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A COUNTABLE PRODUCT THEOREM FOR ANTI-PONDEROUS SPACES

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Abstract. An anti-ponderous space is a space in which every infinite countably compact subspace has a convergent sequence that is not eventually constant. It is shown that a countable family of anti-ponderous spaces has an anti-ponderous product, with only a modest separation axiom assumed. This still leaves a wide range of uncertainty if the Continuum Hypothesis is not assumed.

This paper continues a theme begun in [1] and [3], on the effects of cardinal functions on the convergent sequences in countably compact spaces, without necessarily assuming that the spaces in question are Hausdorff or better. Often, as in the case of the main new result of this paper (Theorem 7), weaker axioms are adequate for various theorems, and stronger axioms do not seem to lead to stronger results. Theorem 7 is a natural variation on the classical result that the product of countably many sequentially compact spaces is sequentially compact. It involves a natural weakening of sequential compactness, given in Definition 4.

Definition 1. A space is *countably compact* if every countable open cover has a finite subcover. Equivalently, every infinite sequence has a cluster point.

Definition 2. A space is *sequentially compact* if every infinite sequence has a convergent subsequence.

The following property was introduced in [3], and some cardinal invariants associated with it were discussed there.

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Definition 3. A *ponderous* space is an infinite, countably compact space in which every convergent sequence is eventually constant.

Definition 4. An *anti-ponderous* space is one that has no ponderous subspaces.

In other words, every countably compact subspace has a convergent sequence which is not eventually constant. A formally stronger condition will be given in Corollary 17 near the end of this paper, replacing "not eventually constant" with "one to one."

The following modest separation axiom plays a key role in our main theorem.

Definition 5. A space has *Property S* if every convergent sequence has a unique cluster point.

Clearly, every ponderous space is T_1, as is every space with Property S. As is well known, every KC space (that is, a space in which every compact subset is closed) has Property S, and every Hausdorff space is KC.

Well-known examples of ponderous spaces include $\beta\mathbb{N}$ and its countably compact subspaces, including the Novak-Teresaka example of a countably compact Tychonoff space whose square is not countably compact (nor even pseudocompact). The famous Efimov Conjecture, still not completely disproven, is equivalent to the conjecture that every ponderous compact Hausdorff space contains a copy of $\beta\mathbb{N}$.

The following unpublished improvement on the classical theorem mentioned above was discovered by Jan Pelant and Petr Simon, and independently by the author.

Theorem 6. *The small uncountable cardinal $\mathfrak{h}$ is the least cardinality of a family of sequentially compact spaces whose product is not sequentially compact.*

Here $\mathfrak{h}$ stands for the least height of a splitting tree on ω [2]. A recent characterization of $\mathfrak{h}$ related to the above theorem is the result that $\mathfrak{h}$ is the least cardinality (also the least net weight) of a countably compact space that is not sequentially compact, and a KC example was given of a space that witnesses this [3].

On the other hand, new ideas are needed for any strengthening of the following result along these lines.

Theorem 7. *Let $\{X_n : n \in \omega\}$ be a countable family of spaces with Property S. If none of the X_n contains a ponderous subspace, then neither does their product.*

Since Property S is productive, Corollary 8 immediately follows.

Corollary 8. *The class of anti-ponderous spaces with Property S is countably productive.*

This is as far as we are able to go at present.

Problem 9. *Is there a model of set theory in which every product of $\aleph_1$ anti-ponderous spaces with Property S is anti-ponderous?*

This is in marked contrast to the characterizations of $\mathfrak{h}$ related above. For example, Martin's Axiom implies $\mathfrak{h} = \mathfrak{c}$, and Martin's axiom is compatible with $\mathfrak{c}$ being "arbitrarily large." We are somewhat better off in the opposite direction.

Problem 10. *Is it consistent that there is a family of $< \mathfrak{s}$ anti-ponderous spaces with property S, whose product contains a ponderous subspace?*

Here $\mathfrak{s}$ is the splitting number, which satisfies $\mathfrak{h} \leq \mathfrak{s} \leq \mathfrak{c}$ (see [2], [4], [7]). Ponderous compact subsets of $2^{\mathfrak{s}}$ have been constructed in models of $\aleph_1 = \mathfrak{s} < \mathfrak{c}$ (see [5], [6]). In particular, they were shown to exist in any model obtained by adding random reals to a model of CH in the usual way [6]; hence, $\mathfrak{c}$ could be arbitrarily large.

