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NETS DEFINING ENDS OF TOPOLOGICAL SPACES

MATT INSALL AND MAŁGORZATA ANETA MARCINIAK

Abstract. This article uses nets to define ends of topological spaces, modifying the sequence-based idea introduced in 1931 by Hans Freudenthal. We connect the lack of ends with the compactness of a topological space more firmly than did Freudenthal's original presentation. Specifically, we prove the following generalization of Freudenthal's related result:

Main Theorem: *If X is a connected, exquisitely remotely locally connected, and locally boundary compact topological space, then $e(X) = 0$ implies that X is compact.*

1. Introduction

The concept of an end formally captures the intuitive notion one might informally call "a hole at infinity" of a topological space. The following definition of ends was introduced by Hans Freudenthal in [1] and used by Heinz Hopf in [3]. This concept has been applied successfully in the theory of complex manifolds; see, for example, [2] and [4].

Definition 1.1 (Ends of a topological space)**.** Let X be a connected topological space. Let $\mathcal{F}$ be the family of all monotone decreasing sequences $\sigma = \{U_s\}_{s\in\mathbb{N}} : \mathbb{N} \to \tau$, where τ is the given topology on X, such that

(1) each $U \in \sigma[\mathbb{N}]$ is nonempty connected and has compact boundary, and

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(2) $\bigcap \overline{\sigma}[\mathbb{N}] = \emptyset$, where $\overline{\sigma}$ is the closed set-valued mapping defined by $\overline{\sigma}(j) = \overline{\sigma(j)}$, for $j \in \mathbb{N}$.

Freudenthal introduced the relation $\sim$ on $\mathcal{F}$ given by $\{U_n\}_{n\in\mathbb{N}} \sim \{V_m\}_{m\in\mathbb{N}}$ if and only if for every $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $U_n \subseteq V_m$. The set $\mathcal{F}/\sim$ of $\sim$-sections (see [5]) of this relation is the set of Freudenthal *ends* of X.

It can be shown that $\sim$ is an equivalence relation (see [1] and [4]). In fact, this relation can be defined equivalently as follows: $\{U_n\}_{n\in\mathbb{N}} \sim \{V_m\}_{m\in\mathbb{N}}$ if and only if for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $V_m \subseteq U_n$. (Thus, in particular, the $\sim$-sections are $\sim$-equivalence classes, and the set $\mathcal{F}/\sim$ is a quotient of $\mathcal{F}$ by the equivalence relation $\sim$.) The number of ends in the sense intended by Freudenthal, i.e., the cardinality of the set $\mathcal{F}/\sim$, will herein be denoted by $e_F(X)$.

This article reformulates the definition given by Freudenthal, using nets instead of sequences. We denote by $e(X)$ the cardinality of the set of ends of X, according to our revised definition. We also introduce new local ("at infinity") notions related to connectedness and compactness, and prove, under mild assumptions, that having no ends implies compactness.

Theorem 1.2. *If X is a connected, exquisitely remotely locally connected, and locally boundary compact topological space, then $e(X) = 0$ implies that X is compact.*

2. Definitions and Examples

As indicated above, the following definition of ends is a modification of that introduced by Freudenthal. The main difference is in replacing sequences of open sets by nets of open sets. For the sequel, let us say that a net ν in τ is *nested* provided that for any indices α and β in the domain of ν, if α precedes β, then $\nu(\beta) \subseteq \nu(\alpha)$.

Definition 2.1 (Ends of a topological space)**.** Let X be a connected topological space, and let $\mathcal{F}$ be the family of all nested nets $\nu = \{U_\lambda\}_{\lambda\in\Lambda} : \Lambda \to \tau$, where Λ is a directed set, such that

(1) each $U \in \nu[\Lambda]$ is nonempty and connected, with compact boundary, and

(2) $\displaystyle\bigcap_{\lambda\in\Lambda} \overline{U}_\lambda = \emptyset$.

In a manner similar to Freudenthal's approach, we define the equivalence relation $\sim$ on $\mathcal{F}$ as follows: Let $\mu = \{U_\lambda\}_{\lambda\in\Lambda}$ and $\nu = \{V_\gamma\}_{\gamma\in\Gamma}$ be members of $\mathcal{F}$, with domains Λ and Γ, respectively. Then $\mu \sim \nu$ if and only if, for every $\gamma \in \Gamma$, there exists $\lambda \in \Lambda$ such that $\mu(\lambda) \subseteq \nu(\gamma)$. The elements of the set of equivalence classes $\mathcal{F}/\sim$ are the *ends* of X. The number of ends is denoted by $e(X)$.

