TOPOLOGY PROCEEDINGS

Volume 40, 2012

Pages 311–318

http://topology.auburn.edu/tp/

A Shorter Proof of a Theorem on Hereditarily Orderable Spaces

by

Harold Bennett and David Lutzer

Electronically published on March 5, 2012

Topology Proceedings

Web: http://topology.auburn.edu/tp/

Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA

E-mail: topolog@auburn.edu

ISSN: 0146-4124

http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS
Volume 40 (2012)
Pages 311-318

E-Published on March 5, 2012

A SHORTER PROOF OF A THEOREM ON HEREDITARILY ORDERABLE SPACES

HAROLD BENNETT AND DAVID LUTZER

Abstract. We give a shorter proof of a result of S. Purisch and Yasushi Hirata and Nobuyuki Kemoto that any subspace of any space of ordinals is a LOTS (under some linear ordering).

1. Introduction

A topological space is orderable if it is homeomorphic to some linearly ordered topological space (LOTS) $(X, <, \mathcal{L}(<))$ where $<$ is a linear ordering of X and $\mathcal{L}(<)$ is the usual open interval topology of $<$. As the subspace $[0, 1] \cup (2, 3)$ of the usual space $\mathbb{R}$ of real numbers shows, a subspace of a LOTS may fail to be orderable, as may a topological sum of two LOTS (no matter what linear ordering is used).

In [3], Yasushi Hirata and Nobuyuki Kemoto showed that any subspace of any space of ordinal numbers must be orderable (under some ordering), a result that follows from an earlier paper by S. Purisch [4] [5]. In this paper we give a new proof that is shorter than the proofs given by Purisch or by Hirata and Kemoto, and we raise some questions about hereditary orderability, where we say that a space X is *hereditarily orderable* if each of its subspaces is an orderable space.

Recall that a *generalized ordered* (GO) space is a triple $(X, <, \tau)$ where $<$ is a linear ordering of X and where τ is a Hausdorff topology on X that has a basis consisting of order-convex (possibly degenerate) sets.

2010 *Mathematics Subject Classification.* Primary = 54F05, Secondary= 03E04.

The research of the first author was supported by a Texas Tech University Faculty Development Leave.

2. Ordinals Are Hereditarily Orderable

For any linearly ordered set $(X,<)$, the symbol $(X,<)^*$ denotes the set X with the reverse ordering $<^*$. It is easy to see that the LOTS $(X,<,\mathcal{L}(<))$ is homeomorphic to the LOTS $(X,<^*,\mathcal{L}(<^*))$. For a given linearly ordered set X, we sometimes write X^* for $(X,<^*,\mathcal{L}(<^*))$.

Suppose $(X_1,<)$ and $(X_2,\prec)$ are disjoint linearly ordered sets. We use the symbol $(X_1,<)\frown(X_2,\prec)$ to mean the set $X_1\cup X_2$ with the ordering defined by $a\ll b$ if either $a,b\in X_1$ and $a<b$, or $a\in X_1$ and $b\in X_2$, or $a,b\in X_2$ with $a\prec b$. We sometimes write $\ll=<\frown\prec$. The relation $\ll$ is always a linear ordering, but if $\mathcal{L}(<)$ and $\mathcal{L}(\prec)$ are the usual open interval topologies on X_1 and X_2, respectively, then the open interval topology $\mathcal{L}(\ll)$ might not be the topology of the topological sum $(X_1,\mathcal{L}(<))\oplus(X_2,\mathcal{L}(\prec))$. For an example, let $X_1=[0,1]$ and $X_2=(2,3)$ have their usual orderings. However, there are times when the topological sum of two or more LOTS is guaranteed to be a LOTS.

