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TOPOLOGICAL STRUCTURES OF THE SPACE OF CONTINUOUS FUNCTIONS ON A NON-COMPACT SPACE WITH THE FELL TOPOLOGY

ZHONGQIANG YANG, SHAORUI HU AND GUO WEI

Abstract. For a non-compact locally compact separable and metrizable space X, let $\downarrow\mathrm{USC}(X)$ and $\downarrow\mathrm{C}(X)$ denote the collections of the hypographs of all upper semi-continuous maps and of all continuous maps from X to $[0,1]$, respectively. We shall give the topological structure of the pair $(\downarrow\mathrm{USC}(X), \downarrow\mathrm{C}(X))$ with the Fell topology.

1. Introduction and Main Results

One of the most important results in infinite-dimensional topology is to give the topological structure of function space $\mathrm{C}(X,Y)$, where $\mathrm{C}(X,Y)$ denotes the set of all continuous functions from a topological space X to another space Y, that is, to prove that $\mathrm{C}(X,Y)$ with some natural topologies are homeomorphic to certain classical topological spaces. For example, Kadec-Anderson Theorem states that, for the real line $\mathbb{R}$, $\mathrm{C}_u(X,\mathbb{R})$, the set $\mathrm{C}(X,\mathbb{R})$ equipped with the topology of uniform convergence, is homeomorphic to the Hilbert space $l_2 \approx \mathbb{R}^\omega$ ($\approx$ means homeomorphic) if X is an infinite compact metric space, see [8, 1]. Let $Q = [-1,1]^\omega$ be the Hilbert cube and let $\Sigma = \{(x_n) \in Q : \sup|x_n| < 1\}$ and $c_0 = \{(x_n) \in \Sigma : \lim\limits_{n\to\infty} x_n = 0\}$ be its two subspaces. Dobrowolski-Marciszewski-Mogilski Theorem states that $\mathrm{C}_p(X,\mathbb{R})$, the set $\mathrm{C}(X,\mathbb{R})$ equipped with the topology of pointwise convergence, is homeomorphic to c_0 if X is a countable non-discrete metric space, [4, 5], cf., [11, Theorem 6.12.15].

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In [13], the first author of the present paper considered some natural topologies on $\mathrm{C}(X) = \mathrm{C}(X, \mathbf{I})$, where $\mathbf{I} = [0,1]$ equipped with the usual topology and order. In fact, we considered a larger set $\mathrm{USC}(X) = \mathrm{USC}(X, \mathbf{I})$ consisting of all upper semi-continuous maps from X to $\mathbf{I}$. At first, for every $f \in \mathrm{USC}(X)$, let

$$\downarrow f = \{(x,t) \in X \times \mathbf{I} : t \leq f(x)\}$$

be the *hypograph* of f, which is a closed set in the product space $X \times \mathbf{I}$. Hence, for every $A \subset \mathrm{USC}(X)$,

$$\downarrow A = \{\downarrow f : f \in A\}$$

is a subset of the set $\mathrm{Cld}(X \times \mathbf{I})$ of all non-empty closed sets of $X \times \mathbf{I}$. It is well-known that, on the set $\mathrm{Cld}(X \times \mathbf{I})$, many natural topologies have been defined. The Fell topology and the topology induced by the Hausdorff metric, which will be defined in the last part of this section, are two of those. Therefore, we may introduce those topologies to the set $\downarrow A$ for $A \subset \mathrm{USC}(X)$. Consequently, we are able to investigate the pair $(\downarrow\mathrm{USC}(X), \downarrow\mathrm{C}(X))$ with various topologies. Use X_0 to denote the set of isolated points in X. When X is a compact metric space, the Fell topology on $\mathrm{Cld}(X \times \mathbf{I})$ can be induced by the Hausdorff metric. In [16][1] and [14], we proved that, for a compact metric space X,

$$(\downarrow\mathrm{USC}(X), \downarrow\mathrm{C}(X)) \approx \begin{cases} (\mathbf{I}^{|X|}, \mathbf{I}^{|X|}) & \text{if } X \text{ is finite;} \\ (Q, c_0) & \text{if } \overline{X_0} \neq X; \\ (Q, c_0 \cup (Q \setminus \Sigma)) & \text{otherwise,} \end{cases}$$

where $|X|$ denotes the cardinal number of X and $\overline{(\cdot)}$ is the closure-operator, and $(X,Y) \approx (A,B)$ means that there exists a homeomorphism $h : X \to A$ such that $h(Y) = B$. When X is a non-compact metric space, the Hausdorff metric may not be a metric on $\mathrm{Cld}(X \times \mathbf{I})$ since it is possible that $d_{\mathrm{H}}(A,B) = \infty$ for non-empty closed sets A, B in $X \times \mathbf{I}$. In [15], we considered the Fell-topology on $\mathrm{Cld}(X \times \mathbf{I})$ for a locally compact separable metric space X. Thus we may investigate the spaces $\downarrow\mathrm{USC}_F(X)$ and $\downarrow\mathrm{C}_F(X)$, which mean that the sets $\downarrow\mathrm{USC}(X)$ and $\downarrow\mathrm{C}(X)$ are equipped with the Fell topology. In [15], it was proved that $(\downarrow\mathrm{USC}_F(X), \downarrow\mathrm{C}_F(X)) \approx (Q, c_0)$ if X is a locally compact separable and metrizable space with $\overline{X_0} \neq X$. In the present paper, we consider the case that the set of isolated points is dense in X. The main result is:

[1] In fact, in [16], replacing c_0, $c_0^Q = \{(x_n) \in Q : \lim x_n = 0\}$ was used. But, by Lemma 2.8 in the present paper, $(Q, c_0^Q) \approx (Q, c_0)$.