In the original definition, condition (1) is accompanied by the additional assumption that each U_λ has nonempty boundary. This assumption is not necessary, since it is implied by the connectedness of X, as shown in Lemma 3.7 below.

The short proof of the following proposition justifies that $\sim$ is an equivalence relation.

Proposition 2.2. *The relation $\sim$ is an equivalence relation.*

Proof. The relation $\sim$ is reflexive, since for every net $\{U_\lambda\}_{\lambda\in\Lambda}$ and any index $\lambda \in \Lambda$, $U_\lambda \subseteq U_\lambda$. If $\{U_\lambda\}_{\lambda\in\Lambda} \sim \{V_\gamma\}_{\gamma\in\Gamma}$ and $\{V_\gamma\}_{\gamma\in\Gamma} \sim \{W_\kappa\}_{\kappa\in K}$, then for any κ there exists γ so that $V_\gamma \subseteq W_\kappa$. For this γ, there further exists λ so that $U_\lambda \subseteq V_\gamma$. In particular, for any κ, there exists λ so that $U_\lambda \subseteq W_\kappa$, which proves the transitivity of $\sim$.

For symmetry, we must prove that if $\{U_\lambda\}_{\lambda\in\Lambda} \sim \{V_\gamma\}_{\gamma\in\Gamma}$, then $\{V_\gamma\}_{\gamma\in\Gamma} \sim \{U_\lambda\}_{\lambda\in\Lambda}$. That is, we assume that, for every $\gamma \in \Gamma$, there exists $\lambda \in \Lambda$ so that $U_\lambda \subseteq V_\gamma$, and we must show that, for every $\lambda \in \Lambda$, there exists $\gamma \in \Gamma$ such that $V_\gamma \subseteq U_\lambda$. Assume otherwise; i.e., there exists λ_0 so that for all γ, V_γ is not a subset of U_{λ_0}, which implies that $V_\gamma \setminus U_{\lambda_0} \neq \emptyset$. Notice that for each γ

$$\partial(V_\gamma \setminus U_{\lambda_0}) \subseteq \left[(\partial U_{\lambda_0}) \cap \overline{V}_\gamma\right] \cup \left[(\partial V_\gamma) \cap (X \setminus U_{\lambda_0})\right].$$

Consequently, $(\partial U_{\lambda_0}) \cap \overline{V}_\gamma$ is a compact set in X for any γ, and for $\gamma_2 \succeq \gamma$ we have $(\partial U_{\lambda_0}) \cap \overline{V}_{\gamma_2} \subseteq (\partial U_{\lambda_0}) \cap \overline{V}_\gamma$. Therefore,

$$\bigcap_{\gamma\in\Gamma} \left[(\partial U_{\lambda_0}) \cap \overline{V}_\gamma\right] \neq \emptyset.$$

Then

$$\emptyset \neq \bigcap_{\gamma\in\Gamma} \left[(\partial U_{\lambda_0}) \cap \overline{V}_\gamma\right] \subseteq \bigcap_{\gamma\in\Gamma} \overline{V}_\gamma = \emptyset,$$

which is a contradiction. □

Definition 2.3. Two nets $\{U_\lambda\}_{\lambda\in\Lambda}$ and $\{V_\gamma\}_{\gamma\in\Gamma}$ are disjoint if $U_\lambda \cap V_\gamma = \emptyset$ for all $\lambda \in \Lambda$ and $\gamma \in \Gamma$.

The proof of the following theorem is analogous to the proof of Theorem 2.3.3 in [4].

