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CONNECTEDNESS OF THE REPRESENTATION SPACE FOR CONTINUA

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Abstract. In *On representation spaces*, a forthcoming article, José G. Anaya, Félix Capulín, Włodzimierz J. Charatonik, and Fernando Orozco-Zitli have introduced the representation space $\mathcal{C}$ of all continua (up to homeomorphism). Here, we reproduce the argument that it is a topological space, and then we investigate its connectedness properties. Specifically, we show that $\mathcal{C}$ has exactly two components, and we demonstrate that the subspace $\mathcal{N}$, consisting of all nondegenerate continua, is itself connected and even path connected. Moreover, we show that there exists a single continuum L such that $\mathcal{N}$ is the closure of the class of $\{L\}$.

1. Introduction

In topology of metric spaces, ε-maps occur naturally. For instance, they appear as projections of inverse sequences and in the definitions or in the important characterizations of the properties, such as dimension, arc-likeness, tree-likeness, etc. If a space X admits, for each $\varepsilon > 0$, an ε-map onto some member of a class $\mathcal{P}$, then we think of X being "near" $\mathcal{P}$. This sense of being "near" has been proven in [1] to yield a topological structure on the representation space $\mathcal{C}$ of all continua, that is, the collection of the equivalence classes of mutually homeomorphic continua. We reproduce this argument in section 2 for the convenience of the reader. In section 3, we show that $\mathcal{C}$ has exactly two components: the collection $\mathcal{N}$ of the equivalence classes of all nondegenerate continua, and the isolated class that includes all singletons. The collection $\mathcal{N}$ is shown

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to be path connected and to be the closure of the class of a single locally connected continuum.

The representation space for continua has been recently studied by various authors. It was investigated at the First Polish-Mexican Workshop in Continuum Theory by José G. Anaya, Félix Capulín, Włodzimierz J. Charatonik, and Fernando Orozco-Zitli [1]. Félix Capulín, Raúl Escobedo, Fernando Orozco-Zitli, and Isabel Puga [3] have examined the very interesting related concept of an ε-property – in our context, these properties are the closed subsets of $\mathcal{C}$. In an unpublished paper, Jorge Bustamante, Raúl Escobedo, and Janusz R. Prajs [2], in 2003, defined the closure operator we investigate here and studied some of its properties.

2. Definitions and Notation

In this section, we set the stage for the construction of the representation space for continua as presented in [1], and we reproduce the argument that it is a topological space in order to investigate, in section 3, its connectedness properties.

In this article, the term “mapping” means “continuous function.”

Let $I = [0,1]$ be the unit interval and let (I^ω, d) denote the Hilbert cube, i.e., the product of denumerably many copies of I endowed with the product metric. Let t_d denote the topology defined on the Hilbert cube by this metric, d. It is well known that the metrizable space (I^ω, t_d) is universal in the class of all (metrizable) continua; that is, if (X, ρ) is any metric continuum, then there is a topological embedding from X into I^ω. Let $\cong$ denote the equivalence relation on the power set of I^ω that is defined by the notion of homeomorphism; i.e., two subspaces of the Hilbert cube are equivalent under $\cong$ if and only if they are homeomorphic. Then, by $\mathcal{C}_\cong$, we mean the factor set $\{C \subseteq I^\omega \,|C \text{ is a continuum}\}/\cong$, and by $\mathcal{C}$, we generally mean any specific set of representatives for $\cong$; for our purposes, $\mathcal{C}$ is such a set of representatives of all subcontinua of the Hilbert cube.

Definition 2.1. Let $\mathcal{P}$ be a subset of $\mathcal{C}$ and let $X \in \mathcal{C}$. Then we write $X \in \mathrm{Cl}(\mathcal{P})$ to mean that

$$(\forall \varepsilon > 0)\,(\exists X_\varepsilon \in \mathcal{P})\,(\exists f_\varepsilon : X \to X_\varepsilon)\,[f_\varepsilon \text{ is a surjective } \varepsilon\text{-mapping}]$$

We will prove that Cl is a (topological) closure operator, but to do so, we will use the following result.

