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KLEIN BOTTLES AND MULTIDIMENSIONAL FIXED POINT SETS

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Abstract. We show that for any action of a finite p-group on a torus, all components of the fixed point set have the same dimension. In contrast, we give two systematic constructions of families of group actions on other aspherical manifolds whose fixed point sets contain components of differing dimensions.

1. Introduction

The starting point for this note is the observation that a linear action of a finite group on a torus has the property that all components of the fixed point set have the same dimension.

This led to a search for a topological reason for this phenomenon and we show that, for any action of a finite p-group on a torus, all components of the fixed point set have the same dimension. At its heart the reason is that the torus is aspherical and has abelian fundamental group.

The form of the result then suggested the possibility of actions on other aspherical manifolds, with nonabelian fundamental group, for which the fixed point set contains components of differing dimensions.

The most well-known example of such an action is given by an involution on the Klein bottle with fixed point set consisting of two isolated points and a circle. It is somewhat more difficult to find similar orientation-preserving actions and actions of larger periods.

We generalize the construction of the Klein bottle to produce a variety of orientation-preserving actions on orientable aspherical manifolds with fixed point set containing components of differing dimensions. Such examples arise only beginning in dimension 4. We give two different families

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of such examples. They all occur on manifolds that are virtually fibered, that is on manifolds that are finite quotients of a direct product of lower dimensional manifolds. In that sense we view them as generalized Klein bottles.

We give a second construction of such actions on aspherical manifolds with multidimensional fixed point sets via a process of equivariant hyperbolization. These exist in abundance but have a somewhat artificial feel to them. While the generalized Klein bottles are perhaps too standard, these hyperbolized examples are perhaps too exotic.

It appears to be an open problem to produce orientation-preserving actions of a finite cyclic group of prime order on a closed orientable hyperbolic manifold such that the fixed point set contains components of differing dimensions.

2. Linear Actions on the Torus

One can examine explicit linear actions on a torus by bare-hands calculation of their fixed point sets. But the simplest approach to the problem of dimension seems to be through a more abstract result.

Proposition 2.0.1. *Let a group G act on a Lie group L via a homomorphism $G \to Aut\, L$. Let $x \in L^G$. Then left translation $\lambda_x : L \to L$ is a G-homeomorphism taking the identity $e \in L$ to x.*

The proof is straightforward and may be left to the reader.

Corollary 2.0.2. *If a finite group G acts on a torus T^n via a homomorphism $G \to GL(n, \mathbb{Z})$, then all components of $Fix(G, T^n)$ have the same dimension.*

In the next section we sketch a proof of a topological analog for p-group actions on the torus. Crucial ingredients are Smith theory and the fact that $\pi_1(T^n)$ is abelian.

3. Group Actions on Aspherical Manifolds

We begin with a summary of some generalities about lifting a group action through a covering map. See [3, pp. 230–234] for a fuller discussion. For simplicity, we assume our spaces are manifolds. The first results are immediate from elementary covering space theory.

3.1. Extending a group action to the universal covering.

Proposition 3.1.1 (Fundamental Extension)**.** *If G acts on connected manifold X, then there is an extension*

$$1 \to \pi \to E \to G \to 1$$

where π is the group of deck transformations of the universal covering $\widetilde{X}$ and E is the group of all lifts of all elements of G to $\widetilde{X}$.

3.2. Lifting a group action to a covering space.

Suppose G acts on a connected manifold X with universal covering $\mathrm{pr} : \widetilde{X} \to X$. Let $x \in F = \mathrm{Fix}(G, X)$. Then the action of G on X induces an action on $\pi_1(X, x)$.

Proposition 3.2.1 (Lifting Criterion)**.** *For any $y \in pr^{-1}(x)$ there is a unique action of G on the universal covering $\widetilde{X}$ fixing y such that $pr : \widetilde{X} \to X$ is a G-map.*

Corollary 3.2.2 (Splitting Criterion)**.** *The choice of $y \in pr^{-1}(x)$ determines a splitting of the fundamental extension $1 \to \pi \to E \to G \to 1$.*

For $x \in F$, let F_x denote the path component of x in F. Lift the group action to the universal covering by choosing a fixed point y over x, as above. Let $\mathrm{Fix}(G, \widetilde{X})_y$ denote the path component corresponding to y in $\mathrm{Fix}(G, \widetilde{X})$.