Theorem 2.4. *Let X be a connected and locally connected topological space and, for each $\iota \in I$, let $\nu_\iota = \{U^{(\iota)}_\lambda\}_{\lambda\in\Lambda_\iota}$ be a net that defines an end in X. Further assume that this family $(\nu_\iota)_{\iota\in I}$ satisfies the following properties:*

(1) *If* $\iota_1, \iota_2 \in I$ *and* $\iota_1 \neq \iota_2$, *then* ν_{ι_1} *is disjoint with* ν_{ι_2}.
(2) *For any choice function* $\phi : I \to \bigcup_{\iota \in I} \Lambda_\iota$, *the set* $X \setminus \bigcup\{U^{(\iota)}_{\phi(\iota)} : \iota \in I\}$ *is compact.*

Then $e(X) = |I|$; *i.e., the collection of ends of* X *has the same cardinality as* I.

Proof. By way of contradiction, let $\zeta = \{U_\lambda\}_{\lambda\in\Lambda}$ be a net that defines an end in X and suppose that ζ is not equivalent to any ν_ι, $\iota \in I$. From the definition of nonequivalent nets there exists a choice function $\phi : I \to \bigcup_{\iota\in I} \Lambda_\iota$ such that for any $\lambda \in \Lambda$ and any $\iota \in I$, U_λ is not a subset of $U^{(\iota)}_{\phi(\iota)}$. We claim that for each $\lambda \in \Lambda$,

$$U_\lambda \setminus \left(\bigcup_{\iota\in I} U^{(\iota)}_{\phi(\iota)}\right) \neq \emptyset.$$

Assume otherwise; i.e., for some $\widetilde{\lambda} \in \Lambda$

$$U_{\widetilde{\lambda}} \setminus \left(\bigcup_{\iota\in I} U^{(\iota)}_{\phi(\iota)}\right) = \emptyset.$$

Then for this $\widetilde{\lambda} \in \Lambda$, we have

$$U_{\widetilde{\lambda}} \subseteq \bigcup_{\iota\in I} U^{(\iota)}_{\phi(\iota)}.$$

If $|I| = 1$, then the contradiction is immediate since ζ is not equivalent to ν_ι, where $I = \{\iota\}$. If $|I| > 1$, then assumption (1) implies that $U_{\widetilde{\lambda}}$ is not connected or it is a subset of exactly one set $U^{(\iota)}_{\phi(\iota)}$. In both cases, we obtain a contradiction. Thus, for all $\lambda \in \Lambda$, $U_\lambda \setminus \left(\bigcup_{\iota\in I} U^{(\iota)}_{\phi(\iota)}\right) \neq \emptyset$. Then $W_\lambda = \overline{U}_\lambda \setminus \left(\bigcup_{\iota\in I} U^{(\iota)}_{\phi(\iota)}\right)$ is a closed nonempty subset of a compact set $X \setminus \left(\bigcup_{\iota\in I} U^{(\iota)}_{\phi(\iota)}\right)$. Note that $\{W_\lambda\}_{\lambda\in\Lambda}$ is a nested net of nonempty closed subsets of a compact set. Since $\emptyset \neq \bigcap_{\lambda\in\Lambda} W_\lambda \subseteq \bigcap_{\lambda\in\Lambda} \overline{U}_\lambda = \emptyset$, we obtain a contradiction with the assumption that $\zeta = \{U_\lambda\}_{\lambda\in\Lambda}$ defines an end. □

Let us consider a few examples of connected, noncompact spaces and their ends.

Example 2.5. Clearly, from Theorem 2.4, the complex plane $\mathbb{C}$ has exactly one end described by the net consisting of the sets of the form $U_r = \{z \in \mathbb{C} : |z| > r\}$ with $r > 0$. Let $X = \mathbb{C} \setminus \mathbb{Z}$, then each integer $a \in \mathbb{Z}$ yields another end described by $U^a_\lambda = \{z \in X : |z - a| < \lambda\}$, where

$0 < \lambda < \frac{1}{2}$. Assumption (2) of Theorem 2.4 holds; in particular, the sets of the form $X \setminus \left(U_r \cup \bigcup_{a \in \mathbb{Z}, \lambda \in \Lambda} U_\lambda^a \right)$ are compact. Clearly, assumption (1) of Theorem 2.4 holds, since $U_\lambda^{a_1} \cap U_\lambda^{a_2} = \emptyset$ for $a_1 \neq a_2$, so $e(X) = \aleph_0$.

