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## EQUIVARIANT SELECTIONS OF CONVEX-VALUED MAPS

by

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## EQUIVARIANT SELECTIONS OF CONVEX-VALUED MAPS

SERGEY A. ANTONYAN AND NATALIA JONARD-PÉREZ

**ABSTRACT.** Let  $G$  be a locally compact Hausdorff group,  $L$  a linear  $G$ -space, and  $Y \subset L$  a metrizable convex subset of  $L$  endowed with a proper action of  $G$ . Let  $X$  be a paracompact proper  $G$ -space with a paracompact orbit space. We give conditions for  $Y$  in order that every equivariant lower semicontinuous multivalued map  $\phi : X \rightrightarrows Y$  with complete convex values admits an equivariant near selection and equivariant selection.

### 1. INTRODUCTION

The classic Michael selection theorem [10] states that every lower semicontinuous (l.s.c.) multivalued map from a paracompact space into the non-empty closed convex sets of a Banach space admits a continuous selection. Following the same method used in [10], it was proved in [13, Theorem 1.4.9] that every l.s.c. map from a paracompact space into the non-empty complete convex sets of an arbitrary normed linear space admits a continuous selection. The proof of this result consists in finding first an  $\varepsilon$ -near selection for every positive  $\varepsilon$ . Then the required selection appears as the limit of a carefully constructed sequence of  $2^{-n}$ -near selections, for  $n = 1, 2, \dots$

In [5], an equivariant version of Michael's theorem was proved: If  $G$  is a compact group,  $X$  a paracompact  $G$ -space, and  $Y$  a Banach  $G$ -space, then every l.s.c. multivalued equivariant map from  $X$  into the non-empty

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convex closed subsets of  $Y$  admits an equivariant selection. In the proof of this result, the authors used the vector-valued integral with respect to the normalized Haar measure for averaging a non equivariant selection in order to obtain the desired equivariant selection. Because the normalized Haar vector-valued integral was used, the completeness of the codomain  $Y$  and the compactness of the group  $G$  seem to be essential in that proof.

In the present paper, we will give an equivariant version of Michael's selection theorem which is valid for proper actions of arbitrary locally compact groups. The idea is to "equivariantize" Michael's original proof: First, we obtain an equivariant near selection (Theorem 5.3), and then we use the same method as in [13, Theorem 1.4.9] to obtain an equivariant selection (see Proposition 6.2 and Corollary 6.3).

## 2. PRELIMINARIES

We refer the reader to monographs [7] and [11] for basic notions of the theory of  $G$ -spaces. However, we will recall here some special definitions and results that will be used throughout this paper.

If  $G$  is a topological group and  $X$  is a  $G$ -space, for any  $x \in X$ , we denote by  $G_x$  the stabilizer of  $x$ , i.e.,  $G_x = \{g \in G \mid gx = x\}$ . For a subset  $S \subset X$  and a subgroup  $H \subset G$ ,  $H(S)$  denotes the  $H$ -saturation of  $S$ , i.e.,  $H(S) = \{hs \mid h \in H, s \in S\}$ . If  $H(S) = S$ , then we say that  $S$  is an  $H$ -invariant set. In particular,  $G(x)$  denotes the  $G$ -orbit of  $x$ , i.e.,  $G(x) = \{gx \in X \mid g \in G\}$ . The orbit space is denoted by  $X/G$ . For any subgroup  $H \subset G$ , we will denote by  $G/H$  the  $G$ -space of cosets  $\{gH \mid g \in G\}$  equipped with the action of  $G$  induced by left translations.

A map  $f : X \rightarrow Y$  between two  $G$ -spaces is called *equivariant*, or a  *$G$ -map*, if  $f(gx) = gf(x)$  for every  $x \in X$  and  $g \in G$ .

A  $G$ -space  $X$  is called *proper* (in the sense of Palais) if every point  $x \in X$  has a neighborhood  $U$  such that for any other point  $y \in X$  there exists a neighborhood  $V$  of  $y$  such that the set  $\{g \in G \mid gU \cap V \neq \emptyset\}$  has compact closure in  $G$ .

Each orbit in a proper  $G$ -space is closed and each stabilizer is compact [12, Proposition 1.1.4].

We will denote by  $G\text{-}\mathcal{P}$  the class of all paracompact proper  $G$ -spaces with a paracompact orbit space. It is an open problem if the class  $G\text{-}\mathcal{P}$  coincides with the class of all paracompact proper  $G$ -spaces, i.e., if the orbit space of any paracompact proper  $G$ -space is paracompact (see [6]). By  $G\text{-}\mathcal{M}$ , we will denote the class of all proper  $G$ -spaces that are metrizable by a  $G$ -invariant metric.