Lemma 2.1. *Let $(X_1,<)$ and $(X_2,\prec)$ be disjoint linearly ordered sets and let $\ll$ be the order $<\frown\prec$.*

(1) *If the LOTS $(X_1,<,\mathcal{L}(<))$ contains a right end point and $(X_2,\prec,\mathcal{L}(\prec))$ contains a left end point, then the topological sum $X_1\oplus X_2$ is a LOTS under the order $\ll$.*
(2) *If the LOTS $(X_1,<,\mathcal{L}(<))$ contains no right endpoint and if the LOTS $(X_2,\prec,\mathcal{L}(\prec))$ contains no left end point, then the topological sum $X_1\oplus X_2$ is a LOTS under the order $\ll$.*

Lemma 2.2. *Suppose that $(X,<,\tau)$ is a GO-space having a right endpoint $b\in X$, and suppose that b is a τ-limit point of the set $Y=X-\{b\}$. Suppose that $\prec$ is a linear ordering of the set Y such that the open interval topology $\mathcal{L}(\prec)$ on Y coincides with $\tau|_Y$, and suppose that a subset $C\subseteq Y$ is cofinal*[1] *in $(Y,<)$ if and only if C is cofinal in $(Y,\prec)$. Extend the linear order $\prec$ to a linear order $\lhd$ on X by making b larger than each point of $(Y,\prec)$. Then $\mathcal{L}(\lhd)=\tau$; i.e., $(X,\lhd,\tau)$ is a LOTS.*

Proof. It is enough to show that the topologies τ and $\mathcal{L}(\lhd)$ agree at the point b because, by hypothesis, they agree at each point of the open set Y. Because b is a limit point of Y, the set $(Y,<)$ contains no right endpoint. Hence, neither does $(Y,\prec)$.

Let U be a τ-neighborhood of b. We may assume U is order convex with respect to $<$. Because b is a τ-limit of Y, we may choose $a',a\in U\cap Y$ and $a'<a<b$. Then $a\in\{x\in X : a'<x<b\}\subseteq U$. We will show

[1]A subset S of a linearly ordered set $(X,<)$ is *cofinal* if for each $x\in X$ there is some $s\in S$ with $x\le s$.

that there is some $c \in X$ with $c \lhd b$ and $\{x \in X : c \lhd x\} \subseteq \{x \in X : a \leq x\} \subseteq \{x \in X : a' < x\}$. If that is not true, then for each $c \in Y$, there is some $d(c) \in Y$ with $c \lhd d(c)$ and $d(c) \notin \{x \in X : a \leq x\}$. The set $\{d(c) : c \in Y\}$ is cofinal in $(Y, \prec)$, and, therefore, it is also cofinal in $(Y, <)$ so there is some $d(c)$ with $a \leq d(c)$, contrary to the choice of $d(c)$. An analogous argument shows that for each $a \lhd b$, there is some c with $c < b$ and $\{x \in X : c < x\} \subseteq \{x \in X : a \lhd x\}$. Consequently, the two topologies τ and $\mathcal{L}(\lhd)$ agree at b, as required. $\square$

The following example illustrates a key idea in the proof of our main result. Write $2\omega = \omega + \omega$ and $3\omega = \omega + \omega + \omega$. Form a GO-topology τ by isolating the points ω and 2ω in $[0, 3\omega)$. In the usual ordering $<$, this is not a LOTS because the non-limit points ω and 2ω have no immediate predecessors in the order $<$. However, in the linear ordering $\prec$ of

$$[0, \omega) \frown \Big([\omega, 2\omega)^* \frown [2\omega, 3\omega)\Big),$$

the set $\{n : 0 \leq n < \omega\}$ has no supremum, and the points ω and 2ω both have immediate predecessors and immediate successors. Consequently, $([0, 3\omega), \prec, \tau)$ is a LOTS. Flipping the order on subsegments of a GO-space is the key to our next proof.