The proof of Theorem 7 involves the following lemmas, the first two of which are elementary and well known.

Lemma 11. *Every countable, countably compact space is compact and sequentially compact.*

Lemma 12. *A continuous image of a (countably) compact space is (countably) compact.*

Lemma 13. *In any space X with Property S, the range of a convergent sequence with infinite range, together with its (unique) limit point, is a closed copy of $\omega + 1$.*

Proof. Since X is T_1, every finite subset is closed discrete, and the range of a convergent sequence σ, minus its limit point p, is closed discrete in its relative topology. Therefore, any one-to-one map from ω to $ran(\sigma) \setminus \{p\}$ is a homeomorphism, and its obvious extension to $\omega + 1$ is also a homeomorphism since every neighborhood of p contains all but finitely many points in the range of σ, but is not in the closure of any finite subset of $ran(\sigma) \setminus \{p\}$. □

By the way, Property S should not be confused with the weaker property (to which Lemma 13 does not extend) that no sequence can converge to more than one point.

Proof of Theorem 7. Let $X = \Pi_{n=1}^{\infty} X_n$ and suppose no X_n has a ponderous subspace. Let π_n denote the projection of X to X_n, and let Y be an infinite, countably compact subspace of X.

We will show that Y has a one-to-one convergent sequence. Our strategy will be to find countable, compact Hausdorff subspaces $Z_n \subset X_n$ such that $\Pi_{n=1}^{\infty} Z_n =: Z$ contains an infinite subset of Y. This subset Y_ω, in turn, will be the intersection of a descending chain of infinite closed subsets Y_n of Y such that $\pi_i \to Y_{n+1} \subset Z_i$ for all $i = 1, \ldots, n$. Thus, Y_ω is a subset of the compact metrizable space $Z = \Pi_n Z_n$. Let σ be a one-to-one sequence whose range is in Y_ω, and let τ be a subsequence that converges in Z. Since Y is countably compact, the unique limit of this sequence is in Y, and so we will be done.

To ensure that $Y_\omega = \bigcap_{n=1}^{\infty} Y_n$ is infinite, we will choose points $\{p_n : n \in \mathbb{N}\}$ by induction in Y, making sure p_n is in Y_m for all m and n. We will also define $Z_n \subset X_n$ and $Y_n \subset Y_{n-1}$ by induction. Let $Y_1 = Y$. If $\pi_1^{\rightarrow} Y$ is finite, let $Z_1 = \pi_1^{\rightarrow} Y$. Otherwise, let Z_1 be a copy of $\omega + 1$ in $\pi_1^{\rightarrow} Y$. In either case, let $z_1 \in Z_1$ and choose $p_1 \in Y_1$ such that $\pi_1(p_1) = z_1$.

Let $Y_2 = \pi_1^{\leftarrow} Z_1$. Clearly, Y_2 is an infinite closed subspace of Y containing p_1. Let $Z_2 = \pi_2^{\rightarrow} Y_2$ if $\pi_2^{\rightarrow} Y_2$ is finite. Otherwise, let Z_2 be a copy of $\omega + 1$ in $\pi_2^{\rightarrow} Y_2$ that includes $\pi_2(p_1)$. (In spaces satisfying Property S, adding finitely many points to a copy of $\omega + 1$ still produces a copy of $\omega + 1$.) In either case, let p_2 be a point in Y_2 other than p_1.

In general, suppose that we have defined infinite closed, hence countably compact subspaces $Y_i \subset Y$ and $Z_i \subset \pi_i^{\rightarrow} Y_i$ such that $Y_j \supset Y_i$ for $j \le i \le n$ and such that Z_n is either finite and all of $\pi_n^{\rightarrow} Y_n$, or a copy of $\omega + 1$. Also suppose that $\pi_n(p_i) \in Y_n$ for $i \le n$. Let $Y_{n+1} = \pi_n^{\leftarrow} Z_n$. Then either $Y_{n+1} = Y_n$ or Y_{n+1} is the preimage of a copy of $\omega + 1$.

Let $Z_{n+1} = \pi_{n+1}^{\rightarrow} Y_{n+1}$ if this image is finite; otherwise, let Z_{n+1} be a copy of $\omega + 1$ in $\pi_{n+1}^{\rightarrow} Y_{n+1}$ that includes $\pi_{n+1}(p_i)$ for $i = 1, \ldots, n$. In either case, let $p_{n+1} \in \pi_{n+1}^{\leftarrow} Z_{n+1}$, $p_{n+1} \ne p_i$ for $i = 1, \ldots, n$. It is routine to show that the induction hypotheses are satisfied.