Let  $G$  be a topological group and  $X$  a  $G$ -space. A  $G$ -space  $Y$  is called an *equivariant absolute neighborhood extensor* for  $X$ , denoted  $Y \in G\text{-ANE}(X)$ , if for any closed invariant subset  $A \subset X$  and any equivariant

map  $f : A \rightarrow Y$ , there exists an invariant neighborhood  $U$  of  $A$  in  $X$  and an equivariant map  $F : U \rightarrow Y$  such that  $F|_A = f$ . If, in addition, one can always take  $U = X$ , then we say that  $Y$  is an *equivariant absolute extensor* for  $X$ , denoted  $Y \in G\text{-AE}(X)$ .

A  $G$ -space  $Y$  is called an equivariant absolute neighborhood extensor for the class  $G\text{-}\mathcal{P}$ , denoted  $Y \in G\text{-ANE}(\mathcal{P})$ , if  $Y \in G\text{-ANE}(X)$  for every  $X \in G\text{-}\mathcal{P}$ . Similarly, if  $Y \in G\text{-AE}(X)$  for any  $X \in G\text{-}\mathcal{P}$ , then  $Y$  is called an equivariant absolute extensor for the class  $G\text{-}\mathcal{P}$ , denoted  $Y \in G\text{-AE}(\mathcal{P})$ . The classes  $G\text{-ANE}(\mathcal{M})$  and  $G\text{-AE}(\mathcal{M})$  are defined analogously.

Let  $X$  and  $Y$  be topological spaces. A multivalued map  $\phi$  from  $X$  to  $Y$  is a map  $\phi$  from  $X$  into the non-empty subsets of  $Y$ ; i.e., for every  $x \in X$ ,  $\phi(x)$  is a non-empty subset of  $Y$ . Following [13], we will use the symbol

$$\phi : X \rightrightarrows Y$$

to denote a multivalued map from  $X$  to  $Y$ .

A multivalued map  $\phi : X \rightrightarrows Y$  is called l.s.c. if for each open subset  $V \subset Y$ , the set

$$\phi^{\leftarrow}(V) = \{x \in X \mid \phi(x) \cap V \neq \emptyset\},$$

called the inverse image of  $V$  under  $\phi$ , is open in  $X$ .

If  $X$  and  $Y$  are  $G$ -spaces, then a multivalued map  $\phi : X \rightrightarrows Y$  is called equivariant if

$$\phi(gx) = g\phi(x), \text{ where } g\phi(x) = \{gy \mid y \in \phi(x)\},$$

for every  $x \in X$  and  $g \in G$ .

A selection for a multivalued map  $\phi : X \rightrightarrows Y$  is a continuous map  $f : X \rightarrow Y$  such that  $f(x) \in \phi(x)$  for every  $x \in X$ .

If  $X$  and  $Y$  are  $G$ -spaces, a selection  $f : X \rightarrow Y$  is called an *equivariant selection* if  $f$  is a  $G$ -map.

A compatible metric  $d$  on a  $G$ -space  $X$  is called *invariant*, or  *$G$ -invariant*, if  $d(gx, gy) = d(x, y)$  for all  $g \in G$  and  $x, y \in X$ .

By a *linear  $G$ -space*, we shall mean a real topological vector space on which  $G$  acts continuously and linearly, i.e.,  $g(\lambda x + \mu y) = \lambda(gx) + \mu(gy)$ , for every  $g \in G$  and for all scalars  $\lambda$  and  $\mu$ , and  $x, y \in X$ . If  $L$  is a Banach space where a topological group  $G$  acts continuously, linearly, and isometrically, then we will say that  $L$  is a *Banach  $G$ -space*.

We will denote by  $G\text{-}\mathcal{M}$  the class of all proper  $G$ -spaces which admit a  $G$ -invariant metric.

If  $(X, d)$  is a metric space,  $x \in X$ , and  $\varepsilon > 0$ , then we will denote by  $B(x, \varepsilon)$  the  $\varepsilon$ -ball around  $x$ , i.e.,  $B(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}$ .

Let  $Y$  be an invariant convex subset of a normed linear  $G$ -space where  $G$  acts by means of linear isometries. Suppose that  $d$  is the metric in  $Y$

induced by the norm. We will say that  $(Y, d)$  belongs to the class  $G\text{-}\mathcal{L}$  if the action of  $G$  restricted to  $Y$  is proper.