Theorem 2.3. *Let Δ be any ordinal with $\Delta \geq \omega$ and let T be any set of limit ordinals in $[0, \Delta)$. Let $[0, \Delta)_T$ denote the GO-space obtained from the usual ordinal space $[0, \Delta)$ by isolating every element of T. Then the GO-space $[0, \Delta)_T$ is homeomorphic to some LOTS.*

Proof. In this proof, the symbol $\cong$ means "is homeomorphic to" and we use $\leq$ and $<$ to denote the usual well-ordering of $[0, \Delta)$. Interval notation such as $[\alpha, \beta)$ will always refer to intervals in the usual ordinal ordering. For any $\alpha < \Delta$, the symbol $[0, \alpha)_T$ denotes the GO-space obtained from the usual LOTS $[0, \alpha)$ by isolating all points of $T \cap [0, \alpha)$. We will write $\mathcal{S}$ for the GO-topology of $[0, \Delta)_T$ and $\mathcal{S}_{[\alpha,\beta)}$ for the relative topology that $[\alpha, \beta)$ inherits from $[0, \Delta)_T$. We will argue by contradiction. For contradiction, suppose that

$$(*) [0, \Delta)_T \text{ is not homeomorphic to any LOTS.}$$

By an *acceptable pair*, we mean an ordered pair $([0, \alpha), \prec_\alpha)$ where

(1) $\alpha \leq \Delta$;
(2) $\prec_\alpha$ is a linear ordering of the set $[0, \alpha)$;
(3) 0 is the left end point of the linearly ordered set $([0, \alpha), \prec_\alpha)$;
(4) a subset $C \subseteq [0, \alpha)$ is cofinal in $([0, \alpha), \prec_\alpha)$ if and only if C is cofinal in $([0, \alpha), <)$; and
(5) $\mathcal{S}_{[0,\alpha)} = \mathcal{L}(\prec_\alpha)$ where $\mathcal{L}(\prec_\alpha)$ is the usual open interval topology of the linear order $\prec_\alpha$.

Let $\mathcal{P}$ be the set of all acceptable pairs. Then $\mathcal{P}$ is not empty because $([0,2),<)$ is in $\mathcal{P}$. Partially order $\mathcal{P}$ by the rule that $([0,\alpha),\prec_\alpha) \sqsubseteq ([0,\beta),\prec_\beta)$ if and only if the following four statements hold:

(a) $\alpha \le \beta$;
(b) $\prec_\beta |_{[0,\alpha)} = \prec_\alpha$ (so that $\prec_\beta$ extends $\prec_\alpha$);
(c) if $\alpha < \beta$, then $\{x \in [0,\beta) : x \prec_\beta \alpha\} = [0,\alpha)$ (so $\prec_\beta$ adds no points to the domain of $\prec_\alpha$);
(d) $\mathcal{L}(\prec_\alpha) \subseteq \mathcal{L}(\prec_\beta)$.

Suppose $\mathcal{C} = \{([0,\alpha),\prec_\alpha) : \alpha \in A\}$ is a chain in the partially ordered set $(\mathcal{P},\sqsubseteq)$. For some $\gamma \le \Delta$, the set $\bigcup\{[0,\alpha) : \alpha \in A\} = [0,\gamma)$. Define $\prec_\gamma = \bigcup\{\prec_\alpha : \alpha \in A\}$. We will show that $([0,\gamma),\prec_\gamma)$ is an acceptable pair that is an upper bound for $\mathcal{C}$ in $(\mathcal{P},\sqsubseteq)$.

It is clear that $\prec_\gamma$ is a linear ordering of $[0,\gamma)$ and that its left end point is 0. If $\gamma = \alpha$ for some $\alpha \in A$, then $([0,\gamma),\prec_\gamma) = ([0,\alpha),\prec_\alpha)$ is an acceptable pair that is an upper bound for the chain $\mathcal{C}$, so assume that $\alpha < \gamma$ for all $\alpha \in A$. Consequently, γ is a limit ordinal and the set A is cofinal in the usual ordering of $[0,\gamma)$.

