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LINEAR HOMEOMORPHISMS OF FUNCTION SPACES AND PROPERTIES CLOSE TO COUNTABLE COMPACTNESS

HÉCTOR DAVID RAMÍREZ HERNÁNDEZ AND OLEG OKUNEV

ABSTRACT. Several topological properties of countable compactness type are proved to be preserved by the relations of l-equivalence, M-equivalence and A-equivalence of topological spaces.

All spaces considered in this paper are assumed to be Tychonoff (=completely regular Hausdorff). We use terminology and notation as in [3].

Given a space X, $C_p(X)$ is the linear topological space of all continuous real-valued functions on X equipped with the topology of pointwise convergence, that is, the topology of the subspace of the set $\mathbb{R}^X$ of all real-valued functions on X with the Tychonoff product topology. Two spaces X and Y are called *l-equivalent* if the space $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. See [2] for basic properties and constructions and some essential results in the theory of spaces $C_p(X)$.

The symbols $F(X)$ and $A(X)$ denote the free topological group and the free Abelian topological group of a Tychonoff space X in the sense of Markov [5]. Two Tychonoff spaces X and Y are called *M-equivalent* (respectively *A-equivalent*) if their free topological groups (free Abelian topological groups), are topologically isomorphic. The notions of M-equivalence and A-equivalence were introduced by Graev in [4] where he constructed the first known example of non-homeomorphic M-equivalent spaces and posed a general problem: *What topological properties are preserved by the relation of M-equivalence?* (We say that a topological property $\mathcal{P}$ is preserved by M-, A- or l-equivalence if for any pair X, Y of spaces which are equivalent in the respective sense, X has $\mathcal{P}$ if and only if Y has $\mathcal{P}$).

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M-equivalence of two spaces implies their A-equivalence, and A-equivalence implies l-equivalence [1]. As far as the authors know, it is not known to date whether M-equivalence and A-equivalence are the same. There are several topological properties that are known to be preserved by A-equivalence but not by l-equivalence (such as connectedness and, in the class of polyhedra, homology groups), but covering properties and cardinal invariants tend to behave similarly under these equivalences.

There are many results related to preservation and non-preservation of various topological properties under the relations of M-, A-, l- and other related equivalences (see [2]), however, little is known about preservation or non-preservation by these relations of countable compactness and similar properties; apparently, because of the "non-functional" nature of these properties. While compactness and pseudocompactness are known to be preserved by linear homeomorphisms of function spaces [9], the question about the preservation by l-equivalence of countable compactness [1] is still open. Note that the uncountable version of countable compactness turned out to be somewhat easier: it is known that the existence of a closed uncountable discrete subspace is not preserved by M-equivalence (Example 3.17 in [6]), and the existence of a closed uncountable discrete subspace in some finite power of the space is preserved even by homeomorphisms (not necessarily linear) of the spaces of continuous functions [7].

In this article we prove the preservation of some properties similar to countable compactness by A-equivalence, and of one of them by the relation of l-equivalence. Several arguments in the article rely on the following statement proved by V. Uspenskij in [9]:

Theorem 0.1. *A space X is pseudocompact if and only if $C_p(X)$ is a countable union of its totally bounded subspaces. In particular, if X is pseudocompact, and a space Y is l-equivalent to X, then Y is pseudocompact.*

1. Countable compactness type properties and free topological groups

The following statement summarizes basic properties of the free Abelian topological groups [4].

Theorem 1.1. *Let X be a Tychonoff topological space. The free Abelian topological group over X is the (unique up to a topological isomorphism) topological group $A(X)$ with the following properties:*

(1) $A(X)$ *without topology is the free Abelian group with the set of generators* X*,*
(2) X *is a closed subspace of* $A(X)$*,*
(3) *If* $f\colon X \to G$ *is a continuous mapping of* X *to an Abelian topological group* G*, then the homomorphism* $f^*\colon A(X) \to G$ *extending* f *is continuous.*

Thus, $A(X)$ is the set of all formal linear combinations $k_1x_1+\cdots+k_nx_n$, where $n \in \mathbb{N}$, $k_1,\ldots, k_n$ are integers, and $x_1, \ldots, x_n$ are elements of X, equipped with the natural group operation. The *length* of an element $a \neq 0$ of $A(X)$ is the minimum number $|k_1|+\cdots+|k_n|$ over all representations $a = k_1x_1 + \cdots + k_nx_n$; the length of the neutral element 0 (represented by the empty linear combination), is by definition equal to 0. The sets $A_n(X)$, $n \geq 0$, are defined as the sets of all elements of $A(X)$ of length $\leq n$. It is easy to see that $A_n(X) \subset A_{n+1}(X)$, $A(X) = \bigcup_{n\in\mathbb{N}} A_n(X)$, and that $A_n(X)$ is a continuous image of $(X\oplus(-X)\oplus\{0\})^n$ under the addition mapping; here $-X$ is a homeomorphic copy of X whose elements when applying the mapping are interpreted as inverses of the corresponding elements of X. In particular,