In the trivial case where all but finitely many of the Z_n are one-point spaces, their product is a countable, compact space, and the chain will stabilize at a Y_n where the $|Z_m| = 1$ for all $m > n$, and we will be done, as indicated in the second paragraph. Y_n is homeomorphic to an infinite subset of the metric space $\Pi_{i=1}^{n} Z_i$.

Otherwise, we continue the induction for infinitely many steps. When it is done, we will have ensured that Y_ω is infinite, inasmuch as it contains all the p_i. □

Problem 9 asked whether we can improve on "countable" in Theorem 7. The proof does not seem to lend itself to continuation beyond ω; even going to $\omega + 1$ seems to involve sacrificing an essential part of Y_ω and practically starting over. The following problem is also open.

Problem 14. *Can we weaken the topological conditions on the spaces X_n in Theorem 7?*

The condition on uniqueness of limit points of convergent sequences was used in showing that Y_ω is closed in Y and hence countably compact, that Y_ω is sequentially closed in Z, and in getting Z to be metrizable by making sure that a one-to-one sequence and its limit point form a copy of $\omega + 1$. The third use is not really essential since we could also show Z to be sequentially compact using Lemma 11 and the classical theorem that a countable product of sequentially compact spaces is sequentially compact.

On the other hand, the first two uses do seem to call for something along the lines of Property S. We need to avoid a situation where any bijective sequence in Y_ω that is convergent in Z also has so many cluster points that it has no convergent subsequences in Y. However, the following theorem suggests that a weakening of Property S that does not imply the T_1 property might be usable. We recall the following concept from [3].

Definition 15. A space is *almost ponderous* if it is countably compact and has no convergent one-to-one sequences.

Theorem 16. *Every almost ponderous space contains a ponderous (hence T_1) subspace.*

Proof. First, we show that if X is almost ponderous, it contains an almost ponderous T_0 subspace. Let $x_1 \equiv x_2$ if and only if x_1 and x_2 have the same set of open neighborhoods. This is clearly an equivalence relation on X, and equivalent points are "topologically indistinguishable." In particular, if x is a cluster point of $\langle x_n \rangle$ and $y_n \equiv x_n$ for all n, then x is also a cluster point of $\langle y_n \rangle$, as is any other member of the equivalence class of x.

Now, if some equivalence class $[x]$ modulo $\equiv$ were infinite, then any one-to-one sequence from ω to $[x]$ would converge to each point of $[x]$, a contradiction. So the axiom of choice for finite sets can be applied to get an infinite subspace Y of X that meets each equivalence class in exactly one point. Clearly, Y is T_0 and almost ponderous by the comments at the end of the preceding paragraph.

The proof will be finished once we find an almost ponderous T_1 subspace of Y, because every almost ponderous T_1 space is ponderous. This is because, in a T_1 space, a convergent sequence cannot repeat more than one point infinitely many times.

For this, we use the following order relation: $z \leq y$ if and only if $z \in c\ell\{y\}$. The minimal closed sets (if any) in a space are of the form $c\ell\{y\}$

for some y, and if Y is T_0, then these sets are singletons, and $\leq$ is a partial order on Y. If Y is almost ponderous, then no point is below infinitely many others, and any descending ω-sequence with respect to $\leq$ would have a cluster point to which the sequence converges, a contradiction. So every point is above some minimal point, and the set of minimal points is infinite, and they are easily seen to constitute a closed subspace Z which is, thus, almost ponderous. Finally, Z is T_1 because singletons are closed. □

Corollary 17. *A space is anti-ponderous if and only if every infinite countably compact subspace contains a convergent one-to-one sequence.*

Proof. The "if" part is obvious. Conversely, by Theorem 16, no anti-ponderous space can contain an almost ponderous subspace. □

We close with two problems which may be easier to answer than the previous ones. The second is reminiscent both of Problem 10 and of the Efimov Conjecture.

Problem 18. *Is there a countable family of anti-ponderous, countably compact spaces whose product is not countably compact?*

Problem 19. *Is it consistent that the product of fewer than $\mathfrak{c}$ spaces can contain a copy of $\beta\mathbb{N}$ without any of the factors containing one?*

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