We claim that $([0,\gamma),\prec_\gamma)$ satisfies part (4) in the definition of an acceptable pair. We first show that the set A is cofinal in the ordering $\prec_\gamma$. Let $x \in [0,\gamma)$ and choose $\alpha,\beta \in A$ with $x < \alpha < \beta$. In the light of (c), we have $x \in [0,\alpha) = \{y \in [0,\beta) : y \prec_\beta \alpha\}$, so that $x \prec_\gamma \alpha$. Hence, A is cofinal in the order $\prec_\gamma$. Now suppose that C is a cofinal subset of $([0,\gamma),<)$. Fix $(\alpha,\prec_\alpha) \in \mathcal{C}$. Choose $x \in C$ with $\alpha < x$, and then choose $\beta \in A$ with $\alpha < x < \beta$. By (c), we have $x \notin [0,\alpha) = \{y \in [0,\beta) : y \prec_\beta \alpha\}$, so that $\alpha \preceq_\beta x$. Therefore, $\alpha \preceq_\gamma x$, showing that C is cofinal in the ordering $\prec_\gamma$. Next suppose that C is cofinal in the ordering $\prec_\gamma$. If C is not cofinal in the usual ordering $<$ of $[0,\gamma)$, then there is some $\alpha \in A$ with $C \subseteq [0,\alpha)$. Then for each $\beta \in A$ with $\alpha < \beta$, we have $C \subseteq [0,\alpha) = \{y \in [0,\beta) : y \prec_\beta \alpha\}$, so that $x \prec_\beta \alpha$ for each $x \in C$, and therefore, $x \prec_\gamma \alpha$ for each $x \in C$. But that is impossible because C is cofinal in the ordering $\prec_\gamma$.

We next show that $([0,\gamma),\prec_\gamma)$ satisfies $\mathcal{S}_{[0,\gamma)} = \mathcal{L}(\prec_\gamma)$, which is part (5) in the definition of an acceptable pair. First note that the collection $\mathcal{B} := \bigcup\{\mathcal{L}(\prec_\alpha) : \alpha \in A\}$ is a base for the topology $\mathcal{L}(\prec_\gamma)$ and that $\mathcal{B}' := \bigcup\{\mathcal{S}_{[0,\alpha)} : \alpha \in A\}$ is a base for $\mathcal{S}_{[0,\gamma)}$. Because we know that $\mathcal{L}(\prec_\alpha) = \mathcal{S}_{[0,\alpha)}$ for each $\alpha \in A$, we see that $\mathcal{B} = \mathcal{B}'$ which gives $\mathcal{L}(\prec_\gamma) = \mathcal{S}_{[0,\gamma)}$ as required.

Now that we have $([0,\gamma),\prec_\gamma) \in \mathcal{P}$, we must show that $([0,\alpha),\prec_\alpha) \sqsubseteq ([0,\gamma),\prec_\gamma)$ for each $\alpha \in A$. Clearly (a) and (b) are satisfied. For (c), note that for each $\alpha < \beta$ in the set A, we have

$$[0,\alpha) = \{y \in [0,\beta) : y \prec_\beta \alpha\} \subseteq \{y \in [0,\gamma) : y \prec_\gamma \alpha\}.$$

To prove that $\{y \in [0,\gamma) : y \prec_\gamma \alpha\} \subseteq [0,\alpha)$, suppose $y < \gamma$ satisfies $y \prec_\gamma \alpha$. Choose $\beta \in A$ so large that $\{\alpha, y\} \subseteq [0,\beta)$. Then $y \in \{z \in [0,\beta) : z \prec_\beta \alpha\} = [0,\alpha)$, as required. To verify (d), note that the collection $\bigcup\{\mathcal{L}(\prec_\alpha) : \alpha \in A\}$ is a basis for the topology $\mathcal{L}(\prec_\gamma)$.