Lemma 2.2. *Let $\varepsilon > 0$ and let $f : X \to Y$ be an ε-mapping between compact metric spaces X and Y. Then*

$$(\exists \delta > 0)\,(\forall A \subseteq Y)\left[\mathrm{diam}(A) < \delta \Longrightarrow \mathrm{diam}\left(f^{-1}(A)\right) < \varepsilon\right]. \tag{2.1}$$

Proof. Assume not, and then for each n, let $x_n, y_n \in X$ with $d_X(x_n, y_n) \geq \varepsilon$, but $d_Y(f(x_n), f(y_n)) \leq \frac{1}{n}$. Because of compactness, by taking subsequences, we may assume that $x, y \in X$ satisfy $x = \lim_n x_n$ and $y = \lim_n y_n$. Then $d_X(x, y) \geq \varepsilon$ and $d_Y(f(x), f(y)) = 0$. That is, $f(x) = f(y)$, contrary to the fact that f is an ε-mapping. $\square$

In fact, it is not difficult to see that any function between compact spaces that satisfies (2.1) must be an ε-mapping, so Lemma 2.2 provides an alternative characterization of ε-mappings, loosely phrased as the following principle: "pre-images of small sets are small." Now, as promised, we use this in proving part (d) of the following theorem.

Theorem 2.3. *The operator* Cl *is a topological closure operator. That is, for all* $\mathcal{A}, \mathcal{B} \subseteq \mathcal{C}$, *we have*

(a) $\mathrm{Cl}(\emptyset) = \emptyset$;
(b) $\mathcal{A} \subseteq \mathrm{Cl}(\mathcal{A})$, *(i.e.,* Cl *is extensive);*
(c) $\mathrm{Cl}(\mathcal{A} \cup \mathcal{B}) = \mathrm{Cl}(\mathcal{A}) \cup \mathrm{Cl}(\mathcal{B})$, *(i.e.,* Cl *preserves finite unions);*
(d) $\mathrm{Cl}(\mathrm{Cl}(\mathcal{A})) = \mathrm{Cl}(\mathcal{A})$, *(i.e.,* Cl *is idempotent).*

It follows that $\tau = \{\mathcal{C} \setminus \mathrm{Cl}(\mathcal{A}) | A \subseteq \mathcal{C}\}$ *is a topology on* $\mathcal{C}$.

Proof. The argument showing that (a) holds is trivial.

For (b), use $X = X_\varepsilon$ and set f_ε equal to the identity on $X \in \mathcal{A}$.

For (c), first note that Cl is *monotone*; i.e., $\mathcal{A} \subseteq \mathcal{B}$ implies $\mathrm{Cl}(\mathcal{A}) \subseteq \mathrm{Cl}(\mathcal{B})$. It then follows that $\mathrm{Cl}(\mathcal{A}) \cup \mathrm{Cl}(\mathcal{B}) \subseteq \mathrm{Cl}(\mathcal{A} \cup \mathcal{B})$. For the reverse inclusion, let P be any member of $\mathrm{Cl}(\mathcal{A} \cup \mathcal{B})$, so that if $\varepsilon > 0$, then there are $P_\varepsilon \in \mathcal{A} \cup \mathcal{B}$ and a surjective ε-mapping $f_\varepsilon : P \to P_\varepsilon$. Choose a sequence $\varepsilon_n > 0$ that converges to 0, and let $\{n_k\}_{k \in \mathbb{N}}$ a strictly increasing infinite sequence such that either for each k, $P_{\varepsilon_{n_k}} \in \mathcal{A}$ or for each k, $P_{\varepsilon_{n_k}} \in \mathcal{B}$. By doing this, we see that $P \in \mathrm{Cl}(\mathcal{A})$ or $P \in \mathrm{Cl}(\mathcal{B})$. Part(c) follows.

Finally, to see that (d) is satisfied, note first that (b) implies that $\mathrm{Cl}(\mathcal{A}) \subseteq \mathrm{Cl}(\mathrm{Cl}(\mathcal{A}))$. For the reverse inclusion, let $X \in \mathrm{Cl}(\mathrm{Cl}(\mathcal{A}))$, and let $\varepsilon > 0$ be given. Let $X_\varepsilon \in \mathrm{Cl}(\mathcal{A})$, and let $f_\varepsilon : X \to X_\varepsilon$ be a surjective ε-mapping, and let $\delta > 0$ be given as in Lemma 2.2. Because $X_\varepsilon \in \mathrm{Cl}(\mathcal{A})$, let $X_\delta \in \mathcal{A}$ and let $g : X_\varepsilon \to X_\delta$ be a surjective δ-mapping. Let $z \in X_\delta$. Then $\mathrm{diam}\left(\mathrm{g}^{-1}(\mathrm{z})\right) < \delta$, so that $\mathrm{diam}\left(f^{-1}[g^{-1}(z)]\right) < \varepsilon$. We are done. $\square$

Now, let us describe a characterization of the interior operator for the resulting topological space.