Proposition 1.2. *Let* $\mathcal{P}$ *be a topological property invariant with respect to finite unions, finite powers and continuous images. If* X *has* $\mathcal{P}$*, then for every* $n \in \mathbb{N}$*,* $A_n(X)$ *has* $\mathcal{P}$*.*

The following fact is proved in [4]:

Theorem 1.3. *Let* K *be a compact subspace of* $A(X)$*. Then for some* $n \in \mathbb{N}$*,* $K \subset A_n(X)$*.*

We need the following slight generalization of this:

Proposition 1.4. *Let* X *be a topological space,* $A(X)$ *the free Abelian topological group over* X*, and* Y *a pseudocompact subspace of* $A(X)$*. Then there is* $n \in \omega$ *such that* $Y \subset A_n(X)$*.*

Proof. Let $g\colon X \hookrightarrow \beta X$ be the inclusion of X into its Stone-Čech compactification. Consider the homomorphism $h\colon A(X) \to A(\beta X)$ that extends g. Note that h is one-to one and continuous, and $h(A_n(X)) = A_n(\beta X) \cap \langle X\rangle$ where $\langle X\rangle$ is the subgroup of $A(\beta X)$ generated by X. Since Y is pseudocompact, the image $h(Y)$ is pseudocompact. On the other hand, since βX is compact, $F(\beta X)$ is σ-compact. Thus, the closure of $h(Y)$ in $A(\beta X)$ is σ-compact and pseudocompact, hence compact. By Theorem 1.3, there is an $n \in \omega$ such that $h(Y) \subset A_n(\beta X)$. Thus, $Y \subset A_n(X)$. □

As a corollary, we obtain

Theorem 1.5. *Let $\mathcal{P}$ be a topological property invariant with respect to continuous images, closed subspaces, finite products and finite unions. Then the property $\mathcal{P}$ is preserved by A-equivalence within the class of pseudocompact spaces.*

Proof. Let X be a space with a property $\mathcal{P}$ as in the statement of the theorem. Let Y be a pseudocompact space A-equivalent to X, and let $i\colon A(Y) \to A(X)$ be a topological isomorphism. Then by Theorem 1.4, there is $n \in \omega$ such that $i(Y) \subset A_n(X)$, and by Proposition 1.2, $A_n(X)$ has $\mathcal{P}$. Since Y is a closed subspace of $A(Y)$, $i(Y)$ is a closed subspace of $A_n(X)$, so $i(Y)$ has $\mathcal{P}$. It follows that Y has $\mathcal{P}$. □

Recall definitions of some countable compactness type properties [10].

Definition 1.6. A space X is *totally countably compact* if every infinite set in X contains an infinite subset with compact closure,

Definition 1.7. A space X is called *ω-bounded* if every countable set in X has compact closure.

Definition 1.8. Let p be a free ultrafilter on ω. A space X is *p-compact* if every sequence in X has a p-limit point in X (a point z is a *p-limit* of a sequence $\{\, x_n : n \in \omega \,\}$ if for every neighborhood U of z, the set $\{n \in \omega : x_n \in U\,\}$ is an element of p).

Definition 1.9. A space X is called *sequentially compact* if every sequence in X has a convergent subsequence.

Obviously, each of the above properties implies countable compactness, in particular, pseudocompactness.

Definition 1.10. A topological space X is called *initially κ-compact* if every open cover $\mathcal{U}$ of X with $|\mathcal{U}| \le \kappa$ has a finite subcover.

The following fact is proved, e.g., in [10].

Theorem 1.11. *The properties of total countable compactness, ω-boundedness, p-compactness, sequential compactness and initial κ-compactness are invariant with respect to continuous images, closed subspaces and finite unions.*

Recall that a cardinal number κ is said to be a *strong limit cardinal number* if $2^\lambda < \kappa$ whenever $\lambda < \kappa$. The following fact is proved, *e.g.*, in [8].