Example 2.6. Let now $X = \mathbb{C} \setminus \mathbb{Q}$. Then X has an end defined by the net given by $U_r = \{z \in X : |z| > r\}$, where r ranges over all positive irrational numbers. Other ends are determined by the nets consisting of the sets of the form $U_\lambda^a = \{z \in X : |z - a| < \lambda\}$ for $a \in \mathbb{Q}$ and where λ ranges over all positive irrational numbers. Note that for fixed λ_1 and λ_2 and any a_1 there exists a_2 so that $U_{\lambda_1}^{a_1} \cap U_{\lambda_2}^{a_2} \neq \emptyset$, so assumption (1) of Theorem 2.4 is not fulfilled. However, for any fixed a_1 and λ_1, there exist a_2 and λ_2 so that $U_{\lambda_1}^{a_1} \cap U_{\lambda_2}^{a_2} = \emptyset$. Clearly $X \setminus \left(U_r \cup \bigcup_{a \in \mathbb{Q}, \lambda \in \Lambda} U_\lambda^a \right)$ is compact; thus, assumption (2) of Theorem 2.4 holds. Hence, $e(X) \geq \aleph_0$.

Example 2.7. Let $X = \mathbb{C} \setminus \mathbb{K}$, where $\mathbb{K}$ is the Cantor set in the interval $[0, 1]$. Then X has an end defined by the net consisting of the sets of the form $U_r = \{z \in X : |z| > r\}$ with $r > 2$. Other ends are defined by the nets consisting of the sets $U_\lambda^a = \{z \in X : |z - a| < \lambda\}$, where $a \in \mathbb{K}$ and $(a \pm \lambda) \notin \mathbb{K}$. The nets retain similar properties as in Example 2.6, but in this case, $e(X) \geq \mathfrak{c}$.

Since the sets $\overline{U}_\lambda$ from the definition of ends occur frequently hereafter, we introduce terminology to help us consistently refer to them.

Definition 2.8 (Truffles and Morsels)**.** By a *truffle* in a topological space X, we mean a closed set with compact boundary, whose interior is connected, but not compact. A *morsel* in X is a set whose closure is a truffle.

3. Local Boundary Compactness

The usual definition of local compactness can be relaxed to a version that focuses on the boundary.

Definition 3.1. Let X be a topological space and let p be any point of X. Then X is *locally boundary compact* at p if it has a basis of neighborhoods with compact boundaries at this point. We say that X is locally boundary compact if it is locally boundary compact at each of its points.

Spaces that are locally compact are locally boundary compact. But spaces that are locally boundary compact need not be locally compact. Topological spaces $\mathbb{C} \setminus \mathbb{Q}$ and $\mathbb{C} \setminus \mathbb{K}$ as described in Example 2.6 and Example 2.7 are locally boundary compact but not locally compact. Note that irrational real numbers in $\mathbb{C}\setminus\mathbb{Q}$ do not have compact neighborhoods. Numbers from $[0,1]$ that are not in the Cantor set in $\mathbb{C}\setminus\mathbb{K}$ do not have compact neighborhoods.

Boundary compact sets admit properties that are crucial for the construction of nets defining ends.

Lemma 3.2. *Let X be a noncompact space and let $\mathcal{C}$ be an open cover of X that consists of boundary compact sets and does not have a finite subcover. Then, for any $U \in \mathcal{C}$, we have $\overline{U} \neq X$.*

Proof. Assume that $\overline{U} = X$. Since ∂U is compact and $\mathcal{C}$ is an open cover, we can choose a finite subcover $C_1,\ldots,C_n$ for ∂U. Then $C_1,\ldots,C_n,U$ is a finite subcover of $\mathcal{C}$ that covers $\overline{U} = X$. Thus, $\mathcal{C}$ has a finite subcover which contradicts the assumption. □

Definition 3.3. Let $(P,\preceq)$ be an ordered set and let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a net in P. Also, let $\{y_\alpha\}_{\alpha\in A}$ be a net in P, where A is a directed subset of Λ. We say that $\{y_\alpha\}_{\alpha\in A}$ is a *refinement* of $\{x_\lambda\}_{\lambda\in\Lambda}$ provided that for each $\lambda\in\Lambda$, there is $\alpha\in A$ such that $y_\alpha \preceq x_\lambda$.