The following two simple lemmas will be used in the proof of Theorem 5.3.

**Lemma 2.1.** *Let  $G$  be a locally compact Hausdorff group and  $X$  a  $G$ -space such that  $X \in G\text{-}\mathcal{P}$ . If  $\{U_\alpha\}_{\alpha \in \mathcal{A}}$  is an open invariant covering of  $X$ , then there exists a locally finite open invariant refinement of  $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ ; i.e., there exists a locally finite refinement of  $\{U_\alpha\}_{\alpha \in \mathcal{A}}$  consisting of open invariant subsets.*

*Proof.* Consider the orbit projection  $p : X \rightarrow X/G$ . It is continuous and open. Since  $X/G$  is paracompact, the open covering  $\{p(U_\alpha)\}_{\alpha \in \mathcal{A}}$  of the space  $X/G$  has a locally finite open refinement  $\{V_\lambda\}_{\lambda \in \Lambda}$ .

Now, the invariant open covering  $\{p^{-1}(V_\lambda)\}_{\lambda \in \Lambda}$  of  $X$ , obviously, is a locally finite refinement of  $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ . This proves the lemma.  $\square$

**Lemma 2.2.** *Let  $G$  be a locally compact Hausdorff group. For any open invariant covering  $\{U_\alpha\}_{\alpha \in \mathcal{A}}$  of a proper  $G$ -space  $X$  such that  $X \in G\text{-}\mathcal{P}$ , there exists an invariant locally finite partition of unity  $\{p_\alpha\}_{\alpha \in \mathcal{A}}$  subordinated to the covering  $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ . That is to say, each  $p_\alpha : X \rightarrow [0, 1]$  is an invariant continuous map and  $\overline{p_\alpha^{-1}((0, 1])} \subset U_\alpha$  and the family  $\{p_\alpha^{-1}((0, 1]) \mid \alpha \in \mathcal{A}\}$  is locally finite.*

*Proof.* We consider the natural projection  $p : X \rightarrow X/G$  to the orbital space  $X/G$ . Since  $p$  is open, the family  $\{p(U_\alpha)\}_{\alpha \in \mathcal{A}}$  is an open covering of the paracompact space  $X/G$ . Thus, there exists a partition of unity  $\{q_\alpha\}_{\alpha \in \mathcal{A}}$  subordinated to the covering  $\{p(U_\alpha)\}_{\alpha \in \mathcal{A}}$ . So, for every  $\alpha \in \mathcal{A}$ , the closure  $\overline{q_\alpha^{-1}((0, 1])}$  is contained in  $p(U_\alpha)$ .

For each  $\alpha \in \mathcal{A}$ , let us consider the composition  $p_\alpha = q_\alpha \circ p$ . It is evident that  $p_\alpha$  is an invariant map. Furthermore, since every  $U_\alpha$  is invariant, it happens that  $p^{-1}(p(U_\alpha)) = U_\alpha$ . Hence, we have

$$\overline{p_\alpha^{-1}((0, 1])} = \overline{p^{-1}(q_\alpha^{-1}((0, 1]))} \subset p^{-1}(\overline{q_\alpha^{-1}((0, 1]))} \subset p^{-1}(p(U_\alpha)) = U_\alpha.$$

Clearly, the family  $\{p_\alpha\}_{\alpha \in \mathcal{A}}$  is the required one.  $\square$

### 3. A $G$ -FIXED POINT THEOREM

Let  $G$  be a compact group and let  $K \subset L$  be a complete convex and invariant subset of a locally convex metrizable linear  $G$ -space  $L$ .

By  $C(G, K)$ , we denote the space of all continuous maps from  $G$  into  $K$  equipped with the compact-open topology. One can define a continuous action  $G \times C(G, K) \rightarrow C(G, K)$  as follows:

$$(g, f) \rightarrow g * f, \quad (g * f)(h) = gf(h) \text{ for every } h \in G.$$

For each  $f \in C(G, K)$  and  $g \in G$ , let  ${}_g f \in C(G, K)$  be the map defined by the following formula:

$${}_g f(h) = f(gh), \quad h \in G.$$

Similarly, we will denote by  $f_g$  the element of  $C(G, K)$  defined by

$$f_g(h) = f(hg), \quad h \in G.$$

We shall need the following result proved in [2, Lemma 2].

**Proposition 3.1.** *There exists a continuous map  $\int : C(G, K) \rightarrow K$ , such that*

- (1)  $\int {}_g f = \int f = \int f_g$ , for all  $g \in G$  and  $f \in C(G, K)$ ;
- (2)  $\int g * f = g \int f$ , for all  $g \in G$  and  $f \in C(G, K)$ ;
- (3) if  $f(g) = x_0 \in K$  for every  $g \in G$ , then  $\int f = x_0$ .