Theorem 1.12. *Let $\{X_a : a \in A\}$ be a family of initially κ-compact spaces, where κ is a singular, strong limit cardinal number. Then $X = \Pi\{X_a : a \in A\}$ is initially κ-compact.*

Theorem 1.13. *The properties of total countable compactness, ω-boundedness, p-compactness, sequential compactness and initial κ-compactness, with κ a singular strong limit cardinal number, are invariant with respect to finite products.*

Thus,

Theorem 1.14. *The properties of total countable compactness, ω-boundedness, p-compactness, sequential compactness and initial κ-compactness, with κ a singular strong limit cardinal, are preserved by A-equivalence.*

Proof. Let $\mathcal{P}$ be one of the above properties, X a space with the property $\mathcal{P}$, and Y a space A-equivalent to X. By Uspenskij's theorem [9], the space Y is pseudocompact. Now from Theorems 1.5 and 1.11 follows that Y has $\mathcal{P}$. □

Similarly,

Theorem 1.15. *If all finite powers of X are countably compact, and Y is a space A-equivalent to X, then all finite powers of Y are countably compact.*

Theorem 1.16. *Let κ be an infinite cardinal. If X^κ is countably compact, and Y is a space A-equivalent to X, then Y^κ is countably compact.*

2. l-EQUIVALENCE AND ω-BOUNDEDNESS

In this section we prove the following theorem.

Theorem 2.1. *Let X and Y be l-equivalent spaces. If X is ω-bounded, then Y is ω-bounded.*

Recall that for a Tychonoff space X, $L_p(X)$ is the dual space of the linear topological space $C_p(X)$ with the weak topology (in different words, $L_p(X)$ is the subspace of $C_p(C_p(X))$ consisting of all continuous linear functions on $C_p(X)$).

We need the following properties of $L_p(X)$ (see *e.g.* [2]):

Proposition 2.2. *Let X be a Tychonoff space. Then*

(1) *$C_p(X)$ is the dual space of $L_p(X)$ with the weak topology;*
(2) *The function $\hat{}\colon X \to L_p(X)$ defined by the rule $\hat{x}(f) = f(x)$ for all $x \in X$ and $f \in C_p(X)$ is an embedding of X into $L_p(X)$; moreover, $\hat{X}$ is a closed Hamel base for $L_p(X)$, and every linear function $h\colon L_p(X) \to \mathbb{R}$ is continuous if and only if the restriction $h|\hat{X}$ is continuous.*
(3) *If F is a closed subspace of X, then the linear subspace of $L_p(X)$ generated by $\hat{F}$ is closed in $L_p(X)$.*

From (1) it follows that two spaces X and Y are l-equivalent if and only $L_p(X)$ and $L_p(Y)$ are linearly homeomorphic.

We will identify X and $\hat{X}$; thus, X is a closed topological subspace of $L_p(X)$, and every element of $L_p(X)$ has a unique up to the order of summands representation of the form $\phi = \lambda_1 x_1 + \cdots + \lambda_k x_k$ where $\lambda_1, \ldots, \lambda_k \in \mathbb{R} \setminus \{0\}$ and $x_1, \ldots x_k$ are distinct points of X. We denote $\operatorname{supp} \phi = \{x_1, \ldots, x_k\}$, and, for a set $S \subset L_p(X)$, $\operatorname{supp} S = \bigcup\{\operatorname{supp} \phi : \phi \in S\}$. Thus, $\operatorname{supp} S$ is the smallest subset A of X such that S is contained in the linear span of A in $L_p(X)$. It is obvious that for a countable S, $\operatorname{supp} S$ is at most countable.

Proposition 2.3. [2] *Let L be a linear topological space that contains a Tychonoff subspace X. Then there is a linear homeomorphism of L onto $L_p(X)$ fixing the points of X if and only if X is a Hamel base of L and every continuous function $f\colon X \to \mathbb{R}$ has a linear continuous extension over L.*

Recall that a linear topological space is called *weak* if its topology is generated by continuous linear real functions on X. In particular, $L_p(X)$ is a weak space.

Proposition 2.4. [2] *If L is a weak linear topological space, then every continuous function $f\colon X \to L$ has a unique linear continuous extension to $L_p(X)$.*

In particular, for every continuous mapping $f\colon X \to Y$ there is a unique continuous linear mapping $f^{\#}\colon L_p(X) \to L_p(Y)$ such that $f^{\#}|X = f$.