Theorem 1.1. *If X is a non-compact, locally compact, separable and non-discrete metrizable space with $\overline{X_0} = X$, then*

$$(\downarrow\mathrm{USC}_F(X),\ \downarrow\mathrm{C}_F(X)) \approx (Q, c_0 \cup (Q \setminus \Sigma)).$$

By [16, Theorem 3], [14, Theorem 1], [15, Theorem 1] and Theorem 1.1 above, we have the following immediate corollary, which clarifies the topological structure of the pair $(\downarrow\mathrm{USC}_F(X),\ \downarrow\mathrm{C}_F(X))$ for every locally compact separable metric space X:

Corollary 1.2. *Let X be a locally compact separable metrizable space. Then*

$$(\downarrow\mathrm{USC}_F(X), \downarrow\mathrm{C}_F(X)) \approx \begin{cases} (\mathbf{I}^{|X|}, \mathbf{I}^{|X|}) & \textit{if } X \textit{ is discrete}; \\ (Q, c_0) & \textit{if } \overline{X_0} \neq X; \\ (Q, c_0 \cup (Q \setminus \Sigma)) & \textit{otherwise.} \end{cases}$$

For a metric space (X, d), let Cld(X) be the collection of all non-empty closed subsets of X. For each $U \subset X$, let

$$U^- = \{A \in \mathrm{Cld}(X) : A \cap U \neq \emptyset\},\ U^+ = \{A \in \mathrm{Cld}(X) : A \subset U\}.$$

The *Fell topology* on $\mathrm{Cld}(X)$, by definition, is generated by the subbase

$$\{U^-, (X \setminus K)^+ : U \subset X \text{ is open}, \mathrm{K} \subset \mathrm{X} \text{ is compact}\}.$$

For $A \in \mathrm{Cld}(X)$ and $\varepsilon > 0$, let

$$B(A, \varepsilon) = \{x \in X : d(x, a) < \varepsilon \text{ for some } a \in A\}.$$

The *Hausdorff metric* d_{H} on Cld(X) is defined by

$$d_{\mathrm{H}}(A, C) = \inf\{\varepsilon > 0 : A \subset B(C, \varepsilon) \text{ and } C \subset B(A, \varepsilon)\}, A, C \in \mathrm{Cld(X)}.$$

Although it is possible that $d_{\mathrm{H}}(A, C) = +\infty$, d_H may induce a topology on Cld(X).

2. Preliminaries

In this section, firstly, we give some concepts and facts on infinite-dimension topology. For more information, we refer to [10] and [11]. Secondly, some results occurred in bibliographies are given. Thirdly, we show some results on the Fell topology. Let $\omega = \{1, 2, \cdots\}$ be the set of all natural numbers and, as usual, let ω also denote the cardinality of this set. We always endow the Hilbert cube Q with the following metric:

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n}|x_n - y_n|$$

for $x = (x_n), y = (y_n) \in Q$. Let X, Y be any spaces, $k : Y \to \mathrm{USC}(X)$ a map. We define $\downarrow k : Y \to \downarrow\mathrm{USC}(X)$ by $\downarrow k(y) = \downarrow(k(y))$ for $y \in Y$. Similarly, we may define $\downarrow k : \downarrow\mathrm{USC}(Y) \to \downarrow\mathrm{USC}(X)$ for a map $k : \mathrm{USC}(Y) \to \mathrm{USC}(X)$. For a map $k : Y \times X \to \mathbf{I}$, we may use the same symbol k to represent the induced map $k : Y \to \mathbf{I}^X$.

Definition 2.1. Let X, Y be two separable and metrizable spaces and $\mathcal{U}$ an open cover of Y. Two continuous functions $f, g : X \to Y$ are said to be *$\mathcal{U}$-close* if for each $x \in X$, there exists $U \in \mathcal{U}$ such that $f(x), g(x) \in U$. A homotopy $H : X \times \mathbf{I} \to Y$ is called *$\mathcal{U}$-limited* if for each $x \in X$, there exists $U \in \mathcal{U}$ such that $H(\{x\} \times \mathbf{I}) \subset U$. Two continuous functions $f, g : X \to Y$ are called *$\mathcal{U}$-homotopic*, if there is a homotopy $H : X \times \mathbf{I} \to Y$ which is limited by $\mathcal{U}$ and connects f and g, i.e., $H_0 = f, H_1 = g$. If (Y, d) is a metric space and $\mathcal{U} = \{B(y, \frac{\varepsilon}{2}) : y \in Y\}$ for a fixed $\varepsilon > 0$, then the above $\mathcal{U}$-close and $\mathcal{U}$-homotopic are also called *ε-close* and *ε-homotopic*, respectively. f and g are ε-close is denoted by $d(f, g) < \varepsilon$.

Definition 2.2. A closed subset A of a separable metric space X is said to be a *Z-set* provided that for every open cover $\mathcal{U}$ of X and every function $f \in \mathrm{C}(Q, X)$ there is a function $g \in \mathrm{C}(Q, X)$ such that

(a) f and g are $\mathcal{U}$-close, and
(b) $g(Q) \cap A = \emptyset$.

A set in the space X is called a *Z_σ-set* if it can be written as a countable union of Z-sets. The collection of Z-sets and Z_σ-sets in X are denoted by $Z(X)$ and $Z_\sigma(X)$, respectively. An embedding $f : X \to Y$ is called a *Z-embedding* if $f(X) \in Z(Y)$.

Definition 2.3. Let $\mathcal{M}_0$ denote the class of compact metrizable spaces. For a topological class $\mathcal{C}$ of separable metrizable spaces, let $(\mathcal{M}_0, \mathcal{C})$ denote the class of all pairs (M, C) such that $M \in \mathcal{M}_0$, $C \in \mathcal{C}$ and C is a subspace of M. Let X be a subspace of Q. We give the following definitions:

(1) X is said to be *$\mathcal{C}$-universal* in Q (or say that (Q, X) is *$(\mathcal{M}_0, \mathcal{C})$-universal*) if for every pair $(M, C) \in (\mathcal{M}_0, \mathcal{C})$, there exists an embedding $f : M \to Q$ such that $f^{-1}(X) = C$.

