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CONNECTEDNESS IN SPACES OF PSEUDOQUOTIENTS

PIOTR MIKUSIŃSKI AND STEVEN A. PURTEE

ABSTRACT. A space of pseudoquotients, denoted by $\mathcal{B}(X,S)$, is defined as equivalence classes of pairs (x,f), where x is an element of a non-empty set X, f is an element of S, a commutative semigroup of injective maps from X to X, and $(x,f)\sim(y,g)$ if $gx=fy$. A topology on X induces a topology on $\mathcal{B}(X,S)$. We show that connectedness of X implies connectedness of $\mathcal{B}(X,S)$.

1. Introduction

The construction of pseudoquotients (generalized quotients) is a general abstract construction that allows us to build extensions of spaces while preserving their basic structure.

Let X be a set and let S be an abelian semigroup of injections acting on X. In $X\times S$, we introduce the following relation

$$(x,f)\sim(y,g) \quad \text{if} \quad gx=fy.$$

Using the fact that elements of S are injective maps and that they commute, it is easy to show that this is an equivalence relation. Equivalence classes in $X\times S$ will be called *pseudoquotients*. The space of all pseudoquotients will be denoted by $\mathcal{B}(X,S)$, or simply $\mathcal{B}$. The equivalence class $[(x,f)]$ will be denoted by $\frac{x}{f}$. Thus, $\frac{x}{f}=\frac{y}{g}$ means $(x,f)\sim(y,g)$.

In this note, we assume that X is a topological space and that S is an abelian semigroup of continuous injections acting on X. Moreover, S is equipped with its own topology such that the map $g\mapsto fg$ is continuous for every $f\in S$. Note that this condition is always satisfied if the topology of S is discrete. Topologies on X and S induce a natural topology on $\mathcal{B}$;

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we first define the product topology on $X \times S$ and then the quotient topology in $\mathcal{B}$.

There is growing evidence that this method of constructing extensions of spaces is a useful tool.

In [1], a space of pseudoquotients is constructed which allows us to generalize Bochner's theorem so that all Radon measures on a locally compact group are in a one-to-one correspondence with elements of that space of pseudoquotients. In this case, X is the cone of positive definite functions on a locally compact group G with the topology of uniform convergence on compact subsets of G and

$$S = \left\{\varphi \in L^1(G) : \widehat{\varphi}(\xi) > 0 \text{ for all } \xi \in \widehat{G}\right\},$$

(where $\widehat{G}$ is the dual group and $\widehat{\varphi}$ is the Fourier transform of φ) with the discrete topology.

The space of pseudoquotients constructed in [2] is isomorphic to the space of tempered distributions on $\mathbb{R}^N$. In this case, X is the space of complex valued functions of the form $f = pg$, where p is a polynomial and $g \in L^2(\mathbb{R}^N)$, with the inductive limit topology induced by all maps $\Lambda_p : L^2 \to X$, where p is a polynomial and $\Lambda_p f = pf$. The semigroup S is generated by the function

$$E(x) = e^{-(|x_1|+|x_2|+\cdots+|x_N|)},$$

that is,

$$S = \{E^n : n = 1, 2, 3, \dots\},$$

where E^n denotes the n-fold convolution $E * \cdots * E$. Elements of S act on X by convolution. The topology of S is the discrete topology.

In our opinion, there is value in studying general topological properties of pseudoquotients. Some questions concerning the topological structure of pseudoquotients have been studied in [5], [3], [4], and [6]. In [5], the question of whether separation properties of X are inherited by $\mathcal{B}$ is considered. In particular, it is shown that, if X has a Hausdorff topology, the topology of $\mathcal{B}$ need not be Hausdorff. Conditions under which it happens are discussed. In [3] and [4], pseudoquotients are studied in the framework of the category of convergence spaces.

In this short note, we show that various types of connectedness are preserved in extensions to pseudoquotients.

2. Preliminaries

We start with some simple but useful observations:

(1) $(x, f) \sim (gx, gf)$ for all $g \in S$.

(2) If $(x, f) \sim (y, f)$ for some $f \in S$, then $x = y$.