Proposition 3.2.3. *The restriction of the given covering map $pr : \widetilde{X} \to X$ to $Fix(G, \widetilde{X})_y \to F_x$ is a regular covering map with deck transformation group identified with $Im[\pi_1(F_x, x) \to \pi_1(X, x)] \subset \pi_1(X, x)^G$.*

Proof sketch. A deck transformation δ preserving $\mathrm{Fix}(G, \widetilde{X})$ corresponds to the class $[\mathrm{pr}(\lambda)]$ in $\pi_1(X, x)$, where λ is a path in $\widetilde{X}$ from y to $\delta(y)$. If $\delta(y)$ lies in $\mathrm{Fix}(G, \widetilde{X})_y$, then the path λ can be chosen to lie in $\mathrm{Fix}(G, \widetilde{X})_y$ as well. Further details of the verification may be left to the reader. □

Remark 3.2.4. If $\mathrm{Fix}(G, \widetilde{X})$ is path-connected then the path λ above can be chosen within $\mathrm{Fix}(G, \widetilde{X})$. It follows that $\pi_1(X, x)^G = \mathrm{Im}[\pi_1(F_x, x) \to \pi_1(X, x)]$, a fact we shall use below.

3.3. Application of basic Smith theory.

Recall that a manifold is said to be *aspherical* if its universal covering space is contractible. Smith theory gives restrictions on actions of p-groups on contractible or acyclic spaces.

We begin by recalling the main global and local statement about p-group actions on contractible manifolds. For simplicity we assume here that the group actions under consideration are locally linear. Alternatively, one must deal with Čech cohomology and possible local pathology. See, for example, the analysis in [4].

Theorem 3.3.1 (Existence and properties of fixed points)**.** *Suppose G is a finite p-group and acts on a $\mathbb{Z}_p$-acyclic n-manifold Y.*

(1) *$F = Fix(G, Y)$ is mod p acyclic; in particular, F is nonempty and connected, and*
(2) *F is a (cohomology) manifold of even codimension (if the action of G is orientation-preserving).*

See [1, Chapter 3] for a detailed treatment of Smith theory.

Next, we apply basic Smith theory to a lifted action.

Proposition 3.3.2. *Let a finite p-group G act on an aspherical manifold X with nonempty fixed point set F. For $x \in F$, let F_x denote the path component of x in F, and let the group action be lifted to the universal covering by choosing a fixed point over x. Then $Fix(G, \widetilde{X})$ is $\mathbb{Z}_p$-acyclic.*

Because the domain of the covering map $\mathrm{Fix}(G, \widetilde{X}) \to F_x$ is $\mathbb{Z}_p$-acyclic with deck transformation group $\pi_1(X, x)^G$, it follows that

$$H^*(\pi_1(X, x)^G; \mathbb{Z}_p) = H^*(F_x; \mathbb{Z}_p).$$

And since F_x is a compact (cohomology) n-manifold without boundary for some n, we have that $H^n(\pi_1(X, x)^G; \mathbb{Z}_p) \neq 0$ and $H^k(\pi_1(X, x)^G; \mathbb{Z}_p) = 0$ for $k > n$. Therefore, the dimension of F_x is given by $\mathrm{cd}_{\mathbb{Z}_p}(\pi_1(X, x)^G)$, the mod p cohomological dimension of $\pi_1(X, x)^G$. (See [2] for basic information about the cohomological dimension of discrete groups.) We state this interpretation more formally as follows.

Theorem 3.3.3 (Dimension Formula)**.** *If G is a finite p-group acting on a closed, orientable, aspherical manifold X and $x \in Fix(G, X)$, then we have a $\mathbb{Z}_p$-acyclic covering $\tilde{F} \to F_x$ with deck transformation group $\pi_1(X, x)^G$; hence, $\dim(F_x) = cd_{\mathbb{Z}_p}(\pi_1(X, x)^G)$.*

Corollary 3.3.4. *If a finite p-group G acts topologically on a torus T^n, then all (nonempty) components of the fixed point set have the same dimension, given by $Rank\, H^1(T^n)^G$.*

Proof. This follows from the Dimension Formula and the universal coefficient theorem. □

4. The Klein Bottle

We view the Klein bottle as the quotient space $K^2 = S^1 \times_{C_2} S^1$, where the generator of C_2 acts freely, reversing orientation, on the torus $S^1 \times S^1$ by the formula

$$(z, w) \mapsto (\bar{z}, -w),$$

where z and w are complex numbers of unit modulus.