Definition 3.4. A topological space X is *remotely locally boundary compact* provided that for each nested net $\mathcal{N} = \{U_\lambda\}_{\lambda\in\Lambda}$ of open sets with nonempty boundary such that $\bigcap_{\lambda\in\Lambda}\overline{U}_\lambda = \emptyset$, there is a refinement of $\mathcal{N}$ consisting of nonempty boundary compact open sets.

In general, local boundary compactness and remote local boundary compactness are not equivalent properties; however, it is also not clear whether one property implies the other under some natural additional assumptions about the topological space. If X is a connected topological space and $E(X)$ is the collection of all ends of X, then the space $Y = X\cup E(X)$ has the following natural topology: For each end $e\in E(X)$, a neighborhood base at e is the collection of all sets of the form $\{e\}\cup U$, where U is a member of some net that defines e. For each point $p\in X$, a neighborhood base at p is the collection of all open $U\subseteq X$ for which $p\in U$.

Theorem 3.5. *The topological space $Y = X\cup E(X)$ is locally boundary compact if and only if X is both locally boundary compact and remotely locally boundary compact.*

Proof. If X is locally boundary compact and remotely locally boundary compact, then each point in Y has a base of neighborhoods that consists of boundary compact sets. Thus, Y is locally boundary compact. If Y is locally boundary compact, then X, as a topological subspace, is locally boundary compact as well. Moreover, since each point $e\in E(X)\subseteq Y$ has a base of neighborhoods that is boundary compact, X is remotely locally boundary compact. □

Here is an example of a locally boundary compact space which is not remotely locally boundary compact.

Example 3.6. Let $X = [0,1] \times (0,1]$. Then X is locally boundary compact, since it is locally compact. Let us choose the net consisting of the sets of the form $U_\lambda = (0,\lambda) \times (0,\lambda)$ for $\lambda \in (0,1)$. Then $\bigcap_{\lambda\in\Lambda} \overline{U}_\lambda = \emptyset$, but the net $\{U_\lambda\}_{\lambda\in\Lambda}$ does not admit a refinement consisting of nonempty open sets with compact boundaries. Then X is not remotely locally boundary compact.

In the sequel, we shall use this concept of local boundary compactness to generalize a result of Freudenthal that relates the number of ends to compactness of the space. To do so, we need a lemma that helps us find connected (open) subsets with compact boundary of open sets with compact boundary. In fact, it is through the application of this lemma that the value of defining the notion of a truffle will become observable.

Let us recall that a subset U of a topological space is not connected if there exist mutually separated sets A and B (i.e., $\overline{A} \cap B = A \cap \overline{B} = \emptyset$) in X so that $U = A \cup B$ (Theorem 26.4 [6]).

Lemma 3.7. *Let U be an open set with nonempty compact boundary. If $U = A \cup B$, where A and B are mutually separated, then A and B have compact boundaries and at least one has nonempty boundary. In particular, a proper nonempty open subset of a connected space has nonempty boundary.*

Proof. If $U = A \cup B$ with $\overline{A} \cap B = A \cap \overline{B} = \emptyset$, then

$$\partial U = \overline{U} \setminus U = (\overline{A} \cup \overline{B}) \setminus (A \cup B) = (\overline{A} \setminus (A \cup B)) \cup (\overline{B} \setminus (A \cup B)) =$$
$$(\partial A \setminus B) \cup (\partial B \setminus A)) = \partial A \cup \partial B,$$

which proves that ∂A and ∂B are closed subsets of a nonempty compact set ∂U. Thus, they are compact, and at least one of them is nonempty.

For the second part of the lemma, note that if a nonempty open set A has empty boundary, then $\overline{A} = A$, and $X = A \cup (X \setminus A)$ is a decomposition of X into disjoint nonempty open sets, which is not possible in a connected space. □

Even if U is open, there is no guarantee that at least one of its connected components is open. If such is the case, however, the following remark applies.

Remark 3.8. Let an open set U have nonempty compact boundary. Then all open connected components of U have compact boundaries. If, in addition, the topological space X is connected, then each open connected component of U has a nonempty compact boundary. Note that these open connected components of U are morsels.

4. Connectedness Properties

Any refinement of a net of open sets retains the property of empty intersection of the closures.