(3) For all $f, g \in S$, $(fx, f) \sim (gx, g)$.
(4) $(x, f) \sim (y, g)$ if and only if $(x, hf) \sim (y, hg)$ for every $h \in S$ if and only if $(x, hf) \sim (y, hg)$ for some $h \in S$.

Note that, by (3), we may consider X as a subset of $\mathcal{B}$ by identifying $x \in X$ with $\frac{gx}{g}$ for any $f \in S$.

Proposition 2.1. *The mapping* $\tilde{g} : \mathcal{B} \to \mathcal{B}$ *given by*

$$\tilde{g}\left(\frac{x}{f}\right) = \frac{gx}{f}$$

is a well-defined homeomorphism for every $g \in S$.

Proof. It is easy to check that $\tilde{g}$ is well defined and that it is injective. Since, for any $g \in S$ and any $\frac{x}{f} \in \mathcal{B}$,

$$\tilde{g}\left(\frac{x}{gf}\right) = \frac{gx}{gf} = \frac{x}{f}\,,$$

$\tilde{g}$ is surjective and

$$\tilde{g}^{-1}\left(\frac{x}{f}\right) = \frac{x}{gf}\,.$$

For $g \in S$, define $\varphi_g : X \times S \to X \times S$ by $\varphi_g(x, f) = (gx, f)$. Consider the diagram

$$\begin{array}{ccc} X \times S & \xrightarrow{\varphi_g} & X \times S \\ \downarrow q & & \downarrow q \\ \mathcal{B}(X, S) & \xrightarrow[\tilde{g}]{} & \mathcal{B}(X, S) \end{array}$$

where q is the quotient map $q(x, f) = \frac{x}{f}$. Note that this diagram commutes and $q \circ \varphi_g$ is continuous. By a standard result (see, for example, Theorem 22.2 in [7]), $\tilde{g}$ is continuous.

Now, for $g \in S$, define $\psi_g : X \times S \to X \times S$ by $\psi_g(x, f) = (x, fg)$. Then the diagram

$$\begin{array}{ccc} X \times S & \xrightarrow{\psi_g} & X \times S \\ \downarrow q & & \downarrow q \\ \mathcal{B}(X, S) & \xrightarrow[\tilde{g}^{-1}]{} & \mathcal{B}(X, S) \end{array}$$

commutes and $q \circ \psi_g$ is continuous. Consequently, $\tilde{g}^{-1}$ is continuous. □

We will not distinguish between $g \in S$ and its extension $\tilde{g}$, but will use g in both cases.

Corollary 2.2. *The semigroup S can be extended to a group $\widehat{S}$ of homeomorphisms on $\mathcal{B}$.*

The following proposition is often useful in constructing examples of spaces of pseudoquotients with desired properties.

Proposition 2.3. *Let X be a topological space and let $f : X \to X$ be a homeomorphism. Let Y be an open subset of X such that $f(Y) \subset Y$ and*

$$X = \bigcup_{n=1}^{\infty} f^{-n}(Y)\,. \tag{2.1}$$

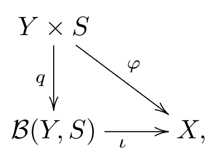
If $S = \{f^n : n \in \mathbb{N}\}$, then X and $\mathcal{B}(Y, S)$ are homeomorphic.

Proof. We will show that the map $\iota : \mathcal{B}(Y, S) \to X$, defined by

$$\iota\left(\frac{y}{f^n}\right) = f^{-n}y,$$

is a homeomorphism. First, note that $\frac{y}{f^n} = \frac{z}{f^m}$ implies $f^m y = f^n z$ and consequently, $f^{-n}y = f^{-m}z$, which shows that ι is well defined. In a similar way we can show that ι is injective. Surjectivity follows from (2.1).

Now consider the diagram



where the function $\varphi : X \times S \to X$ is defined by $\varphi(y, f^n) = f^{-n}y$. Since the diagram commutes and φ is continuous, ι is continuous.

If $U \in \mathcal{B}(Y, S)$ is open, then

$$U = q\left(\bigcup_{n=1}^{\infty} U_n \times \{f^n\}\right)$$

for some open $U_n \subset Y$. Hence,

$$\iota(U) = \varphi\left(\bigcup_{n=1}^{\infty} U_n \times \{f^n\}\right) = \bigcup_{n=1}^{\infty} f^{-n}U_n,$$

which shows that ι is an open map. □

3. Main Results

In this section we show that the extension to $\mathcal{B}$ preserves path connectedness, connectedness, local connectedness, and local path connectedness.