(2) X is said to be *strongly $\mathcal{C}$-universal* in Q (or say that (Q, X) is *strongly $(\mathcal{M}_0, \mathcal{C})$-universal*) provided for any $(M, C) \in (\mathcal{M}_0, \mathcal{C})$, any continuous map $f : M \to Q$, any closed subset K of M such that $f|_K : K \to Q$ is a Z-embedding and any open cover $\mathcal{U}$ of Q, there is a Z-embedding $g : M \to Q$ such that $g|_K = f|_K$, $g^{-1}(X) \setminus K = C \setminus K$ and g and f are $\mathcal{U}$-close.

(3) X is said to be *reflexively universal* in Q (or (Q, X) is *reflexively universal*) if for any continuous map $f : Q \to Q$, any closed subset K of Q such that $f|_K : K \to Q$ is a Z-embedding and any open cover $\mathcal{U}$ of Q, there is a Z-embedding $g : Q \to Q$ such that $g|_K = f|_K$, $g^{-1}(X) \setminus K = X \setminus K$ and g and f are $\mathcal{U}$-close.

Definition 2.4. (Q, X) is called *$(\mathcal{M}_0, \mathcal{C})$-absorbing* if

(a) $X \in \mathcal{C}$;
(b) X is contained in a Z_σ-set of Q, and
(c) X is strongly $\mathcal{C}$-universal in Q.

Definition 2.5. A subset A of a space X is called *homotopy dense* in X if there exists a homotopy $h : X \times \mathbf{I} \to X$ such that $h_0 = id_X$ and $h_t(X) \subset A$ for every $t > 0$.

In [3], the authors proved the following lemma (cf. Theorem 2.1 and Lemma 6.2 of [3]).

Lemma 2.6. *Let $\mathcal{C}$ be a topological class of separable metrizable spaces.*

(1) For a subspace X of Q, X is strongly $\mathcal{C}$-universal in Q if and only if X is both $\mathcal{C}$-universal and reflexively universal in Q.

(2) For a homotopy dense set $X \ni 0$ in Q, (Q, X) is reflexively universal if there exists a homeomorphism $\Phi : Q \to Q^\omega$ such that

$$\{x \in (X)^\omega : x_i = 0 \textit{ for all but finitely many } i\} \subset \Phi(X) \subset X^\omega.$$

(3) If (Q, X_1) and (Q, X_2) are $(\mathcal{M}_0, \mathcal{C})$-absorbing, then $(Q, X_1) \approx (Q, X_2)$.

Definition 2.7. Let X be a separable metrizable space. We call X an *absolute $F_{\sigma\delta}$-space* if X is an $F_{\sigma\delta}$-set in any separable metrizable space containing it as a subspace. We use $\mathcal{F}_{\sigma\delta}$ to denote the class of all absolute $F_{\sigma\delta}$-spaces.

In Section 1 we defined Q, Σ and c_0. To show our theorem, we need to define

$$Q_u = \mathbf{I}^\omega, \quad c_1 = \{(x_n) \in Q_u : \lim x_n = 1\}, \quad \text{and}$$
$$c_0^Q = \{(x_n) \in Q : \lim x_n = 0\}.$$

The following lemma describes the relation between c_0 and c_0^Q.

Lemma 2.8. *(Q, c_0) and (Q, c_0^Q) are $(\mathcal{M}_0, \mathcal{F}_{\sigma\delta})$-absorbing and hence $(Q, c_0) \approx (Q, c_0^Q)$.*

Proof. In [3, Theorem 6.3], it was proved that (Q, c_0) is $(\mathcal{M}_0, \mathcal{F}_{\sigma\delta})$-absorbing. Note that $c_0^Q = \bigcap_{k=1}^\infty \Sigma_k$, where

$$\Sigma_k = \bigcup_{i=1}^\infty [-1,1]^i \times [-\frac{1}{k}, \frac{1}{k}]^{\omega \setminus \{1,2,\cdots,i\}}.$$

Hence c_0^Q satisfies (a) in Definition 2.4. $c_0^Q \subset c_0 \cup B(Q)$, where $B(Q) = Q \setminus (-1,1)^\omega \in Z_\sigma(Q)$, implies that c_0^Q satisfies (b) in Definition 2.4 since c_0 satisfies (b) in this definition. Define $H : Q \times \mathbf{I} \to Q$ as follows

$$H(x,t)(n) = t^n x(n).$$

Then $H(-,1) = id_Q$ and $H([0,1) \times \mathbf{I}) \subset c_0^Q$. Therefore, c_0^Q is homotopy dense in Q. Moreover, let $\Phi : Q = [-1,1]^\omega \to Q^\omega = [-1,1]^{\omega \times \omega}$ be any map that simply rearranges coordinates. Then Φ satisfies the conditions of (2) in Lemma 2.6 for $X = c_0^Q$. It follows that (Q, c_0^Q) is reflexively universal.

It remains to verify that c_0^Q is $\mathcal{F}_{\sigma\delta}$-universal. At first, for any $\varepsilon \in (0,1)$, choose an odd N_0 such that $\frac{1}{2^{N_0}} < \varepsilon$. Define $f : Q \to Q$ as follows

$$f(x)(n) = \begin{cases} x_n & \text{if } n \leq N_0, \\ 0 & \text{if } n = N_0 + 1, \\ x_{N_0 + \frac{n-N_0}{2}} & \text{if } n > N_0 + 1 \text{ and } n \text{ is odd}, \\ (\max\{|x_1|, |x_2|, \cdots, |x_n|\})^n & \text{otherwise.} \end{cases}$$

Then f is an embedding with $d(f, id_Q) < \varepsilon$. Moreover, $f(x) \in c_0$ if $x \in c_0$ and $f(x) \in Q \setminus c_0^Q$ if $x \in Q \setminus c_0$. It follows from c_0 being $\mathcal{F}_{\sigma\delta}$-universal that c_0^Q is $\mathcal{F}_{\sigma\delta}$-universal. □

We also have the following connection between c_0 and c_1.