Lemma 4.1. *Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be a nested net of nonempty open sets with $\bigcap_{\lambda\in\Lambda}\overline{U}_\lambda=\emptyset$, and let $\{V_\alpha\}_{\alpha\in A}$ be a refinement of the given net. Then $\bigcap_{\alpha\in A}\overline{V}_\alpha=\emptyset$.*

Proof. Since, for any $\lambda\in\Lambda$, there exists $\alpha\in A$ such that $V_\alpha\subseteq U_\lambda$, we have $\bigcap_{\alpha\in A}\overline{V}_\alpha\subseteq\bigcap_{\lambda\in\Lambda}\overline{U}_\lambda=\emptyset$. □

There are cases in which such a refinement does not consist solely of connected sets.

Definition 4.2. A topological space X is *remotely locally connected* if any nested net of nonempty open sets $\{U_\lambda\}_{\lambda\in\Lambda}$, such that $\bigcap_{\lambda\in\Lambda}\overline{U}_\lambda=\emptyset$, has a nested refinement $\{V_\alpha\}_{\alpha\in A}$, which consists of nonempty connected open sets.

Still, such refinements might not consist of boundary-compact sets, even if the original net had this property.

Definition 4.3. A topological space X is *exquisitely remotely locally connected* if any nested net of nonempty open sets $\{U_\lambda\}_{\lambda\in\Lambda}$, such that $\bigcap_{\lambda\in\Lambda}\overline{U}_\lambda=\emptyset$, has a nested refinement $\{V_\alpha\}_{\alpha\in A}$, which consists of nonempty open morsels.

Remotely locally connected spaces need not be locally connected. However, locally connected spaces are always remotely locally connected since any connected component of an open set is then open (Theorem 27.9 [6]). We have the following result.

Theorem 4.4. *Let X be a connected topological space and let $\{U_\lambda\}_{\lambda\in\Lambda}$ be a nested net of nonempty open, boundary compact sets, such that $\bigcap_{\lambda\in\Lambda}\overline{U}_\lambda=\emptyset$. If one of the sets U_{λ_0} is locally connected, then there exists a nested refinement $\{V_\alpha\}_{\alpha\in A}$ of the net $\{U_\lambda\}_{\lambda\in\Lambda}$, which consists of nonempty open morsels.*

Proof. If U_{λ_0} is locally connected, then all connected components of all sets U_λ for $\lambda\succeq\lambda_0$ are open by Theorem 27.9 [6] and have compact boundaries by Remark 3.8. Thus, it is possible to choose a nested refinement $\{V_\alpha\}_{\alpha\in A}$ consisting of nonempty open morsels. □

Example 4.5. Let W denote the Warsaw circle. The space $W \setminus \{p\}$, obtained by removing a given point p on the limit bar of W, is not remotely locally connected since there is no base of connected neighborhoods of p. The space $W \setminus \{x\}$, obtained from the Warsaw circle by removing a given point x on the arc of W, is remotely locally connected but not locally connected. However, any nested net of open, boundary compact sets $\{U_\lambda\}_{\lambda\in\Lambda}$ in $W \setminus \{x\}$, such that $\bigcap_{\lambda\in\Lambda} \overline{U}_\lambda = \emptyset$, admits a refinement consisting of nonempty open morsels. Hence, this space is exquisitely remotely locally connected.

Definition 4.6. Let X be a topological space and let $\mathfrak{A}$ be the collection of nested nets, $\mu = \{U_\lambda\}_{\lambda\in\Omega}$, of nonempty open sets in X such that $\bigcap_{\lambda\in\Omega} \overline{U}_\lambda = \emptyset$. Define on $\mathfrak{A}$ the binary relation

$$\mu = \{U_\lambda\}_{\lambda\in\Omega} \;\sim\; \nu = \{V_\gamma\}_{\gamma\in\Gamma}$$

if and only if, for all $\gamma \in \Gamma$, there exists $\lambda \in \Omega$ such that $U_\lambda \subseteq V_\gamma$. Then the $\sim$-sections are called *semi-ends* of X and the set $\mathfrak{A}$ is the set of semi-ends of X.

Note that by not requiring compact boundaries in the above definition, we, in general, lose symmetry of the relation $\sim$. However, we do have the following result.