There is an induced action of C_2 on K^2 given by

$$[z, w] \mapsto [\bar{z}, \bar{w}].$$

It is easy to determine the fixed point set consists of two isolated points and a simple closed curve.

What makes this example work is that there are two commuting actions of C_2 on the circle, one of which has isolated fixed points and the other of which acts freely.

In our first families of higher dimensional examples, we produce spaces X admitting two commuting actions of a group C_p such that one acts with isolated fixed points and the other acts freely. These yield our generalized Klein bottles.

5. Generalized Klein Bottles

Suppose that $G = C_p$ acts in two ways on a space X. Choosing a generator of G, we denote one action by $x \mapsto a(x)$ and the other by $x \mapsto b(x)$. Suppose $b(x) \neq x$ for all $x \in X$ and suppose that $ab = ba$. We may interpret this as saying that a and b generate an action of the larger group $C_p \times C_p$, which in practice will be effective.

Let Y be another space on which G acts with fixed points, with the generator acting by $y \mapsto g(y)$. Then G acts freely on $Y \times X$ via the diagonal action $g \times b$. We now define the associated generalized Klein bottle to be the quotient space $Z = Y \times_G X$ by this action, with the action of G induced by $g \times a$. We write $[y, x] \mapsto [g(y), a(x)]$.

In this case the fixed point set has a description as the following disjoint union:

$$\mathrm{Fix}(G, Z) = \bigsqcup_{i=0}^{p-1} \{[y, x] : gy = g^i y, ax = b^i x\}.$$

Each of these p pieces is the image of the corresponding $(g \times a)$-invariant subspace of $Y \times X$.

The part corresponding to $i = 0$ can be described as

$$Y^g \times_G X^a.$$

This is the part of the fixed point set that comes from fixed points in $Y \times X$.

The part when $i = 1$ is of the form

$$Y \times_G X^{a^{-1}b}$$

since in these cases, the equation $gy = gy$ implies no condition on y and $ax = bx$ is equivalent to the condition $a^{-1}bx = x$.

When $2 \leq i \leq p-1$, the corresponding part of the fixed point set is of the form similar to that when $i = 0$,

$$Y^g \times_G X^{a^{-1}b^i},$$

since the condition $gy = g^i y$ is equivalent to the condition $gy = y$ in this case, because p is prime.

In the simplest cases we use $Y = X$ with the action $g = a$. The usefulness of the slightly more general construction is that it allows for iterating the construction.

5.1. Application I.

Here we produce examples of actions of C_p, for all prime periods p, on $2n$-manifolds, where $n = p - 1$, by making use of commuting actions on a torus. In these examples n grows with p.

Theorem 5.1.1. *For any prime p there is a smooth action of the group C_p on a flat manifold M^{2n}, where $n = p-1$, with fixed point set consisting of $p(p-1)$ isolated points, together with a copy of the n-torus T^n.*

Proof. Fix a prime p and set $n = p-1$. Then there is an action of $C_p \times C_p$ on the n-torus T^n, which, using a convenient model of the torus suggested by the referee, we describe as follows.

Let V be the $\mathbb{R}$-algebra $\mathbb{R}[t]/I$, where I is the principal ideal $\Phi\mathbb{R}[t]$ where $\Phi = 1 + t + t^2 + \cdots + t^{p-1}$. Let ζ denote the class of t in V. Let L denote the lattice $\mathbb{Z}[\zeta] \subset V$ and identify $T^n = V/L$.