Lemma 2.9. *(Q_u, c_1) is $(\mathcal{M}_0, \mathcal{F}_{\sigma\delta})$-absorbing and hence $(Q_u, c_1) \approx (Q, c_0) \approx (Q, c_0^Q)$.*

Proof. Using a similar argument like in the above lemma, we may show that (Q_u, c_1) satisfies (a) and (b) in Definition 2.4 and is reflexively universal. To verify that it is $\mathcal{F}_{\sigma\delta}$-universal, define $f : Q \to Q_u$ as follows

$$f(x)(n) = \begin{cases} 1 - |x_{\frac{n+1}{2}}| & \text{if } n \text{ is odd and } x_{\frac{n+1}{2}} \geq 0, \\ 1 & \text{if } n \text{ is odd and } x_{\frac{n+1}{2}} < 0, \\ 1 - |x_{\frac{n}{2}}| & \text{if } n \text{ is even.} \end{cases}$$

Then f is an embedding and $f^{-1}(c_1) = c_0^Q$. Since (Q, c_0^Q) is $\mathcal{F}_{\sigma\delta}$-universal, so is (Q_u, c_1). □

We call a sequence $(T_i)_i$ in Q a *tower* if $T_i \subset T_{i+1} \subset Q$ for any $i \in \omega$.

Definition 2.10. Let ρ be an admissible metric on the space Q. An element $A \in Z_\sigma(Q)$ is called a *capset* provided that A can be written as the union of a tower $\{A_i\}$ of Z-sets in Q such that the following absorption property holds:

for every $\varepsilon > 0, n \in \omega$ and $K \in Z(Q)$, there exist $m > n$ and a homeomorphism $h : Q \to Q$ such that

(a) $\rho(h, id_Q) < \varepsilon$,
(b) $h|_{A_n} = id|_{A_n}$,
(c) $h(K) \subset A_m$.

We need the following theorem (see [14, Theorem 3]) and associated operation properties in $\downarrow\mathrm{USC}_F(X)$.

Theorem 2.11. *Let (X, Y, W) be a triple of spaces. Then $(X, Y, W) \approx (Q, \Sigma, c_0)$ if and only if the following conditions hold:*

(1) $X \approx Q$,
(2) Y *can be written as the union of a tower* $(Y_n)_n$ *such that*
  (a) $Y_n \in Z(Y_{n+1}) \cap Z(X)$ *for every* n,
  (b) $(Y_n, Y_n \cap W)$*is strongly* $(\mathcal{M}_0, \mathcal{F}_{\sigma\delta})$*-universal for every* n, *and*
  (c) $\bigcup_{n=1}^{\infty} Y_n$ *is a capset in* X.

Lemma 2.12. *Let X be a Hausdorff space and Y an arbitrary space.*

(1) If $k : Y \times X \to \mathbf{I}$ is a continuous map and $h : Y \to \mathrm{USC}(X)$ is a map such that $\downarrow h : Y \to \downarrow \mathrm{USC}_F(X)$ is continuous, then $\downarrow(k \cdot h) : Y \to \downarrow \mathrm{USC}_F(X)$ is continuous, where $(k \cdot h)(y)(x) = k(y,x)h(y)(x)$ for $(y,x) \in Y \times X$.

(2) Let $k : Y \times X \to \mathbf{I}$ be a continuous map and $h : Y \to \mathrm{USC}(X)$ a map such that $\downarrow h : Y \to \downarrow \mathrm{USC}_F(X)$ is continuous. If $k(y,x) + h(y)(x) \leq 1$ for all $(y,x) \in Y \times X$, then $\downarrow(k+h) : Y \to \downarrow \mathrm{USC}_F(X)$ is continuous, where $(k+h)(y)(x) = k(y,x) + h(y)(x)$ for $(y,x) \in Y \times X$. Consequently, if $k : Y \times X \to \mathbf{I}$ be a continuous map, then $\downarrow k : Y \to \downarrow \mathrm{C}_F(X)$ is continuous.

(3) If $k, g : Y \times X \to \mathbf{I}$ are continuous maps and $h : Y \to \mathrm{USC}(X)$ is a map such that $\downarrow h : Y \to \downarrow \mathrm{USC}_F(X)$ is continuous, then $\downarrow(k \cdot g + (1-k) \cdot h) : Y \to \downarrow \mathrm{USC}_F(X)$ is continuous. We call the map $k \cdot g + (1-k) \cdot h$ is a mean of g and h with the weight k. Or simply, $k \cdot g + (1-k) \cdot h$ is a weighed mean of g and h.

(4) Let $X = X_1 \cup X_2$, where X_i is closed for $i = 1, 2$. If $h_i : Y \to \mathrm{USC}(X_i)$ is a map such that $\downarrow h_i : Y \to \downarrow \mathrm{USC}_F(X_i)$ is continuous for each $i = 1, 2$ and $h_1(y)(x) = h_2(y)(x)$ for every $(y,x) \in Y \times (X_1 \cap X_2)$, then $\downarrow h : Y \to \downarrow \mathrm{USC}_F(X)$ is continuous, where

$$h(y)(x) = \begin{cases} h_1(y)(x) & \text{if } x \in X_1, \\ h_2(y)(x) & \text{if } x \in X_2. \end{cases}$$

Here h is called the join of h_1 and h_2.

(5) Let U be an open set in X and $U \subset A \subset \overline{U}$. Then

$$\downarrow \mathrm{C}_F(X) \ni \downarrow f \mapsto \downarrow f|A \in \downarrow \mathrm{C}_F(A)$$

is continuous. Consequently, if $\downarrow h : Y \to \downarrow \mathrm{C}_F(X)$ is continuous, then $\downarrow h_A : Y \to \downarrow \mathrm{C}_F(A)$ is also continuous, where $h_A(y)(x) = h(y)(x)$ for every $y \in Y$ and $x \in A$.