Proposition 2.4. *Let* int *denote the interior operator in the topological space* $\mathcal{C}_{\cong}$, *with the topology determined by the closure operator* Cl. *Then, for any space* X *and any property* $\mathcal{P}$, *we have*

$$X \in \mathrm{int}(\mathcal{P}) \Longleftrightarrow \exists \varepsilon > 0 \left[\textit{all } \varepsilon - \textit{images of } X \textit{ are in } \mathcal{P} \right].$$

Consequently, every continuous invariant, i.e., any property preserved by continuous surjections, is an open property.

Proof. To see this, formally negate the definition of the closure of the complement of $\mathcal{P}$. We leave the details to the reader. $\square$

Remark 2.5. The topology τ on $\mathcal{C}$ from Theorem 2.3 does not satisfy any of the usually considered separation axioms, not even the T_0 axiom. In particular, it leads to unusual three-element paths in $\mathcal{C}$ described in Corollary 3.5 in the next section. Nevertheless, this topology occurs naturally on the representation space of some of the most intuitive classes of metric spaces, the class of continua. Thus, we have an uncommon case when a space, which would be considered by some authors "pathological," captures essential qualities of a class of very intuitive and applicable metrizable topological spaces.

3. Components of $\mathcal{C}$

Here, we show that $\mathcal{C}$ has exactly two components, and we demonstrate that, in $\mathcal{C}$, the subspace of all nondegenerate continua is itself pathwise connected.

First, we make an observation, as follows.

Remark 3.1. For any property $\mathcal{P} \subseteq \mathcal{C}$, let $\mathcal{P}_{\mathcal{I}}$ be the property of being an inverse limit of members of $\mathcal{P}$ with surjective bonding maps. Some authors call members of $\mathcal{P}_{\mathcal{I}}$ "$\mathcal{P}$ representable continua." Then the property $\mathcal{P}_{\mathcal{I}}$ is a subset of the closure of $\mathcal{P}$.

Remark 3.2. The class of polyhedra is dense in $\mathcal{N}$. This follows from the fact that every non-degenerate continuum can be represented as an inverse limit of polyhedra with surjective bonding mappings. See [7] and [5].

For the following proof of our main theorem, we define a continuum L as follows. First, note that there are only countably many mutually non-homeomorphic polyhedra. Enumerate a class of representatives of all polyhedra, $\mathcal{P} = \{P_1, P_2, ...\}$, and let L be the wedge union of these spaces, $L = \bigcup_{j=1}^{\infty} P_j$, with junction point p. Precisely, we assume that $P_i \cap P_j = \{p\}$ for $i \neq j$ and $\lim_{n\to\infty} \operatorname{diam}(P_n) = 0$. Clearly, L is a locally connected continuum.

Theorem 3.3. *If X is a nondegenerate continuum, then X is L-like.*

Proof. We will show that for every $\varepsilon > 0$, there is a surjective ε-mapping $f : X \to L$. Thus, fix $\varepsilon > 0$. Let j be a positive integer, and, by [7], let $g : X \to P_j$ be a surjective ε-mapping, and choose δ so that the conclusion of Lemma 2.2 is satisfied for g in place of f_ε. Let A be a closed δ-neighborhood of the point p in P_j which is an absolute retract, and let I be an interval with $I \cap P_j = \{p\}$. Since $A \cup I$ is also an absolute extensor, let $e : A \to A \cup I$ be a surjective map, which fixes the boundary of A. Let s map I onto the closure of the complement of P_j in L. Define h on P_j by setting $h(x) = s(e(x))$ if $x \in A$, but so that h fixes $P_j \setminus A$. Then $h \circ g$ is the desired ε-mapping from X onto L. □

Corollary 3.4. *The set $\{L\}$ is dense in $\mathcal{N}$. Thus the density of $\mathcal{N}$ is 1.*

Corollary 3.5. *The space $\mathcal{N}$ is path connected. Indeed, for any nondegenerate continua X and Y, there is a three-element path from X to Y through L.*

Proof. Let $\alpha(0) = X$, $\alpha(t) = L$ for $0 < t < 1$, and $\alpha(1) = Y$. Then α is such a path. □

Remark 3.6. The results of Theorem 3.3 and corollaries 3.4 and 3.5 can also be reproduced for any collection $\mathcal{P}'$ of polyhedra such that for each finite set of members of $\mathcal{P}'$, its wedge is in $\mathcal{P}'$. Indeed, a construction similar to that of L leads to a locally connected continuum $L_{\mathcal{P}'}$ in $\mathrm{Cl}(\mathcal{P}')$ such that every member in $\mathrm{Cl}(\mathcal{P}')$ is $L_{\mathcal{P}'}$-like. In particular, the class of cell-like continua, the class of continua of dimension less than or equal to n, and the intersection of these two classes, namely the class of cell-like continua of dimension less than or equal to n, have density 1 and are path connected.

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