**Corollary 3.2.** *Let  $G$  be a compact topological group and  $L$  a locally convex, metrizable linear  $G$ -space. If  $K \subset L$  is a  $G$ -invariant complete convex subset, then there exists a  $G$ -fixed point in  $K$ , i.e., a point  $a \in K$  such that  $ga = a$  for all  $g \in G$ .*

*Proof.* Pick an arbitrary point  $z \in K$  and define  $f : G \rightarrow K$  as follows:

$$f(g) = gz, \quad g \in G.$$

Let  $\int f$  be the map defined in Proposition 3.1. We claim that the point  $a = \int f \in K$  is the desired one. Indeed, if  $g$  and  $h$  are arbitrary elements of  $G$ , then we have

$$(g * f)(h) = gf(h) = ghz = f(gh) = {}_g f(h),$$

so that  $g * f = {}_g f$  for each  $g \in G$ .

It follows from Proposition 3.1 that

$$ga = g \int f = \int g * f = \int {}_g f = \int f = a$$

for any element  $g \in G$ . This completes the proof.  $\square$

#### 4. AN EQUIVARIANT EXTENSION THEOREM

**Theorem 4.1.** *Let  $G$  be a locally compact group,  $Y \in G\text{-}\mathcal{L}$ , and  $X \in G\text{-}\mathcal{P}$ . Then*

- (1)  $Y \in G\text{-AE}(\mathcal{P})$  if  $Y$  is complete;
- (2)  $Y \in G\text{-ANE}(\mathcal{M})$  if  $G$  is a Lie group.

The proof of this theorem is preceded by two lemmas.

**Lemma 4.2.** *Let  $G$  be a locally compact group. Then*

- (1)  $T \in G\text{-AE}(\mathcal{P})$  whenever  $T \in K\text{-AE}(\mathcal{P})$  for every compact subgroup  $K \subset G$ ;  
 (2)  $T \in G\text{-ANE}(\mathcal{M})$  whenever  $T \in K\text{-ANE}(\mathcal{M})$  for every compact subgroup  $K \subset G$ .

*Proof.* The first statement is due to Herbert Abels [1], who, in Theorem 4.4, states the result only for  $G\text{-AE}(\mathcal{M})$ , while the proof is valid also for  $G\text{-AE}(\mathcal{P})$ . The second statement is proved in [4, Theorem 5].  $\square$

**Lemma 4.3.** *Let  $G$  be a compact group and  $Y \in G\text{-}\mathcal{L}$ . If  $Y$  is complete, then  $Y \in G\text{-AE}(\mathcal{P})$ .*

*Proof.* Let  $A \subset X$  be a closed subset of  $X$  and  $f : A \rightarrow Y$  a  $G$ -map. By [10],  $Y$  is an  $\text{AE}(X)$  yielding that there exists a continuous map  $F : X \rightarrow Y$  such that  $F|_A = f$ . Consider now the map  $\Phi : X \rightarrow C(G, Y)$  defined by  $\Phi(x)(g) = g^{-1}F(gx)$ ,  $x \in X$ ,  $g \in G$ . Then  $\Phi$  is continuous (see [9, p. 95]). Finally, we set  $\phi(x) = \int \Phi(x)$ ,  $x \in X$ , where  $\int$  is the map from Proposition 3.1.

We claim that  $\phi$  is the desired map. First,  $\phi$  is continuous being the composition of two continuous maps.

For every  $a \in A$  and  $g \in G$ , we have

$$\Phi(a)(g) = g^{-1}F(ga) = g^{-1}f(ga) = g^{-1}(gf(a)) = f(a).$$

This means that  $\Phi(a) \in C(G, Y)$  is a constant map. By Proposition 3.1, then, we get  $\phi(a) = \int \Phi(a) = f(a)$  which shows that  $\phi|_A = f$ .

It remains to prove that  $\phi$  is equivariant. First, we observe that

$$\begin{aligned} \Phi(hx)(g) &= g^{-1}F(ghx) = h(gh)^{-1}F(ghx) \\ &= h(\Phi(x)(gh)) = (h * \Phi(x))(gh), \end{aligned}$$

for every  $h, g \in G$  and  $x \in X$ . Therefore,  $\Phi(hx) = (h * \Phi(x))_h$ . Finally, by Proposition 3.1, we have

$$\phi(hx) = \int \Phi(hx) = \int (h * \Phi(x))_h = \int h * \Phi(x) = h \int \Phi(x) = h\phi(x).$$

This proves that  $\phi$  is equivariant and completes the proof.  $\square$

*Proof of Theorem 4.1.* (1) is a combination of lemmas 4.2(1) and 4.3.

