollary 4.7. *Let X be a continuum with the property of Kelley, let n be a positive integer, and let $B \in \mathcal{C}_n(X)$. If $A \in \mathcal{OA}_n(B, X)$, then $\mathcal{OA}_n(B, A)$ is a deformation retract of $\mathcal{C}_n(X)$.*

Proof. Since X has the property of Kelley, $\mathcal{C}_n(X)$ is contractible [9, 6.1.16]. The corollary now follows from Corollary 4.6. □

From Proposition 2.1 and Corollary 4.7, we have the following.

Corollary 4.8. *If X is an irreducible continuum between x_1 and x_2, with the property of Kelley, $B \in \mathcal{C}_2(X) \setminus \mathcal{C}_1(X)$, and $\{x_1, x_2\} \subset B$, then $\mathcal{C}_2(B, X)$ is a deformation retract of $\mathcal{C}_2(X)$.*

As a consequence of Proposition 2.2 and Corollary 4.7, we obtain the following corollary.

Corollary 4.9. *Let X be an indecomposable continuum with the property of Kelley, let n be an integer greater than or equal to 2, and let $B \in \mathcal{C}_n(X) \setminus \mathcal{C}_{n-1}(X)$. If the components of B are contained in n different composants of X, then $\mathcal{C}_n(B, X)$ is a deformation retract of $\mathcal{C}_n(X)$.*

Question 4.10. Let X be a continuum, let n be a positive integer, let $B \in \mathcal{C}_n(X)$, and let $A \in \mathcal{OA}(B, X)$. When are the sets $\mathcal{OA}_n(B, A)$ and $\mathcal{C}_n(B, X)$ strong deformation retracts of $\mathcal{C}_n(X)$?

5. Hilbert Cubes

In this section we extend the results of Eberhart in [5] to n-fold hyperspaces to obtain Hilbert cubes. We present the appropriate changes in the proofs for the convenience of the reader.

Theorem 5.1. *Let X be a continuum, let n be a positive integer, and let $B \in \mathcal{C}_n(X)$. If for each $\varepsilon > 0$, there exists $A \in \mathcal{OA}_n(B,X)$ such that $\mathcal{H}(A,B) < \varepsilon$ and $\mathcal{OA}_n(A,X)$ is a Z-set in $\mathcal{OA}_n(B,X)$, then $\mathcal{OA}_n(B,X)$ is homeomorphic to the Hilbert cube.*

Proof. Let $\varepsilon > 0$ and let $A \in \mathcal{OA}_n(B,X)$ be such that $\mathcal{H}(A,B) < \varepsilon$. Note that, by [10, (1.49)], the map $\xi \colon \mathcal{OA}_n(B,X) \to \mathcal{OA}_n(A,X)$ given by $\xi(D) = D \cup A$ is continuous. By hypothesis, ξ is a Z-map. Note that for each $D \in \mathcal{OA}_n(B,X)$, $\mathcal{H}(D,\xi(D)) < \varepsilon$. Hence, the identity map of $\mathcal{OA}_n(B,X)$ is a uniform limit of Z-maps. Since, by Theorem 4.3, $\mathcal{OA}_n(B,X)$ is an absolute retract, we have that $\mathcal{OA}_n(B,X)$ is homeomorphic to the Hilbert cube by Theorem 3.1. □

Theorem 5.2. *Let X be a continuum, let n be a positive integer, and let $\{a_1,\ldots,a_\ell\} \in \mathcal{F}_n(X)$. Suppose that for each $j \in \{1,\ldots,\ell\}$, X is locally connected at each point of an open set containing a_j. Then $\mathcal{OA}_n$ $(\{a_1,\ldots,a_\ell\},X)$ is homeomorphic to the Hilbert cube if and only if there exists $j \in \{1,\ldots,\ell\}$ such that a_j is not in the interior (relative to X) of a finite graph.*

Proof. First, assume that for each $j \in \{1,\ldots,\ell\}$, there exists a finite graph G_j in X such that $a_j \in Int_X(G_j)$. With a similar argument to the one given in the proof of [9, 6.1.10], we may construct a k-cell $\mathcal{K}$ in $\mathcal{OA}_n(\{a_1,\ldots,a_\ell\},X)$ such that $\{a_1,\ldots,a_\ell\} \in Int_{\mathcal{C}_n(X)}(\mathcal{K})$, for some positive integer k. Therefore, we have that $\mathcal{OA}_n(\{a_1,\ldots,a_\ell\},X)$ is not homeomorphic to the Hilbert cube.