Now define mappings $a : V \to V$ and $b : V \to V$ by

$$a(v) = \zeta v \text{ and } b(v) = v + v_0,$$

where

$$v_0 := (\zeta - 1)^{-1} = \frac{1}{p}(\zeta + 2\zeta^2 + 3\zeta^3 + \cdots (p-1)\zeta^{p-1}).$$

Then one may check that a and b preserve the lattice L, and that for any $v \in V$ one has $ab(v) - ba(v) = 1 \in L$. It follows that a and b induce commuting homeomorphisms $\alpha, \beta : T^n \to T^n$, and that α is a group isomorphism of T^n, while β is a translation of T^n, each of order p.

Of course, $\mathrm{Fix}(\beta, T^n) = \emptyset$. We will observe that $\mathrm{Fix}(\alpha, T^n)$ is a subgroup of T^n of order p. And $\mathrm{Fix}(\beta^{-k}\alpha, T^n)$ is a translate of $\mathrm{Fix}(\alpha, T^n)$.

We may compute directly that

$$\begin{aligned}\mathrm{Fix}(\alpha, T^n) &= \{[v] \in T^n : (\zeta - 1)v \in L\} = (\zeta - 1)^{-1}L/L \\ &\cong L/(\zeta - 1)L \approx \mathbb{Z}[t]/\langle t - 1, \Phi\rangle = \mathbb{Z}[t]/\langle t - 1, p\rangle \\ &\cong \mathbb{Z}/p\mathbb{Z}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Fix}(\beta^{-k}\alpha, T^n) &= \{[v] \in T^n : (\zeta - 1)v - kv_0 \in L\} \\ &= \{[v] \in T^n : v - kv_0^2 \in (\zeta - 1)^{-1}L\} \\ &= k[v_0^2] + \mathrm{Fix}(\alpha, T^n)\end{aligned}$$

In particular, $|\,\mathrm{Fix}(\beta^{-k}\alpha, T^n)| = |\,\mathrm{Fix}(\alpha, T^n)| = p$ for each integer k.

Now C_p acts freely on $T^n \times T^n$ by $\alpha \times \beta$ and we denote the quotient manifold by $M^{2n} = T^n \times_{C_p} T^n$. Then the C_p action on $T^n \times T^n$ given by $\alpha \times \alpha$ induces an action of C_p on M^{2n} by the formula

$$\gamma([v, w]) = [\alpha(v), \alpha(w)].$$

Now $\alpha \times \alpha$ has p^2 isolated fixed points that project to p fixed points in M^{2n}. In addition, there are fixed points in M^{2n} that do not come from such upstairs fixed points. The latter would be the images of points (v, w) such that

$$(\alpha(v), \alpha(w)) = (\alpha^i(v), \beta^i(w)).$$

For $i = 1$, this gives a copy of T^n. For each $i = 2, \ldots, p - 1$, it gives another p isolated points. □

Remark 5.1.2. By iterating this construction, one obtains actions of C_p on manifolds of the form $(\cdots((T^n \times_{C_p} T^n) \times_{C_p} T^n) \times \cdots) \times_{C_p} T^n$ (k factors), where $n = p - 1$ and $k \in \{2, 3, 4, \ldots\}$, having fixed point set containing components of all dimensions $0, n, 2n, \ldots, (k - 1)n$.

Remark 5.1.3. The actions in this section may be viewed as acting by isometries on geometric manifolds whose universal covering is of the form $\mathbb{R}^{kn} = \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n$, where $\mathbb{R}^n$ denotes the standard Euclidean n-space.

5.2. Application II.

Here we produce a rather more flexible family of actions with multidimensional fixed point sets. We make use of a certain action of $C_p \times C_p$ on a 2-dimensional surface.

Lemma 5.2.1. *There is an action of $C_p \times C_p$ on a 2-dimensional surface X^2 with the property that a chosen generator α of the first factor acts with a nonempty set of isolated fixed points and a chosen generator β of the second factor acts freely.*

The smallest examples have orbit surface of genus one and two fixed points.