At this stage, we know that every chain in $(\mathcal{P}, \sqsubseteq)$ has an upper bound in $(\mathcal{P}, \sqsubseteq)$ so that Zorn's Lemma gives us a maximal element $([0,\delta), \prec_\delta)$ of $\mathcal{P}$. We have $\delta \leq \Delta$. If $\delta = \Delta$, then we have contradicted (*) because $([0,\delta), \prec_\delta)$ satisfies part (5) of the definition of acceptable pair, so we have

$$(**)\ \delta < \Delta.$$

Claim 1. We claim that δ must be a limit ordinal. Otherwise, write $\delta = \lambda + n$, where λ is a limit and $n \geq 1$ is an integer. Then $[0,\delta)$ has a right endpoint (namely, $\lambda + (n-1)$) in the usual ordinal ordering, so that $\{\lambda + (n-1)\}$ is a cofinal subset of $[0,\delta)$ in the usual ordering. Therefore, $\{\lambda + (n-1)\}$ is a cofinal subset of $[0,\delta)$ in the linear ordering $\prec_\delta$; i.e., $([0,\delta), \prec_\delta)$ has $\lambda + (n-1)$ as its right endpoint. Because $\delta < \Delta$ by (**), we know that $\delta + 1 \leq \Delta$. Define a linear ordering $\prec_{\delta+1}$ of $[0,\delta+1)$ that agrees with $\prec_\delta$ on $[0,\delta)$ and has $\delta = \lambda + n$ as its right endpoint. Then the LOTS $([0,\delta+1), \prec_{\delta+1}, \mathcal{L}(\prec_{\delta+1}))$ is homeomorphic to the GO-space $[0,\delta+1)_T$ and it is clear that $([0,\delta+1), \prec_{\delta+1})$ belongs to $\mathcal{P}$ and is strictly larger than $([0,\delta), \prec_\delta)$ in the ordering $\sqsubseteq$, contrary to maximality of $([0,\delta), \prec_\delta)$. Therefore, Claim 1 is established and δ must be a limit ordinal.

Two possibilities remain. Either δ is an isolated point of the GO-space $[0,\Delta)_T$, or δ is a limit point of the set $[0,\delta)$ in the space $[0,\Delta)_T$; i.e., either $\delta \in T$ or $\delta \notin T$.

Claim 2. We claim that $\delta \in T$ is impossible. For suppose $\delta \in T$. There are two subcases, depending upon whether $(\delta,\Delta) \cap T$ is or is not empty.

In the first subcase, we have $(\delta,\Delta) \cap T = \emptyset$, and then $[\delta,\Delta)_T$ is identical to the LOTS $[\delta,\Delta)$ with the usual ordering. Consider the linearly ordered set $X = [\delta,\Delta)^*$ obtained by reversing the usual order of $[\delta,\Delta)$, and let $<^*$ denote the reversal of the usual ordering $<$. The linearly ordered set $([\delta,\Delta)^*, <^*)$ has a final point (namely δ), and the linearly ordered set $Y = ([0,\delta), \prec_\delta)$ has 0 as its first point by part (3) of the definition of acceptable pair. Consequently, Lemma 2.1(1) guarantees that the LOTS topology of the linear order $\lhd := <^* \frown \prec_\delta$ on the set $X \oplus Y$ is homeomorphic to the disjoint sum topology of the space $X \oplus Y$. But because $\delta \in T$, we have $[0,\delta)_T \oplus [\delta,\Delta)_T \cong [0,\Delta)_T$ so that

$$X \oplus Y \cong Y \oplus X = \Big([0,\delta), \prec_\delta, \mathcal{L}(\prec_\delta)\Big) \oplus [\delta,\Delta)^* \cong [0,\delta)_T \oplus [\delta,\Delta)$$

$$= [0,\delta)_T \oplus [\delta,\Delta)_T = [0,\Delta)_T,$$

showing that $[0, \Delta)_T$ is a LOTS under the linear ordering $\triangleleft$, contrary to (*). Therefore, the first subcase cannot occur.