(6) For $f, g \in \mathrm{USC}(X)$, define $(f \vee g)(x) = \max\{f(x), g(x)\}$. Then $f \vee g \in \mathrm{USC}(X)$ and $\downarrow \vee : \downarrow \mathrm{USC}_F(X) \times \downarrow \mathrm{USC}_F(X) \to \downarrow \mathrm{USC}_F(X)$ is continuous.

Proof. (1) Let $y_0 \in Y$. We will show that $\downarrow(k \cdot h)$ is continuous at y_0. Firstly, let $U \times (a, 1]$ be open in $X \times (-1, 1]$ and $\downarrow(k \cdot h)(y_0) \cap (U \times (a, 1]) \neq \emptyset$.

Then there exists $x_0 \in U$ such that $k(y_0, x_0)h(y_0)(x_0) \in (a, 1]$. We will verify that there exists a neighborhood G of y_0 such that

$$\downarrow(k \cdot h)(y) \cap (U \times (a, 1]) \neq \emptyset \qquad \text{for every } y \in G. \tag{2.1}$$

This shows that $(\downarrow k \cdot h)^{-1}((U \times (a, 1])^-)$ is open in Y. We consider two cases.

Case A: $k(y_0, x_0) \neq 0$. Then $\frac{a}{k(y_0,x_0)} < h(y_0)(x_0)$. Choose $\varepsilon > 0$ such that

$$\frac{a}{k(y_0, x_0)} + \varepsilon < h(y_0)(x_0). \tag{2.2}$$

Since $k : Y \times X \to \mathbf{I}$ is continuous, there exists a neighborhood $G_1 \times H$ of (y_0, x_0) in $Y \times X$ such that $H \subset U$ and

$$\frac{a}{k(y, x)} < \frac{a}{k(y_0, x_0)} + \varepsilon \qquad \text{for each } (y, x) \in G_1 \times H. \tag{2.3}$$

Moreover, by (2.2), we have $\downarrow h(y_0) \cap (H \times (\frac{a}{k(y_0,x_0)} + \varepsilon, 1]) \neq \emptyset$. It follows from the continuity of $\downarrow h$ there exists a neighborhood $G \subset G_1$ of y_0 such that $\downarrow h(y) \cap (H \times (\frac{a}{k(y_0,x_0)} + \varepsilon, 1]) \neq \emptyset$ for every $y \in G$. That is, for each $y \in G$, there exists $x \in H$ such that $h(y)(x) \in (\frac{a}{k(y_0,x_0)} + \varepsilon, 1]$. Using (2.3), we have $(k \cdot h)(y)(x) \in (a, 1]$. Hence (2.1) holds for the above neighborhood G of y_0.

Case B. $k(y_0, x_0) = 0$. In this case it holds that $a < 0$ and hence (2.1) holds for $G = Y$.

Secondly, let C be a compact set in $X \times \mathbf{I}$ with $\downarrow(k \cdot h)(y_0) \cap C = \emptyset$. Let $\pi : X \times \mathbf{I} \to X$ be the projection and $C_x = \{t \in \mathbf{I} : (x, t) \in C\}$ for $x \in \pi(C)$. Define a map $g : \pi(C) \to \mathbf{I}$ as follows

$$g(x) = \inf C_x.$$

Then $g : \pi(C) \to \mathbf{I}$ is lower semi-continuous and $(k \cdot h)(y_0)(x) < g(x)$ for every $x \in \pi(C)$. Thus, there exists a continuous map $f : \pi(C) \to \mathbf{I}$ such that $(k \cdot h)(y_0)(x) < f(x) < g(x)$ for every $x \in \pi(C)$ (see [6], cf. [7, 5.5.20].) Therefore,

$$\downarrow(k \cdot h)(y_0) \cap \uparrow f = \emptyset \quad \text{and} \quad C \subset \uparrow f, \tag{2.4}$$

where $\uparrow f = \{(x, t) \in \pi(C) \times \mathbf{I} : t \geq f(x)\}$. Let

$$X_1 = \{x \in \pi(C) : k(y_0, x) = 0\}.$$

Then X_1 is compact. By the continuities of f and k, there exist open sets G_1 and H in Y and $\pi(C)$, respectively, such that $y_0 \in G_1$, $X_1 \subset H$ and $k(y, x) < f(x)$ for each $(y, x) \in G_1 \times H$. Let $X_2 = \pi(C) \setminus H$. Then X_2 is also compact and $h(y_0)(x) < \frac{f(x)}{k(y_0,x)}$ for every $x \in X_2$.

Since $h(y_0)(x)$ is upper semi-continuous and $\frac{f(x)}{k(y_0,x)}$ is continuous in the compact set X_2, there exists $\varepsilon > 0$ such that

$$h(y_0)(x) < \frac{f(x)}{k(y_0,x)} - \varepsilon \qquad \text{for every } x \in X_2.$$

It follows from the compactness of X_2 that there exists a neighborhood $G_2 \subset G_1$ of y_0 such that

$$\frac{f(x)}{k(y,x)} > \frac{f(x)}{k(y_0,x)} - \varepsilon \quad \text{for each } (y,x) \in G_2 \times X_2. \tag{2.5}$$

Let

$$K = (X_2 \times \mathbf{I}) \cap \uparrow (\frac{f(x)}{k(y_0,x)} - \varepsilon).$$

Then K is compact and $\downarrow h(y_0) \cap K = \emptyset$. Therefore, there exists a neighborhood $G \subset G_2$ of y_0 such that $\downarrow h(y) \cap K = \emptyset$ for $y \in G$ since $\downarrow h$ is continuous. We will verify that $\downarrow (k \cdot h)(y) \cap \uparrow f = \emptyset$ for $y \in G$. Using this fact and (2.4), we have $\downarrow (k \cdot h)(y) \cap C = \emptyset$ for every $y \in G$. It shows that $\downarrow (k \cdot h)^{-1}(((X \times \mathbf{I}) \setminus C)^+)$ is open in Y.