Suppose now, without loss of generality, that a_1 does not belong to the interior (relative to X) of any finite graph in X. Let $\varepsilon > 0$ be given. For each $j \in \{1,\ldots,\ell\}$, let U_j be a connected open subset of X such that $a_j \in U_j$ and $\mathrm{diam}(U_j) < \frac{\varepsilon}{2}$. Without loss of generality, we assume that X is locally connected at each point of $Cl_X(U_j)$ and $Cl_X(U_j) \cap Cl_X(U_m) = \emptyset$ for all $j,m \in \{1,\ldots,\ell\}$ and $j \neq m$. Note that each $Cl_X(U_j)$ is a continuum. Let $A = \cup_{j=1}^{\ell} Cl_X(U_j)$. Then $A \in \mathcal{OA}_n(\{a_1,\ldots,a_\ell\},X)$ and $\mathcal{H}(\{a_1,\ldots,a_\ell\},A) < \varepsilon$. We show that $\mathcal{OA}_n(A,X)$ is a Z-set in $\mathcal{OA}_n(\{a_1,\ldots,a_\ell\},X)$. To this end, let $\delta > 0$ be such that $\delta < \varepsilon$ and fix $j \in \{1,\ldots,\ell\}$. Let V_j be a connected open subset of X such that $a_j \in V_j \subset Cl_X(V_j) \subset U_j$ and $\mathrm{diam}(V_j) < \frac{\delta}{2}$. Since X is locally connected at each point of U_j, there exist finitely many continua $C_{j1},\ldots,C_{jk_j}$ such that $Bd_X(V_j) \subset \cup_{s=1}^{k_j} C_{js}$, $a_j \notin \cup_{s=1}^{k_j} C_{js}$, and

$$\operatorname{diam}\left(Cl_X(V_j) \cup \left(\cup_{s=1}^{k_j} C_{js}\right)\right) < \delta.$$

Let G_j be a finite graph in V_j that contains a_j and intersects each C_{js} for each $s \in \{1, \ldots, k_j\}$. G_j exists because V_j is arcwise connected. Let $C_j = G_j \cup \left(\cup_{s=1}^{k_j} C_{js}\right)$ and let $B_j = Cl_X(V_j) \cup C_j$. Then both C_j and B_j are continua and $a_j \in C_j \subset B_j$.

Let $B = \cup_{j=1}^{\ell} B_j$ and let $C = \cup_{j=1}^{\ell} C_j$. Then both B and C belong to $\mathcal{OA}_n(\{a_1, \ldots, a_\ell\}, X)$, $C \subset B$, and $\operatorname{mesh}(B) < \delta$. Observe that $B \setminus C$ is open in X since C contains the boundary of each of the sets $V_1, \ldots, V_\ell$. Note that $(B \setminus C) \cap A \neq \emptyset$ because G_1 does not contain a_1 in its interior. Thus, (C, B) is an annular pair of $\mathcal{OA}_n(\{a_1, \ldots, a_\ell\}, X)$ that satisfies the hypotheses of Lemma 3.4. Hence, $\mathcal{OA}_n(A, X)$ is a Z-set in $\mathcal{OA}_n(\{a_1, \ldots, a_\ell\}, X)$. Therefore, $\mathcal{OA}_n(\{a_1, \ldots, a_\ell\}, X)$ is homeomorphic to the Hilbert cube by Theorem 5.1. □

Corollary 5.3. *Let X be a locally connected continuum, let n be a positive integer, and let $\{a_1, \ldots, a_\ell\} \in \mathcal{F}_n(X)$. If there exists $j \in \{1, \ldots, \ell\}$ such that a_j is not in the interior (relative to X) of a finite graph, then $\mathcal{C}_n(\{a_1, \ldots, a_\ell\}, X)$ contains a subset homeomorphic to the Hilbert cube.*

Question 5.4. Let X be a locally connected continuum, let n be a positive integer, and let $\{a_1, \ldots, a_\ell\} \in \mathcal{F}_n(X)$. When is the set $\mathcal{C}_n(\{a_1, \ldots, a_\ell\}, X)$ homeomorphic to the Hilbert cube?