In the second subcase, $(\delta, \Delta) \cap T \neq \emptyset$. Let η be the first element of $(\delta, \Delta) \cap T$. Then η is a limit ordinal (because all members of T are limit ordinals) and $\eta + 1 \leq \Delta$ because $\eta < \Delta$. The LOTS $[\delta, \eta)$ with its usual order $<$ and usual order topology is homeomorphic to the clopen subspace $[\delta, \eta)_T$ of $[0, \Delta)_T$ and hence so is the reversed LOTS $Y = ([\delta, \eta)^*, <^*, \mathcal{L}(<^*))$. Observe that the LOTS $X = ([0, \delta), \prec_\delta, \mathcal{L}(\prec_\delta))$ has no final point and that the LOTS $Y = ([\delta, \eta)^*, <^*, \mathcal{L}(<^*))$ has no first point. According to Lemma 2.1(2), the LOTS topology of the linear order $\triangleleft = \prec_\delta \frown <^*$ on the set $[0, \eta)$ coincides with the topology of the topological sum

$$([0, \delta), \prec_\delta) \oplus [\delta, \eta)^* \cong [0, \delta)_T \oplus [\delta, \eta) \cong [0, \eta)_T.$$

Note that the linear order $\triangleleft$ has a right endpoint, namely, δ. Now extend the linear order $\triangleleft$ on $[0, \eta)$ to the set $[0, \eta]$ by making η greater than each point of $([0, \eta), \triangleleft)$. The set $[0, \eta + 1)$ with this extension of $\triangleleft$ is a member of $\mathcal{P}$ that is strictly larger than $(\delta, \prec_\delta)$ in the partial order $\sqsubseteq$, and that is impossible. Therefore, Claim 2 is established.

Claim 3. We claim that $\delta \notin T$ is also impossible. For suppose $\delta \notin T$. Because δ is a limit ordinal (see Claim 1), the point δ is a limit point of the set $[0, \delta)$ in the space $[0, \Delta)_T$. Because $([0, \delta), \prec_\delta) \in \mathcal{P}$, we know that the orders $<$ and $\prec_\delta$ have exactly the same cofinal subsets of $[0, \delta)$, and then Lemma 2.2 allows us to extend the order $\prec_\delta$ to a linear order $\triangleleft$ of the set $[0, \delta + 1)$ by making the point δ greater than all points of $([0, \delta), \prec_\delta)$ and guarantees that the LOTS topology of $([0, \delta + 1), \triangleleft)$ coincides with the GO topology $[0, \delta + 1)_T$. It is clear that $([0, \delta + 1), \triangleleft) \in \mathcal{P}$ and that is impossible by maximality of $([0, \delta), \prec_\delta)$ in $\mathcal{P}$. Therefore, Claim 3 holds.

In summary, assumption (*) has led us to a maximal element $([0, \delta), \prec_\delta)$ of $\mathcal{P}$, and we have proved that both $\delta \in T$ and $\delta \notin T$ are impossible. Consequently, Theorem 2.3 is proved. □

The hereditary orderability theorem of Purisch, Hirata and Kemoto is an immediate corollary.

Corollary 2.4. *Let Z be an initial segment of the ordinals with the usual topology. Any subspace X of Z is homeomorphic to some LOTS.*

Proof. The set X inherits a well-ordering from Z and we have an order isomorphism h from X onto some set $[0, \Delta)$ of ordinals. Let $\mathcal{S}$ be the topology on $[0, \Delta)$ that makes h a homeomorphism from X onto $([0, \Delta), \mathcal{S})$. The topology $\mathcal{S}$ will fail to be the open interval topology of the usual ordering $<$ of $[0, \Delta)$ if and only if there are limit ordinals $\lambda < \Delta$ such that λ is not a limit of the set $[0, \lambda)$ in the space $([0, \Delta), \mathcal{S})$. Let T be the set of all limit ordinals $\lambda < \Delta$ that are not topological limits of $[0, \lambda)$ in the topology $\mathcal{S}$. Then X is homeomorphic to the GO-space $[0, \Delta)_T$ obtained from the usual ordinal space $[0, \Delta)$ by isolating each point of T. But, from Theorem 2.3, we know that $[0, \Delta)_T$ is homeomorphic to some LOTS, and that completes the proof of the corollary. $\square$

3. Additional Comments

In this section, we use dimension theory definitions from [1]. The following result is part of the folklore.