(2) By [3, Theorem 1],  $Y$  is a  $K\text{-ANE}(\mathcal{M})$  for every compact subgroup  $K$  of  $G$ . Then the result follows from Lemma 4.2(2).  $\square$

5. EQUIVARIANT  $\varepsilon$ -NEAR SELECTIONS

**Definition 5.1** ([8]). Let  $(Y, d)$  be a metric space,  $F : X \rightrightarrows Y$  a multi-valued map, and  $\varepsilon > 0$ . A continuous map  $f : X \rightarrow Y$  is called an  $\varepsilon$ -near selection of  $F$  if, for every  $x \in X$ ,

$$d(f(x), F(x)) = \inf_{y \in F(x)} d(f(x), y) < \varepsilon.$$

**Definition 5.2.** Let  $G$  be a topological group, let  $Y$  be a convex metrizable invariant subset of a linear  $G$ -space, and let  $X$  be an arbitrary  $G$ -space. We say that  $Y$  has the  $G$ -near selection property with respect to  $X$ , denoted  $Y \in G\text{-NSP}(X)$ , if every l.s.c.  $G$ -map  $F : X \rightrightarrows Y$  with complete convex values has, for every  $\varepsilon > 0$ , an equivariant  $\varepsilon$ -near selection.

**Theorem 5.3.** Let  $G$  be a locally compact Hausdorff group,  $(Y, d) \in G\text{-}\mathcal{L}$ , and  $X \in G\text{-}\mathcal{P}$ . If  $Y \in G\text{-ANE}(X)$ , then  $Y \in G\text{-NSP}(X)$ .

Before proving Theorem 5.3, let us establish the following lemma which is an equivariant version of [8, Lemma 3.2].

**Lemma 5.4.** Let  $G$  be a locally compact Hausdorff group, let  $\delta > 0$ , and let  $X$  and  $Y$  be  $G$ -spaces. Suppose that  $Y$  is metrizable by a  $G$ -invariant metric  $d$  and  $\phi : X \rightrightarrows Y$  an l.s.c. equivariant map. Assume that  $X_0$  is an invariant subset of  $X$  for which there exists a continuous equivariant map  $f : X \rightarrow Y$  such that the restriction  $f|_{X_0}$  is an equivariant  $\delta$ -near selection for  $\phi|_{X_0}$ . Then, for every  $\varepsilon > 0$ , there is an invariant neighborhood  $U_\varepsilon$  of  $X_0$  such that  $f|_{U_\varepsilon}$  is an equivariant  $\delta + \varepsilon$ -near selection.

*Proof.* Because  $d$  is invariant and  $\phi$  and  $f$  are equivariant, it is easy to see that the set

$$U_\varepsilon = \bigcup_{x \in X_0} f^{-1}(B(f(x_0, \varepsilon/2)) \cap \phi^{\leftarrow}(B(f(x), \delta + \varepsilon/2)))$$

is an invariant neighborhood of  $X_0$ . By [8, Lemma 3.2], the restriction  $f|_{U_\varepsilon}$  is a  $\delta + \varepsilon$ -near selection. The equivariance of  $f|_{U_\varepsilon}$  follows from the hypotheses.  $\square$

*Proof of Theorem 5.3.* Let  $\phi : X \rightrightarrows Y$  be an l.s.c. equivariant map with complete convex values, and let  $\varepsilon > 0$ . For each  $x \in X$ , the stabilizer  $G_x$  is compact ([12, Proposition 1.1.4]). In addition, since  $\phi$  is equivariant, we have

$$\phi(x) = \phi(gx) = g\phi(x) \text{ for all } g \in G_x.$$

So, compact group  $G_x$  acts continuously and linearly on the complete convex subset  $\phi(x)$  of  $Y$ . By Corollary 3.2, there is a  $G_x$ -fixed point  $a_x \in \phi(x)$ .



Now, we define a map  $\mu_x : G(x) \rightarrow G(a_x) \subset Y$  by  $\mu_x(ga_x) = ga_x$ ,  $g \in G$ .

It is not difficult to see that  $\mu_x$  is an equivariant selection for the restriction  $\phi|_{G(x)}$ .