To show $\downarrow (k \cdot h)(y) \cap \uparrow f = \emptyset$ for every $y \in G$, it suffices to verify that

$$k(y,x)h(y)(x) < f(x) \quad \text{ for } (y,x) \in G \times \pi(C).$$

For every $x \in \pi(C)$, we have $x \in H$ or $x \in X_2$. If $x \in H$, then $k(y,x)h(y)(x) \leq k(y,x) < f(x)$ for every $y \in G \subset G_1$. If $x \in X_2$, we have $h(y)(x) < \frac{f(x)}{h(y_0)(x)} - \varepsilon$ since $\downarrow h(y) \cap K = \emptyset$ for $y \in G$. It follows from $G \subset G_2$ and (2.5) that $h(y)(x) < \frac{f(x)}{k(y,x)}$. That is, $k(y,x)h(y)(x) < f(x)$ for $(y,x) \in G \times X_2$. We are done.

Hence $\downarrow (k \cdot h) : Y \to \downarrow \mathrm{USC}_F(X)$ is continuous.

(2) Let $y_0 \in Y$. We show that $\downarrow (k + h) : Y \to \downarrow \mathrm{USC}(X)$ is continuous at y_0. To this end, firstly, for every open set $U \times (a, 1]$ in $X \times (-1, 1]$ with $\downarrow (k + h)(y_0) \cap (U \times (a, 1]) \neq \emptyset$, there exists $x_0 \in U$ such that $a < k(y_0, x_0) + h(y_0)(x_0)$. Choose $\varepsilon > 0$ such that

$$a + \varepsilon < k(y_0, x_0) + h(y_0)(x_0). \tag{2.6}$$

Since $k : Y \times X \to \mathbf{I}$ is continuous, there exist open sets G_1 and $H \subset U$ in Y and X, respectively, such that $(y_0, x_0) \in G_1 \times H$ and

$$k(y,x) > k(y_0,x_0) - \varepsilon \qquad \text{for } (y,x) \in G_1 \times H. \tag{2.7}$$

By (2.6),

$$\downarrow h(y_0) \cap (H \times (a + \varepsilon - k(y_0,x_0), 1]) \neq \emptyset.$$

It follows from the continuity of $\downarrow h$ that there exists a neighborhood $G \subset G_1$ of y_0 such that

$$\downarrow h(y) \cap (H \times (a + \varepsilon - k(y_0,x_0), 1]) \neq \emptyset \qquad \text{for } y \in G, \tag{2.8}$$

which implies that

$$\downarrow(k+h)(y) \cap (U \times (a,1]) \neq \emptyset \qquad \text{for } y \in G. \tag{2.9}$$

In fact, for $y \in G$, by (2.8), there exists $x \in H$ such that

$$h(y)(x) > a + \varepsilon - k(y_0, x_0).$$

It follows from (2.7) that

$$(k+h)(y,x) = k(y,x) + h(y)(x) > k(y_0,x_0) - \varepsilon + h(y)(x) > a.$$

That is (2.9) holds. This have showed that $\downarrow(k+h)^{-1}((U \times (a,1])^-)$ is open in Y.

Secondly, suppose $\downarrow(k+h)(y_0) \cap C = \emptyset$ for a compact set C in $X \times \mathbf{I}$. Then, similar to the proof in (1), there exists a continuous map $f : \pi(C) \to \mathbf{I}$ such that

$$\downarrow(k+h)(y_0) \cap \uparrow f = \emptyset \quad \text{and} \quad \uparrow f \supset C.$$

There exists $\varepsilon > 0$ such that

$$f(x) - (k(y_0,x) + h(y_0)(x)) > \varepsilon \qquad \text{for every } x \in \pi(C) \tag{2.10}$$

since $f(x) - (k(y_0,x) + h(y_0)(x))$ is lower semi-continuous and $\pi(C)$ is compact. Trivially, there exists a neighborhood G_1 of y_0 such that

$$f(x) - k(y,x) > f(x) - k(y_0,x) - \varepsilon \quad \text{for every } (x,y) \in \pi(C) \times G_1. \tag{2.11}$$

Now let

$$K = \uparrow (f(x) - k(y_0,x) - \varepsilon).$$

Then K is compact in $X \times \mathbf{I}$. It follows from (2.10) that $\downarrow h(y_0) \cap K = \emptyset$. Hence there exists a neighborhood $G \subset G_1$ of y_0 such that $\downarrow h(y) \cap K = \emptyset$ for every $y \in G$. Then

$$\downarrow(k+h)(y) \cap \uparrow f = \emptyset \qquad \text{for } y \in G. \tag{2.12}$$

In fact, if otherwise, there exists $(y,x) \in G \times \pi(C)$ such that

$$k(y,x) + h(y)(x) \geq f(x).$$

That is, $(x, f(x) - k(y,x)) \in \downarrow h(y)$. Moreover, by (2.11) and the definition of K we have $(x, f(x) - k(y,x)) \in K$. This contradicts $\downarrow h(y) \cap K = \emptyset$ for $y \in G$. Hence (2.12) holds. Since $\uparrow f \supset C$, we have $\downarrow(k+h)(y) \cap C = \emptyset$ for $y \in G$. It follows that $\downarrow(k+h)^{-1}(((X \times \mathbf{I}) \setminus C)^+)$ is open in Y.

Hence $\downarrow(k+h) : Y \to \downarrow\mathrm{USC}_F(X)$ is continuous.

(3) follows from (1) and (2).