As a consequence of Theorem 3.1 and Theorem 4.3, we have the following.

Theorem 5.5. *Let X be a continuum, let n be a positive integer, and let $\{a_1, \ldots, a_\ell\} \in \mathcal{F}_n(X)$. Suppose that for each $\varepsilon > 0$, there exist $A \in \mathcal{OA}_n(\{a_1, \ldots, a_\ell\}, X)$ and a map*

$$\xi \colon \mathcal{OA}_n(\{a_1, \ldots, a_\ell\}, X) \to \mathcal{OA}_n(\{a_1, \ldots, a_\ell\}, A)$$

such that $\mathcal{OA}_n(\{a_1, \ldots, a_\ell\}, A)$ is a Z-set in $\mathcal{OA}_n(\{a_1, \ldots, a_\ell\}, X)$ and for each $D \in \mathcal{OA}_n(\{a_1, \ldots, a_\ell\}, X)$, $\mathcal{H}(D, \xi(D)) < \varepsilon$. Then

$$\mathcal{OA}_n(\{a_1, \ldots, a_\ell\}, X)$$

is homeomorphic to the Hilbert cube.

Theorem 5.6. *Let X be a continuum, let n be a positive integer, and let $\{a_1, \ldots, a_\ell\} \in \mathcal{F}_n(X)$. Suppose that for each $\varepsilon > 0$, there exists a map $f \colon X \to X$ such that $d(x, f(x)) < \varepsilon$ for all $x \in X$, $f(X)$ does not cover $\{a_1, \ldots, a_\ell\}$, and $f(a_j) = a_j$ for each $j \in \{1, \ldots, \ell\}$. Then the set $\mathcal{OA}_n(\{a_1, \ldots, a_\ell\}, X)$ is homeomorphic to the Hilbert cube.*

Proof. Let $\varepsilon > 0$ and let $f\colon X \to X$ be a map such that $f(X)$ does not cover $\{a_1, \dots, a_\ell\}$, $f(a_j) = a_j$ for each $j \in \{1, \dots, \ell\}$, and $d(x, f(x)) < \varepsilon$ for all $x \in X$. Let $\mathcal{C}_n(f)\colon \mathcal{C}_n(X) \to \mathcal{C}_n(X)$ be the induced map of f to the n-fold hyperspace of X [3, p. 783]. Note that for each $D \in \mathcal{C}_n(X)$, $\mathcal{H}(D, \mathcal{C}_n(f)(D)) < \varepsilon$ [3, Lemma 37] and $\mathcal{C}_n(f)(\mathcal{OA}_n(\{a_1, \dots, a_\ell\}, X)) \subset \mathcal{OA}_n(\{a_1, \dots, a_\ell\}, X)$. Since for each $j \in \{1, \dots, \ell\}$, $f(a_j) = a_j$, $f(X) \in \mathcal{OA}_n(\{a_1, \dots, a_\ell\}, X)$. Hence, we have that, in fact,

$$\mathcal{C}_n(f)(\mathcal{OA}_n(\{a_1, \dots, a_\ell\}, X)) \subset \mathcal{OA}_n(\{a_1, \dots, a_\ell\}, f(X)).$$

Since $f(X)$ does not cover $\{a_1, \dots, a_\ell\}$, we obtain, by Lemma 3.5, that $\mathcal{OA}_n(\{a_1, \dots, a_\ell\}, f(X))$ is a Z-set in $\mathcal{OA}_n(\{a_1, \dots, a_\ell\}, X)$. Therefore, $\mathcal{OA}_n(\{a_1, \dots, a_\ell\}, X)$ is homeomorphic to the Hilbert cube by Theorem 5.5. □

If X is a compact metric space, then $Sus(X)$ denotes its topological suspension and v_1 and v_2 denote the vertices of $Sus(X)$.