Theorem 3.1. *If X is path connected, then $\mathcal{B}$ is path connected.*

Proof. Let $A, B \in \mathcal{B}$. Without loss of generality, we can assume that $A = \frac{x}{f}$ and $B = \frac{y}{f}$. Since X is path connected, there exists a path $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Define $\alpha : [0,1] \to X \times S$ by $\alpha(t) = (\gamma(t), f)$. Then α is a path in $X \times S$ with $\alpha(0) = (x, f)$ and $\alpha(1) = (y, f)$. Hence, $q \circ \alpha$ is a path in $\mathcal{B}$ from A to B. □

Theorem 3.2. *If X is connected, then $\mathcal{B}$ is connected.*

Proof. Suppose $\mathcal{B}$ is not connected and let $U, V \subset \mathcal{B}$ be a separation of $\mathcal{B}$. Then $q^{-1}(U)$ and $q^{-1}(V)$ form a separation of $X \times S$.

If there exists an $f \in S$ such that $U_f = (X \times \{f\}) \bigcap q^{-1}(U) \neq \emptyset$ and $V_f = (X \times \{f\}) \bigcap q^{-1}(V) \neq \emptyset$, then U_f and V_f form a separation of $X \times \{f\}$ and thus, $\Pi_X U_f$ and $\Pi_X V_f$ form a separation of X (Π_X denotes the projection on X).

Now suppose that there are $f, g \in S$ such that $X \times \{f\} \subset q^{-1}(U)$ and $X \times \{g\} \subset q^{-1}(V)$. Then $\frac{x}{f} \in U$ and $\frac{x}{g} \in V$ for all $x \in X$. Hence, $\frac{gx}{fg} \in U$ and $\frac{fx}{fg} \in V$. This implies $(X \times \{fg\}) \bigcap q^{-1}(U)$, and $(X \times \{fg\}) \bigcap q^{-1}(V)$ is a separation of $X \times \{fg\}$. Then $\Pi_X(X \times \{fg\}) \bigcap q^{-1}(U)$, and $\Pi_X(X \times \{fg\}) \bigcap q^{-1}(V)$ is a separation of X. □

Let $X = \mathbb{R}$ with its standard topology, $f(x) = \frac{x}{3}$, and $Y = (-1, 1) \cup (2, 3)$. If $S = \{f^n : n \in \mathbb{N}\}$, then $\mathcal{B}(Y, S)$ is homeomorphic to $\mathbb{R}$ by Proposition 2.3. This simple example shows that $\mathcal{B}(Y, S)$ can be connected (or path connected) even if Y is not.

We assume that the topology of S is discrete in the next two theorems.

Theorem 3.3. *If X is locally connected, then $\mathcal{B}$ is locally connected.*

Proof. Since every quotient of a locally connected space is locally connected (see, for example, Theorem 27.12 in [8]), it suffices to show that $X \times S$ is locally connected. If the topology of S is discrete, then every $(x, g) \in X \times S$ has a neighborhood of the form $U \times \{g\}$. Since X is locally connected, U can be chosen to be connected. □

In the proof of the next theorem, we use the following well-known property of locally path connected spaces (see, for example, Theorem 25.4 in [7]): *A space Y is locally path connected if and only if for every open $U \subset Y$, each path component of U is open in Y.*

Theorem 3.4. *If X is locally path connected, then $\mathcal{B}$ is locally path connected.*

Proof. Let U be an open subset of $\mathcal{B}$ and let

$$q^{-1}(U) = \bigcup_{n=1}^{\infty} U_n \times \{f^n\},$$

where each U_n is open in X. Let V be a path component of U. We need to show that V is open in $\mathcal{B}$ or, equivalently, that $q^{-1}(V)$ is open in $X \times S$. If

$$q^{-1}(V) = \bigcup_{n=1}^{\infty} V_n \times \{f^n\},$$

then each V_n is the union of path components of U_n. But each U_n is open in X and X is locally path connected, so V_n must open in X. Hence, $q^{-1}(V)$ is open in $X \times S$. □

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