(4) We note the following facts on the Fell topology on $\mathrm{Cld}(X)$:

(a) $\cup : \mathrm{Cld}_F(X) \times \mathrm{Cld}_F(X) \to \mathrm{Cld}_F(X)$ is continuous;

(b) If X_0 is a closed subspace of X, then $\mathrm{Cld}_F(X_0)$ is a subspace of $\mathrm{Cld}_F(X)$.

Using them, (4) follows from $\downarrow h = \downarrow h_1 \cup \downarrow h_2$.

(5) Let $f \in \mathrm{C}(X)$. For every open set V in $A \times \mathbf{I}$, $V \cap (U \times \mathbf{I})$ is open in $X \times \mathbf{I}$. Since f is continuous, $\downarrow f|A \cap V \neq \emptyset$ if and only if $\downarrow f|A \cap V \cap (U \times \mathbf{I}) \neq \emptyset$ if and only if $\downarrow f \cap V \cap (U \times \mathbf{I}) \neq \emptyset$. For a compact set C in $A \times \mathbf{I}$, C is also compact in $X \times \mathbf{I}$ and $\downarrow f|A \cap C = \emptyset$ if and only if $\downarrow f \cap C = \emptyset$. Thus, the map defined is continuous.

(6) Trivially, $f \vee g \in \mathrm{USC}(X)$. It follows from (a) in proof of (4) and $\downarrow(f \vee g) = \downarrow f \cup \downarrow g$ that $\downarrow\vee$ is continuous. □

Remark 2.13. (1). In neither (1) nor (2) of the above lemma, the assumption that $k : Y \times X \to \mathbf{I}$ is continuous can be replaced by $\downarrow k : Y \to \downarrow \mathrm{C}_F(X)$ being continuous, even if where Y and X are compact metrizable spaces and, $\downarrow h : Y \to \downarrow \mathrm{C}_F(X)$ is continuous (see Example 2.14 below). Similarly, in (3) the continuous map $g : Y \times X \to \mathbf{I}$ cannot be replaced by a map $g : Y \to \mathrm{C}(X)$ such that $\downarrow g$ is continuous.

(2). In (5) of the above lemma, if we replace $\mathrm{C}(X)$ by $\mathrm{USC}(X)$, then the statement is not true in general case. See Example 2.15.

(3). $\downarrow\wedge : \downarrow \mathrm{USC}_F(X) \times \downarrow \mathrm{USC}_F(X) \to \downarrow \mathrm{USC}_F(X)$ is not necessarily continuous, where $(f \wedge g)(x) = \min\{f(x), g(x)\}$. See Example 2.14.

Example 2.14. Let $Y = \{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots\}$, i.e., a converging sequence with its limit 0, and let $X = \mathbf{I}$. Define $k, h : Y \to \mathrm{C}(X)$ by

$$k(y)(x) = \begin{cases} \frac{1}{2} & if\ y = 0, \\ \frac{1}{4}(1 + \sin nx) & if\ y = \frac{1}{n}. \end{cases}$$

and

$$h(y)(x) = \begin{cases} \frac{1}{2} & if\ y = 0, \\ \frac{1}{2} - \frac{1}{4}(1 + \sin nx) & if\ y = \frac{1}{n}. \end{cases}$$

Then $\downarrow k, \downarrow h : Y \to \downarrow \mathrm{C}_F(X)$ are continuous but $\downarrow(k + h) : Y \to \downarrow \mathrm{C}_F(X)$ is not continuous. Moreover, $\downarrow 2k, \downarrow 2h : Y \to \downarrow \mathrm{C}_F(X)$ are continuous but $\downarrow(2k \cdot 2h) : Y \to \downarrow \mathrm{C}_F(X)$ is not continuous since $(2k \cdot 2h)(\frac{1}{n})(x) = \frac{1}{4}\cos^2 nx \leq \frac{1}{4}$ but $(2k \cdot 2h)(0)(x) = 1$ for every $n \in \omega$ and $x \in X$. It is not hard to verify that $(k(\frac{1}{n}) \wedge h(\frac{1}{n}))(x) = \frac{1}{4}(1 - |\sin nx|) \leq \frac{1}{4}$ for $x \in X$ and $n \in N$. But $(k(0) \wedge h(0))(x) = \frac{1}{2}$ for $x \in X$. This shows that $\downarrow\wedge : \downarrow \mathrm{USC}_F(X) \times \downarrow \mathrm{USC}_F(X) \to \downarrow \mathrm{USC}_F(X)$ is not continuous.

Example 2.15. Let $X = [0, 2]$ and $U = [0, 1)$. Define $f_n \in \mathrm{USC}(X)$ and $f \in \mathrm{USC}(X)$ by $f_n(x) = 0$ for all $x \in X$ but $f_n(1 + \frac{1}{n}) = 1$, and $f(x) = 0$ for all $x \in X$ but $f_n(1) = 1$. Then $\downarrow f_n \to \downarrow f$ in $\downarrow \mathrm{USC}_F(X)$ but $\downarrow f_n|\mathbf{I} \to \downarrow f|\mathbf{I}$ does not hold in $\downarrow \mathrm{USC}_F(\mathbf{I})$.

3. Proof of Theorem 1.1

In this section, we will always assume X to be a non-compact, non-discrete, locally compact and separable metrizable space with $\overline{X_0} = X$. Note that the Fell topology on $\mathrm{Cld}(X)$ is completely determined by the topology of the underlying space X, i.e., it does not depend on the choice of an admissible metric of X. Thus we may choose an admissible metric d on X such that every bounded closed subset is compact (cf. [12, Lemma 2.2]). Let $X' = X \setminus X_0$. Fix a point $x_\infty \in X'$. Moreover, we may assume that $x_\infty \in X' \setminus (X')_0$ if the set is not empty, that is, we may require that x_∞ is not an isolated point in the subspace X' if this subspace contains a non-isolated point. Let $X_n = \overline{B(x_\infty, n)}$. Then $X = \bigcup_{n=1}^\infty X_n$ and each X_n is compact. We can get a metric on ${\downarrow}\mathrm{USC}_F(X)$ as follows. Define a metric ρ on $X \times \mathbf{I}$ by

$$\rho((x_1,t_1),(x_2,t_2)) = \max\{d(x_1,x_2), |t_1 - t_2|\}.$$

Let $h^*(x) = e^{-d(x,x_\infty)}h(x)$ for any $h \in \mathrm{USC}(X)$. Define

$$\rho'_{\mathrm{H}}({\downarrow}f,\ {\downarrow}g) = \rho_{\mathrm{H}}({\downarrow}f^*,\ {\downarrow}g^*) \qquad \text{for } f,g \in \mathrm{USC}(X).$$