Using the property  $Y \in G\text{-ANE}(X)$ , we can extend the map  $\mu_x$  to a continuous equivariant map  $F_x$  defined on an invariant neighborhood  $W_x$  of  $G(x)$ .

By Lemma 5.4, there exists an invariant neighborhood  $U_x \subset W_x$  of  $G(x)$  such that  $F_x|_{U_x}$  is an equivariant  $\varepsilon$ -near selection for the restriction  $\phi|_{U_x}$ .

Varying  $x \in X$ , we obtain an open invariant cover  $\{U_x\}_{x \in X}$  of  $X$ . Since  $X \in G\mathcal{P}$ , by Lemma 2.1, there exists a locally finite open invariant refinement  $\{O_\alpha\}_{\alpha \in \mathcal{A}}$  of  $\{U_x\}_{x \in X}$ . For each  $\alpha \in \mathcal{A}$ , pick an  $x(\alpha) \in X$  such that  $O_\alpha \subset U_{x(\alpha)}$ . Now, for each  $\alpha \in \mathcal{A}$ , we extend the map  $F_{x(\alpha)}|_{O_\alpha}$  to a map  $\tilde{F}_\alpha : X \rightarrow Y$  as follows:

$$\tilde{F}_\alpha(z) = \begin{cases} F_{x(\alpha)}(z), & \text{if } z \in O_\alpha, \\ y_0, & \text{if } z \in X \setminus O_\alpha, \end{cases}$$

where  $y_0$  is an arbitrary point in  $Y$ . By Lemma 2.2, there exists an invariant partition of unity  $\{p_\alpha\}_{\alpha \in \mathcal{A}}$  subordinated to  $\{O_\alpha\}_{\alpha \in \mathcal{A}}$ . Hence, for every  $\alpha \in \mathcal{A}$ ,

$$\overline{p_\alpha^{-1}((0, 1])} \subset O_\alpha \subset U_{x(\alpha)}.$$

The desired  $\varepsilon$ -near selection  $f : X \rightarrow Y$  can now be defined by the formula

$$f(x) = \sum_{\alpha \in \mathcal{A}} p_\alpha(x) \tilde{F}_\alpha(x), \quad x \in X.$$

For every  $z \in X$ , let  $Q(z)$  be the subset of  $\mathcal{A}$  consisting of all  $\alpha \in \mathcal{A}$  such that  $z \in p_\alpha^{-1}((0, 1])$ . Similarly, define  $Q'(z)$  as the subset of  $\mathcal{A}$  consisting of all  $\alpha \in \mathcal{A}$  such that  $z \in \overline{p_\alpha^{-1}((0, 1])}$ . If  $z \in X$ , it is clear that  $Q(z) \subset Q'(z)$  and both sets are finite. Then, for every  $z \in X$ , we write

$$(5.1) \quad f(z) = \sum_{\alpha \in Q(z)} p_\alpha(z) \tilde{F}_\alpha(z) = \sum_{\alpha \in Q'(z)} p_\alpha(z) \tilde{F}_\alpha(z).$$

To see the continuity of  $f$ , let us fix an arbitrary point  $x_0 \in X$ . Define

$$V = \bigcap_{\alpha \in Q'(x_0)} p_\alpha^{-1}((0, 1]) \setminus \bigcup_{\alpha \notin Q'(x_0)} \overline{p_\alpha^{-1}((0, 1])}.$$

Since the family  $\{p_\alpha^{-1}((0, 1])\}_{\alpha \in \mathcal{A}}$  is locally finite, the union

$$\bigcup_{\alpha \notin Q'(x_0)} \overline{p_\alpha^{-1}((0, 1])}$$

is closed, and therefore,  $V$  is a neighborhood of  $x_0$ . It is evident that for every  $z \in V$ , the set  $Q(z)$  is contained in  $Q'(x_0)$ . Using equality (5.1), we infer that

$$f(z) = \sum_{\alpha \in Q'(x_0)} p_\alpha(z) \tilde{F}_{x(\alpha)}(z), \quad z \in V.$$

However, for every  $\alpha \in Q'(x_0)$ , the restriction  $\tilde{F}_\alpha|_V$  is equal to  $F_{x(\alpha)}|_V$ . Indeed, if  $z \in V$  and  $\alpha \in Q'(x_0)$ , then  $z \in p_\alpha^{-1}(0, 1] \subset O_\alpha$ , which means that  $\tilde{F}_\alpha(z) = F_{x(\alpha)}(z)$ . So, for every  $z \in V$ , we can write  $f(z)$  as follows:

$$(5.2) \quad f(z) = \sum_{\alpha \in Q'(x_0)} p_\alpha(z) F_{x(\alpha)}(z).$$

Thus,  $f|_V$  is a finite sum of continuous maps and, consequently, it is also continuous.