Lemma 3.1. *In any GO-space X, the following three properties are equivalent:*

(a) *$Ind(X) = 0$*
(b) *$ind(X) = 0$*
(c) *a connected subset of X has at most one point; (i.e., X is* totally disconnected*).*

H. Herrlich's theorem [2] (see also [1, Problem 6.3.2]) is the key to understanding hereditary orderability in metrizable spaces.

Proposition 3.2. *Let X be a metrizable space. Then the following are equivalent:*

(i) *$Ind(X) = 0$;*
(ii) *X is orderable and $Ind(X) = 0$;*
(iii) *X is orderable and totally disconnected;*
(iv) *X is hereditarily orderable.*

Proof. Herrlich's theorem is that (i) $\Rightarrow$ (ii), and (ii) and (iii) are equivalent in light of Lemma 3.1. Because X is metrizable, for any subspace $Y \subseteq X$, we have $Ind(Y) \leq Ind(X)$ so that Herrlich's theorem shows that (ii) $\Rightarrow$ (iv). Finally, (iv) $\Rightarrow$ (iii) because if X contains a connected subset C with at least two points, then X contains an infinite connected open interval (a, b) (containing no end points of itself) and a point $c \notin [a, b]$. But then the subspace $Y = (a, b) \cup \{c\}$ is not linearly orderable by any ordering. $\square$

However, outside the class of metrizable spaces, $Ind(X) = 0$ is not enough to make a LOTS hereditarily orderable.

Example 3.3. Let X be the Alexandroff double arrow, i.e., $X = [0, 1] \times \{0, 1\}$ with the lexicographic ordering. Then X is a compact separable LOTS and has $Ind(X) = 0$, but its subspace $S := \{(x, 1) : x \in [0, 1]\}$ is not a LOTS under any ordering, because S has a G_δ-diagonal but is not metrizable.

Question 3.4. Characterize those LOTS that are hereditarily orderable.

There is an important topological characterization of orderability by J. van Dalen and E. Wattel [6]. By a *nest*, van Dalen and Wattel meant a collection that is linearly ordered by set containment. A nest $\mathcal{N}$ is *interlocking* if whenever a member $N_0 \in \mathcal{N}$ has $N_0 = \bigcap\{N \in \mathcal{N} : N \neq N_0 \text{ and } N_0 \subseteq N\}$, then N_0 also satisfies $N_0 = \bigcup\{N \in \mathcal{N} : N \neq N_0, N \subseteq N_0\}$. Van Dalen and Wattel [6] proved the following theorem.

Theorem 3.5. *A T_1 space is orderable if and only if it has a sub-base that is the union of two nests, each of which is interlocking.*

That theorem ought to play a key role in studies of hereditary orderability and should give an even shorter proof of the theorem of Purisch and Hirata and Kemoto.

Acknowledgment. We want to thank the referee whose comments significantly improved this paper.

References

[1] Ryszard Engelking, *General Topology.* Translated from the Polish by the author. 2nd ed. Sigma Series in Pure Mathematics, 6. Berlin: Heldermann Verlag, 1989.

[2] H. Herrlich, *Ordnungsfähigkeit total-diskontinuierlicher Räume*, Math. Ann. **159** (1965), no. 2, 77–80.

[3] Yasushi Hirata and Nobuyuki Kemoto, *Orderability of subspaces of well-orderable topological spaces*, Topology Appl. **157** (2010), no. 1, 127–135.

[4] S. Purisch, *The orderability and suborderability of metrizable spaces*, Trans. Amer. Math. Soc. **226** (1977), 59–76.

[5] ———, *Scattered compactifications and the orderability of scattered spaces. II*, Proc. Amer. Math. Soc. **95** (1985), no. 4, 636–640.

[6] J. van Dalen and E. Wattel, *A topological characterization of ordered spaces*, General Topology and Appl. **3** (1973), 347–354.

(Bennett) Department of Mathematics & Statistics; Texas Tech University; Lubbock, TX 79409

E-mail address: harold.bennett@ttu.edu

(Lutzer) Mathematics Department; College of William and Mary; Williamsburg, VA 23187

E-mail address: djlutz@wm.edu