Theorem 4.7. *Let X be a connected and exquisitely remotely locally connected topological space with a semi-end. Then X has an end.*

Proof. Let $\mu = \{U_\lambda\}_{\lambda\in\Lambda}$ be a net from the definition of a semi-end. Since X is exquisitely remotely locally connected, there is a refinement $\{V_\alpha\}_{\alpha\in A}$ that consists of open morsels and defines an end since $\bigcap_{\alpha\in A} \overline{V}_\alpha = \emptyset$. □

5. Compactness Vs. Ends

Recall that the notation $e(X)$ describes the number of ends of X and the notation $e_F(X)$ describes the number of ends in the sense formalized by Freudenthal. Clearly, $e_F(X) \leq e(X)$, since a sequence is a specific kind of net. In the following theorems, let us focus on topological spaces with $e(X) = 0$.

Theorem 5.1. *If X is compact, then $e(X) = 0$.*

Proof. If X is compact, then there is no nested net of closed sets with empty intersection. Thus, condition (2) of Definition 2.1 cannot be fulfilled. □

Note: The preceding theorem generalizes a result of Freudenthal, in which only ends defined by *sequences* of open sets are considered.

In general, the converse of this theorem is not true. Consider the following example.

Example 5.2. Let $X = (0,1) \times (0,1) \cup \{(0,0)\}$ be an open square with the point $\{(0,0)\}$ attached. Then X is not compact, but $e(X) = 0$, since all nested nets in X that consist of boundary compact open sets have nonempty intersection.

The following proves the converse of this theorem using some assumptions about the space X.

Theorem 5.3. *If X is a connected, exquisitely remotely locally connected, and locally boundary compact topological space, then $e(X) = 0$ implies that X is compact.*

Proof. In the spirit of Freudenthal's work, we shall show that if X is not compact, then $e(X) \neq 0$. Assume that X is not compact, and let $\mathcal{C}$ be an open cover of X that has no finite subcover. If X is locally boundary compact, assume that each member of $\mathcal{C}$ is boundary compact. Let $\mathcal{F} = \{\bigcup_{j=1}^{n} F_j | n < \omega, F_1, \dots F_n \in \mathcal{C}\}$. Note that if X is locally boundary compact, then each member of $\mathcal{F}$ is boundary compact. Note that $X = \bigcup \mathcal{F}$, but $X \notin \mathcal{F}$. Let

$$\mathcal{D} = \{X \setminus \overline{F} | F \in \mathcal{F}\},$$

and note that in case X is locally boundary compact, each member of $\mathcal{D}$ (and each of its open connected components, according to Remark 3.8) is boundary compact and nonempty by Lemma 3.2. Order $\mathcal{D}$ by reverse inclusion, and for all $\delta \in \mathcal{D}$, let $U_\delta = \delta$; then $\{U_\delta\}_{\delta \in \mathcal{D}}$ is a nested net in the topology of X. Now

$$\bigcap_{\delta \in \mathcal{D}} \overline{U}_\delta = \bigcap_{\delta \in \mathcal{D}} (X \setminus F_\delta) = X \setminus \bigcup_{\delta \in \mathcal{D}} F_\delta = \emptyset.$$

We may produce an end of X as follows: Since X is remotely locally connected, let $\{V_\alpha\}_{\alpha \in A}$ be a nested refinement of $\{U_\delta\}_{\delta \in \mathcal{D}}$ which consists of nonempty connected open sets. Since X is locally boundary compact and is exquisitely remotely locally connected, we can assume that $\{V_\alpha\}_{\alpha \in A}$ consists of open morsels. Clearly, since $\bigcap_{\alpha \in A} \overline{V}_\alpha \subseteq \bigcap_{\delta \in \mathcal{D}} \overline{U}_\delta$, we have $\bigcap_{\alpha \in A} \overline{V}_\alpha = \emptyset$, so $e(X) \neq 0$. □

6. Further Research

Recall that since any sequence defining an end in the sense of Freudenthal also defines an end in the sense introduced in this article, $e_F(X) \leq e(X)$. The question arises whether the converse holds under other natural hypotheses. For example, in [1], second countability of the original space

X is assumed, although Freudenthal comments that this assumption can be removed. Here we have removed it; moreover, we have generalized the concept of an end in doing so. But, in fact, one may now ask whether reinstatement of certain relaxed versions of first or second countability axioms can be employed in place of some of the hypotheses in our main theorem. Such investigation would lead to interesting results, in our view.

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