Since  $O_\alpha$  is an invariant subset, we have  $gz \in O_\alpha$  if and only if  $z \in O_\alpha$ . As a consequence, we have that  $Q(z) = Q(gz)$  for every  $z \in X$  and for all  $g \in G$ . Now, using the linearity of the action, we get

$$\begin{aligned} f(gz) &= \sum_{\alpha \in Q(gz)} p_\alpha(gz) F_{x(\alpha)}(gz) = \sum_{\alpha \in Q(z)} p_\alpha(z) g F_{x(\alpha)}(z) \\ &= g \left( \sum_{\alpha \in Q(z)} p_\alpha(z) F_{x(\alpha)}(z) \right) = g f(z). \end{aligned}$$

This proves that  $f$  is equivariant.

We have still to prove that  $f$  is an  $\varepsilon$ -near selection for  $\phi(x)$ . For this purpose, we recall that for every  $z \in X$  and for every  $\alpha \in Q(z)$ , the point  $F_{x(\alpha)}(z)$  belongs to the convex set  $N_\varepsilon(\phi(z)) = \{y \in Y \mid d(y, \phi(z)) < \varepsilon\}$ . So,  $f(z)$  is a convex combination of finitely many  $F_{x(\alpha)}(z)$ 's, each of them lying in the convex set  $N_\varepsilon(\phi(z))$ . This yields that  $f(z) \in N_\varepsilon(\phi(z))$  and completes the proof.  $\square$

Theorem 5.3 and Theorem 4.1 immediately imply the following.

**Corollary 5.5.** *Let  $G$  be a locally compact group, let  $Y \in G\text{-}\mathcal{L}$ , and let  $X \in G\text{-}\mathcal{P}$ . Then  $Y \in G\text{-NSP}(X)$  in each of the following cases:*

- (a)  $Y$  is complete;
- (b)  $G$  is a Lie group and  $X$  is metrizable.

## 6. EQUIVARIANT SELECTIONS

Analogous to the definition of the  $G$ -near selection property, we can define the  $G$ -selection property.

**Definition 6.1.** Let  $G$  be a topological group,  $X$  a  $G$ -space, and  $Y$  a convex metrizable invariant subset of a linear  $G$ -space. We say that  $Y$

has the  $G$ -selection property with respect to  $X$ , denoted  $Y \in G\text{-SP}(X)$ , if every l.s.c. equivariant map  $\phi : X \rightrightarrows Y$  with complete convex values admits an equivariant selection.

**Proposition 6.2.** *Let  $G$  be a locally compact Hausdorff group and  $(Y, d) \in G\text{-}\mathcal{L}$ . If  $Y \in G\text{-NSP}(X)$  for some  $G$ -space  $X$ , then  $Y \in G\text{-SP}(X)$ .*

*Proof.* Let  $\phi : X \rightrightarrows Y$  be an l.s.c. equivariant map with complete convex values. We will construct, by induction, a sequence of equivariant maps  $f_n : X \rightarrow Y$  satisfying, for every  $x \in X$ , the following two properties:

- (a)  $d(f_n(x), f_{n+1}(x)) < 2^{-(n-1)}$ ,  $n = 1, 2, \dots$  and
- (b)  $d(f_n(x), \phi(x)) < 2^{-n}$ ,  $n = 1, 2, \dots$

Since  $Y \in G\text{-NSP}(X)$ , there exists an equivariant 1/2-near selection  $f_1 : X \rightarrow Y$ . This map satisfies (b). Suppose that the equivariant maps  $f_1, \dots, f_n$  have been constructed which satisfy (a) and (b). In order to construct the equivariant map  $f_{n+1}$ , let us define a multivalued equivariant map  $\phi_n : X \rightrightarrows Y$  as follows:

$$\phi_n(x) = \overline{\phi(x) \cap B(f_n(x), 2^{-n})}, \quad x \in X.$$

By [13, Lemma 1.4.6],  $\phi_n$  is an l.s.c. map. In addition, for each  $x \in X$ ,  $\phi_n(x)$  is a closed subset of the complete set  $\phi(x)$ , so  $\phi_n(x)$  is itself complete. Since the balls defined by the metric  $d$  are convex, and since  $\phi(x)$  is also convex, we infer that  $\phi_n(x)$  is a convex subset of  $Y$ .