Proof. We start with a surface W^2 of genus at least one that will be the orbit space of the action. Choose points $w_1, w_2, \dots w_m$ in W^2, $m \geq 2$ (and m even if $p = 2$). Choose a base point $w_0 \in W^2 - \{w_1, w_2, \dots w_m\}$. Let $a_1, b_1, \dots a_h, b_h$ be standard generators of $\pi_1(W^2, w_0)$ and let $c_1, c_2, \dots c_m$ denote generators corresponding to loops going once around the respective punctures, all connected appropriately to a base point w_0, so that

$$[a_1, b_1][a_2, b-2] \cdots [a_h, b_h]\, c_1 c_2 \cdots c_m = 1.$$

Define a surjective homomorphism

$$\pi_1(W^2 - \{w_1, w_2, \dots w_m\}; w_0) \to C_p \times C_p$$

by mapping each $c_i \mapsto \alpha^{j_i}$ to a nontrivial power j_i of α, $0 < j_i < p$ such that the total exponent sum satisfies the condition $\sum_i j_i \equiv 0 \mod p$. Map $a_1 \mapsto \beta$, say, and map all other generators trivially.

The corresponding p^2-fold regular branched cover has the required property. □

For use below we note that the equation $\alpha(x) = \beta^i(x)$, for $1 \leq i \leq p-1$, has a nonempty finite set of solutions. One can see this by considering how the homomorphism

$$\pi_1(W^2 - \{w_1, w_2, \dots w_m\}; w_0) \to C_p \times C_p$$

corresponding to $\beta^{-i}\alpha$ evaluates nontrivially on the generators c_i.

Theorem 5.2.2. *Suppose the finite cyclic group C_p acts on Y^n with fixed point set F. Then C_p acts on $Y^n \times_{C_p} X^2$ with fixed point set consisting of copies of F together with copies of Y^n.*

Proof. Let $\gamma : Y^n \to Y^n$ denote the action of a generator of C_p. We view C_p as acting (freely) on $Y^n \times X^2$ by $(y, x) \mapsto (\gamma(y), \beta(x))$ with quotient $Z^{n+2} = Y^n \times_{C_p} X^2$. Then let C_p act on Z^{n+2} by $[y, x] \mapsto [\gamma(y), \alpha(x)]$.

As in earlier calculations, we see that the fixed point set of C_p on Z^{n+2} contains fixed points $[y, x]$ where $\gamma(y) = y$ and $\alpha(x) = x$, which yield copies of F. In addition, it contains fixed points $[y, x]$ where $\alpha(x) = \beta^i(x)$, $1 \leq i \leq p - 1$, which yield copies of Y^n.

□

Corollary 5.2.3. *The finite cyclic group C_p acts on an orientable aspherical 4-manifold with fixed point set containing both isolated points and surfaces.*

Proof. Let $Z = X^2 \times_{C_p} X^2$, where X^2 is the surface with action of $C_p \times C_p$ described above. □

Iterating the construction one obtains the following.

Corollary 5.2.4. *The finite cyclic group C_p acts on an orientable aspherical $2k$-manifold, with fixed point set containing components of dimensions $0, 2, \dots, 2k-2$ of k different dimensions.*

Remark 5.2.5. The actions in this section may be viewed as acting by isometries on geometric manifolds whose universal covering is of the form $\mathbb{H}^2 \times \mathbb{H}^2 \times \cdots \times \mathbb{H}^2$, where $\mathbb{H}^2$ denotes the hyperbolic plane.

6. Hyperbolization

Another way of producing C_p actions on aspherical manifolds with multidimensional fixed point sets is through the process of hyperbolization due originally to Gromov. We briefly outline this approach which produces orientation-preserving actions with multidimensional fixed point sets for any C_p in any dimension greater than 3. The manifolds constructed this way tend to be a bit more exotic, having universal coverings that are not usually simply connected at infinity.

6.1. Hyperbolization.

We refer the reader to [5] and [6] for more details. A hyperbolization functor $\mathcal{H}$ assigns to each simplicial or cubical complex (perhaps suitably subdivided) an aspherical complex built inductively by replacing all n-cells by aspherical n-manifold building blocks (with π_1-injective boundaries) in the same pattern.

Vertices and edges do not need to be changed. One might choose, for example, $\mathcal{H}(D^2)$ = once-punctured torus (or Möbius band, etc.). Higher-dimensional asphericalized n-cells are built inductively.

The details of the construction need not concern us here. Moreover, there are several variations on the process. While the construction had its origin in considerations of negative curvature, those are not a concern here.