Corollary 5.7. *Let n be an integer greater than or equal to 2 and let X be a compactum such that for each $\varepsilon > 0$, there exists a map $f\colon X \to X$ such that $f(X) \neq X$ and $d(x, f(x)) < \varepsilon$ for all $x \in X$. Then $\mathcal{OA}_n(\{v_1, v_2\}, Sus(X))$ is homeomorphic to the Hilbert cube.*

Proof. Let $\varepsilon > 0$ and let $f\colon X \to X$ be a map such that $f(X) \neq X$ and $d(x, f(x)) < \varepsilon$ for all $x \in X$. Then f induces a map $Sus(f)\colon Sus(X) \to Sus(X)$ given by $Sus(f)(v_j) = v_j$, $j \in \{1, 2\}$, and $Sus(f)((x,t)) = (f(x), t)$ [4, 5.4, p. 127]. Note that for each $\chi \in Sus(X)$, $d(\chi, Sus(\chi)) < \varepsilon$. Since $f(X) \neq X$, $Sus(f)(Sus(X))$ does not cover $\{v_1, v_2\}$. The corollary now follows from Theorem 5.6. □

Observe that if X is either the harmonic sequence $\{0\} \cup \{\frac{1}{k}\}_{k=1}^{\infty}$ or the Cantor set and $n \geq 2$, then, by Corollary 5.7, we have that $\mathcal{OA}_n(\{v_1, v_2\}, Sus(X))$ is homeomorphic to the Hilbert cube.

References

[1] Karol Borsuk, *Theory of Retracts*. Monografie Matematyczne, Tom 44. Warsaw: Panstwowe Wydawnictwo Naukowe, 1967.

[2] Karol Borsuk and Stanislaw Ulam, *On symmetric products of topological spaces*, Bull. Amer. Math. Soc. **37**, (1931), no. 12, 875–882.

[3] Janusz J. Charatonik, Alejandro Illanes, and Sergio Macías, *Induced mappings on the hyperspaces $C_n(X)$ of a continuum X*, Houston J. Math. **28** (2002), no. 4, 781–805.

[4] James Dugundji, *Topology*. Boston, Mass.: Allyn and Bacon, Inc., 1966.

[5] Carl Eberhart, *Intervals of continua which are Hilbert cubes*, Proc. Amer. Math. Soc. **68** (1978), no. 2, 220–224.

[6] John G. Hocking and Gail S.Young, *Topology.* 2nd ed. New York: Dover Publications, Inc., 1988.

[7] Hiroshi Hosokawa, *Induced mappings on hyperspaces*, Tsukuba J. Math. **21** (1997), no. 1, 239–250.

[8] María de J. López and Sergio Macías, *Induced maps on n-fold hyperspaces*, Houston J. Math. **33** (2007), no. 4, 1047–1057.

[9] Sergio Macías, *Topics on Continua.* Pure and Applied Mathematics, Volume 275. Boca Raton, London, New York, Singapore: Chapman & Hall/CRC, Taylor & Francis Group, 2005.

[10] Sam B. Nadler, Jr., *Hyperspaces of Sets. A Text with Research Questions.* Monographs and Textbooks in Pure and Applied Mathematics, Vol. 49. New York-Basel: Marcel Dekker, Inc., 1978. Reprinted in Aportaciones Matemáticas: Textos, 33. México, Sociedad Matemática Mexicana, 2006.

[11] ——, *Continuum Theory. An Introduction.* Monographs and Textbooks in Pure and Applied Mathematics, 158. New York: Marcel Dekker, Inc., 1992.

[12] H. Torunczyk, *On* CE-*images of the Hilbert cube and characterization of Q-manifolds*, Fund. Math. **106** (1980), no. 1, 31–40.

[13] Stephen Willard, *General Topology.* Reading, Mass.: Addison-Wesley Publishing Co., 1970.

Instituto de Matemáticas; Universidad Nacional Autónoma de México; Circuito Exterior, Ciudad Universitaria; México D. F., C. P. 04510, MÉXICO

E-mail address: macias@servidor.unam.mx