Finally, the invariance of the metric  $d$  and the equivariance of the map  $f_n$  imply that

$$\begin{aligned} g\phi_n(x) &= g(\overline{\phi(x) \cap B(f_n(x), 2^{-n})}) = \overline{g\phi(x) \cap gB(f_n(x), 2^{-n})} \\ &= \overline{\phi(gx) \cap B(f_n(gx), 2^{-n})} = \phi_n(gx), \end{aligned}$$

showing that  $\phi_n$  is equivariant.

Now apply the fact that  $Y \in G\text{-NSP}(X)$  to find an equivariant  $2^{-(n+1)}$ -near selection for  $\phi_n$ , say  $f_{n+1} : X \rightarrow Y$ . Since  $\phi_n(x) \subset \phi(x)$ , we have that

$$d(f_{n+1}(x), \phi(x)) \leq d(f_{n+1}(x), \phi_n(x)) < 2^{-(n+1)}.$$

Thus,  $f_{n+1}$  satisfies property (b).

On the other hand,  $\phi_n(x) \subset \overline{B(f_n(x), 2^{-n})}$ . Then

$$\begin{aligned} d(f_{n+1}(x), f_n(x)) &\leq d(f_{n+1}(x), \phi_n(x)) + d(\phi_n(x), f_n(x)) \\ &< 2^{-(n+1)} + 2^{-n} < 2^{-n+1}, \end{aligned}$$

which verifies (a). This completes the inductive step.

We claim that for every  $x \in X$ , the limit  $\lim_{n \rightarrow \infty} f_n(x)$  exists and belongs to  $\phi(x)$ . In order to see this, take  $x \in X$  arbitrary. By (b), for every

$n \in \mathbb{N}$ , there exists a point  $a_n \in \phi(x)$  such that  $d(f_n(x), a_n) < 2^{-n}$ . Let us consider the sequence  $(a_n)_{n \in \mathbb{N}} \subset \phi(x)$ . By (a), we have

$$\begin{aligned} d(a_n, a_{n+1}) &\leq d(a_n, f_n(x)) + d(f_n(x), f_{n+1}(x)) \\ &\quad + d(f_{n+1}(x), a_{n+1}) < 2^{-(n-2)}. \end{aligned}$$

Therefore,  $(a_n)_{n \in \mathbb{N}}$  is a Cauchy sequence contained in the complete subset  $\phi(x)$ . We conclude that  $\lim_{n \rightarrow \infty} a_n$  exists and belongs to  $\phi(x)$ . Since  $d(f_n(x), a_n) < 2^{-n}$  for every  $n$ , we infer that  $\lim_{n \rightarrow \infty} f_n(x)$  also exists and is equal to  $\lim_{n \rightarrow \infty} a_n$ . For every  $x \in X$ , define  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ . This yields that  $f(x) \in \phi(x)$ . By (a), the sequence  $(f_n)_{n \in \mathbb{N}}$  is Cauchy and thus converges uniformly to  $f$ . This implies that  $f$  is continuous.

Finally, for every  $g \in G$  and  $x \in X$ , we have

$$f(gx) = \lim_{n \rightarrow \infty} f_n(gx) = \lim_{n \rightarrow \infty} g f_n(x) = g \left( \lim_{n \rightarrow \infty} f_n(x) \right) = g f(x).$$

This proves that  $f$  is an equivariant selection for  $\phi$ . Thus,  $Y \in G\text{-SP}(X)$ , as required.  $\square$

Proposition 6.2, in combination with Corollary 5.5, yields the following result.

**Corollary 6.3.** *Let  $G$  be a locally compact Hausdorff group,  $Y \in G\text{-}\mathcal{L}$ , and  $X \in G\text{-}\mathcal{P}$ . Then  $Y \in G\text{-SP}(X)$  in each of the following cases:*

- (a)  $Y$  is complete (in particular,  $Y$  is a Banach  $G$ -space).
- (b)  $G$  is a Lie group and  $X$  is metrizable.

**Remark 6.4.** Theorem 5.3, Proposition 6.2, and corollaries 5.5 and 6.3 remain valid if, instead of assuming that  $(Y, d) \in G\text{-}\mathcal{L}$ , we assume that  $(Y, d)$  is an invariant metric convex subset of a locally convex linear  $G$ -space  $L$  satisfying the following conditions:

- (1)  $(Y, d) \in G\text{-}\mathcal{M}$ ;
- (2)  $d(x + z, y + z) = d(x, y)$  for all  $x, y \in Y$  and  $z \in L$  as soon as  $x + y$  and  $x + z$  belong to  $Y$ ;
- (3) all open balls determined by  $d$  are convex sets.

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