Proposition 2.5. [2] *$L_p(X)$ is a countable union of products of finite powers of X with finite powers of the unit interval $[0,1]$. In particular, if X compact, then $L_p(X)$ is σ-compact.*

Proposition 2.6. *Let X be a pseudocompact space, βX its Stone-Čech compactification, and $i\colon X \hookrightarrow \beta X$ the standard embedding. Then the mapping $i^{\#}\colon L_p(X) \to L_p(\beta X)$ is an embedding of linear topological spaces.*

Proof. Let L be the subspace of $L_p(X)$ spanned by X. Then L is weak, because it is a linear topological subspace of the weak space $L_p(\beta X)$, so to prove that $i^{\#}$ is an embedding it is enough to prove that every continuous function $f\colon X \to \mathbb{R}$ has a continuous extension over L. Since X is pseudocompact, f is bounded, so there is a continuous extension $g\colon \beta X \to \mathbb{R}$ of f. Let $g^{\#}\colon L_p(\beta X) \to \mathbb{R}$ be the continuous linear extension of g, then $g^{\#}|L$ is the required continuous linear extension of f. □

Thus, for pseudocompact spaces X, we may view $L_p(X)$ as the subspace of $L_p(\beta X)$ spanned by X.

Proposition 2.7. *Let X and Y be pseudocompact spaces, βX and βY their Stone-Čech compactifications, and $h\colon L_p(X) \to L_p(Y)$ a linear homeomorphism. Then there is a linear homeomorphism $\tilde{h}\colon L_p(\beta X) \to L_p(\beta Y)$ such that $\tilde{h}|L_p(X) = h$.*

Proof. Since βY is compact, the space $L_p(\beta Y)$ is σ-compact. It follows that the closure B in $L_p(\beta Y)$ of the pseudocompact subspace $h(X)$ is compact. Let $h_1\colon \beta X \to B$ be the continuous extension of $h|X$, and $\tilde{h}\colon L_p(\beta X) \to L_p(\beta Y)$ the continuous linear extension of h_1.

Let $g = h^{-1}$. By a similar argument, there is a linear continuous $\tilde{g}\colon L_p(\beta Y) \to L_p(\beta X)$ such that $\tilde{g}|L_p(Y) = g$. Obviously, $(\tilde{g} \circ \tilde{h})|X = (g \circ h)|X = \mathrm{id}_X$, so by the continuity, $(\tilde{g} \circ \tilde{h})|\beta X = \mathrm{id}_{\beta X}$, and since βX is a Hamel base of $L_p(\beta X)$, $\tilde{g} \circ \tilde{h} = \mathrm{id}_{L_p(\beta X)}$. A symmetric argument shows that $\tilde{h}\tilde{g} = \mathrm{id}_{L_p(\beta Y)}$, so $\tilde{h}$ is a linear homeomorphism. □

We are now ready to prove the main theorem of this section.

Proof of Theorem 2.1. Let X and Y be l-equivalent spaces such that X is ω-bounded. Then X is pseudocompact, and by Uspenskij's Theorem [9], Y is also pseudocompact. Since X and Y are l-equivalent, by Proposition 2.7 there is a linear homeomorphism $h\colon L_p(\beta X) \to L_p(\beta Y)$ such that $h(L_p(X)) = L_p(Y)$. Let S be a countable set in Y. Put $B = h^{-1}(S)$. Then $B \subset L_p(X)$ and $\operatorname{supp} B \subset X$ is at most countable. Let K be the closure of $\operatorname{supp} B$ in X. Since X is ω-bounded, K is compact. Let E be the linear subspace of $L_p(X)$ generated by K. Then $B \subset E$ and E is closed in $L_p(\beta X)$, because K is closed in βX.

Put $F = h(E)$, then $F \subset L_p(Y)$, F is closed in $L_p(\beta Y)$, and $S \subset F$. It follows that S is contained in $F \cap \beta Y \subset L_p(Y) \cap \beta Y = Y$. Since F is closed in $L_p(\beta Y)$, $F \cap \beta Y$ is compact, so the closure of S in Y is compact. □

3. Some open problems

We do not know the answers to the following questions:

1. [1] *Let X and Y be l-equivalent spaces. If X is countably compact, must Y be countably compact?*

2. *Let X and Y be M-equivalent spaces. If X is countably compact, must Y be countably compact?*

3. *Let X and Y be l-equivalent spaces. If all finite powers of X are countably compact, must Y be countably compact?*

4. *Let X and Y be l-equivalent spaces. If X is sequentially compact, must Y be sequentially compact?*

5. *Let X and Y be l-equivalent spaces. If X is totally compact, must Y be totally compact?*

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