It suffices to say that there is an hyperbolization functor $\mathcal{H}$ such that for any (triangulated or cubulated) n-manifold X, the space $\mathcal{H}(X)$ is an aspherical n-manifold.

Although one does not explicitly need it here, one can arrange the hyperbolization to have the following additional properties. There is a map $\mathcal{H}(X) \to X$ that is inductively defined, mapping each hyperbolized cell of $\mathcal{H}(X)$ to the corresponding ordinary cell of X. If X is orientable, then one can arrange that $\mathcal{H}(X)$ is also orientable. With sufficient care, one can arrange that this map is surjective on homology and π_1 and, indeed, is a degree 1 normal map in the sense of surgery theory.

One key point is that the local structure of a neighborhood of a point in $\mathcal{H}(X)$ is the same as that of the corresponding point of X. This is what ensures that the result of hyperbolization of a manifold is again a manifold.

6.2. Equivariant hyperbolization.

Here are a couple of immediate consequences.

We consider a PL group action on a PL n-manifold. We understand this to mean that there is an equivariant triangulation or cubulation of the PL manifold. Such a triangulation or cubulation can be assumed to have the property that for any cell and any group element, the group element either moves the open cell off of itself or fixes it pointwise. This can always be achieved by subdividing a cell structure preserved by the group action. If we apply the hyperbolization process to such a triangulation or cubulation, the result is a new aspherical manifold with a group action permuting the hyperbolized cells in exactly the same pattern that the original group action permutes the original cells. The preceding discussion implies the following result.

Theorem 6.2.1 (Equivariant Hyperbolization)**.** *If a finite group G acts simplicially or cubically on a manifold X, then the hyperbolization $\mathcal{H}(X)$ can be constructed with G action so that the hyperbolization map is G-equivariant (indeed, isovariant):*

$$(\mathcal{H}(X), \mathcal{H}(F)) \to (X, F).$$

Moreover, local neighborhoods of points in $\mathcal{H}(X)$ are equivalent to the neighborhoods of corresponding points in X.

6.3. Applications of equivariant hyperbolization.

We conclude with a couple of immediate applications.

Corollary 6.3.1. *All local patterns of fixed points and normal representations that occur for n-manifolds also occur for aspherical n-manifolds.*

Corollary 6.3.2. *There exist C_p actions (any p) on aspherical 4-manifolds with both isolated points and surfaces of fixed points.*

Proof. For example, a standard C_p action on $\mathbb{C}P^2$ with fixed point set consisting of an isolated point and a 2-sphere yields an action of C_p on an aspherical 4-manifold with fixed point set consisting of an isolated point and a surface of some higher genus determined by the number of 2-cells in the fixed 2-sphere in an equivariant triangulation of $\mathbb{C}P^2$. □

Many other examples in higher dimensions can be constructed by hyperbolizing interesting actions on familiar non-aspherical n-manifolds.

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References

[1] Glen E. Bredon, *Introduction to Compact Transformation Groups.* Pure and Applied Mathematics, Vol. 46. New York-London: Academic Press, 1972.

[2] Kenneth S. Brown, *Cohomology of Groups.* Graduate Texts in Mathematics, 87. New York-Berlin: Springer-Verlag, 1982.

[3] P. E. Conner and Frank Raymond, *Actions of compact Lie groups on aspherical manifolds*, in *Topology of Manifolds (Proc. Inst., Univ. of Georgia, Athens, Ga., 1969).* Ed. James C. Cantrell and C. H. Edwards, Jr. Chicago, IL: Markham, 1970. 227–264.

[4] Frank Connolly, James F. Davis, and Qayum Khan, *Topological rigidity and H_1-negative involutions on tori.* Available at arXiv:1102.2660v1 [math.GT].

[5] Michael W. Davis and Tadeusz Januszkiewicz, *Hyperbolization of polyhedra*, J Differential Geom. **34** (1991), no 2, 1991, 347–388.

[6] Michael W. Davis, Tadeusz Januszkiewicz, and Shmuel Weinberger, *Relative hyperbolization and aspherical bordisms: An addendum to "Hyperbolization of polyhedra."* J. Differential Geom. **54** (2001), no. 3, 535–541.

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