By [15, Lemma 4.2], ρ'_{H} is an admissible metric on ${\downarrow}\mathrm{USC}_F(X)$. We put, for every $n = 1, 2, \cdots$,

$$\mathrm{F}_n(X) = \{f \in \mathrm{USC}(X) : \text{there exists } x \in X_n \cap X' \text{ such that } f(x) \geq \tfrac{1}{n}\};$$
$$\mathrm{C}_n(X) = \mathrm{C}(X) \cap \mathrm{F}_n(X);$$
$$\mathrm{C}_0(X) = \{f \in \mathrm{USC}(X) : f(x) = 0 \text{ for all } x \in X'\}.$$

Then it is easy to verify that $\mathrm{C}_0(X) \subset \mathrm{C}(X)$ and $\mathrm{USC}(X) \setminus \mathrm{C}_0(X) = \bigcup_{n=1}^\infty \mathrm{F}_n(X)$.

Using Theorem 2.11, we will focus on

$$({\downarrow}\mathrm{USC}_F(X), {\downarrow}\mathrm{USC}_F(X) \setminus {\downarrow}\mathrm{C}_0(X), {\downarrow}\mathrm{C}_F(X) \setminus {\downarrow}\mathrm{C}_0(X)) \approx (Q, \Sigma, c_0).$$

Then it is not hard to get $({\downarrow}\mathrm{USC}_F(X), {\downarrow}\mathrm{C}_F(X)) \approx (Q, (c_0 \cup (Q \setminus \Sigma)))$.

Lemma 3.1. ${\downarrow}\mathrm{USC}_F(X) \approx Q$.

Proof. Please refer to [15]. □

Lemma 3.2. *${\downarrow}\mathrm{C}_F(X)$ is homotopy dense in ${\downarrow}\mathrm{USC}_F(X)$.*

Proof. By [16, Theorem 2], ${\downarrow}\mathrm{C}_F(\alpha X)$ is homotopy dense in ${\downarrow}\mathrm{USC}_F(\alpha X)$, where $\alpha X = X \cup \{\infty\}$ is the one-point compactification of X. So there exists a homotopy $H : {\downarrow}\mathrm{USC}_F(\alpha X) \times \mathbf{I} \to {\downarrow}\mathrm{USC}_F(\alpha X)$ such that $H_0({\downarrow}f) = {\downarrow}f$ and $H_t({\downarrow}\mathrm{USC}_F(\alpha X)) \subset {\downarrow}\mathrm{C}_F(\alpha X)$ for every $t \in (0,1]$.

Let $k : \mathrm{USC}(\alpha X) \to \mathrm{USC}(X)$ be a map defined by $k(f) = f|_X$. Then, by [15, Corollary 5.5], ${\downarrow}k :\ {\downarrow}\mathrm{USC}_F(\alpha X) \to {\downarrow}\mathrm{USC}_F(X)$ is a continuous surjection. Let $h : \mathrm{USC}(X) \to \mathrm{USC}(\alpha X)$ defined by $h(f) = f \cup \{(\infty, 1)\}$.

Then $\downarrow h : \downarrow\mathrm{USC}_F(X) \to \downarrow\mathrm{USC}_F(\alpha X)$ is a Z-embedding.

Let $H' : \downarrow\mathrm{USC}_F(X) \times \mathbf{I} \to \downarrow\mathrm{USC}_F(X)$ defined by

$$H'(\downarrow f, t) = \downarrow k \circ H(\downarrow h(\downarrow f), t).$$

Then H' is continuous, $H'_0(\downarrow f) = \downarrow f$ and $H'_t(\downarrow f) \in \downarrow\mathrm{C}_F(X)$ for every $t \in (0, 1]$. We are done. □

Lemma 3.3. $\downarrow\mathrm{F}_n(X) \in Z(\downarrow\mathrm{USC}_F(X)) \cap Z(\downarrow\mathrm{F}_{n+1}(X))$.

Proof. Note that $K = (X_n \cap X') \times [\frac{1}{n}, 1]$ is compact in $X \times \mathbf{I}$ and $\downarrow\mathrm{F}_n(X) = \downarrow\mathrm{USC}(X) \setminus (X \times \mathbf{I} \setminus K)^+$. It follows that $\downarrow\mathrm{F}_n(X)$ is closed in both $\downarrow\mathrm{USC}_F(X)$ and $\downarrow\mathrm{F}_{n+1}(X)$.

Fix an $\varepsilon \in (0, 1)$, there exists an $m \in \omega$ such that $e^{-d(x,x_\infty)} < \frac{\varepsilon}{4}$ for every $x \in X \setminus X_m$. By [11, Proposition 4.1.7] and Lemma 3.2, there exists a homotopy $H : \downarrow\mathrm{USC}_F(X) \times \mathbf{I} \to \downarrow\mathrm{USC}_F(X)$ such that

$$H_0 = id_{\downarrow\mathrm{USC}_F(X)},\ H_t(\downarrow\mathrm{USC}_F(X)) \subset \downarrow\mathrm{C}_F(X) \text{ and } \rho'_{\mathrm{H}}(H_t(\downarrow f), \downarrow f) \le t$$

for every $t \in (0, 1]$. Define a map $h : \mathrm{USC}(X) \to \mathrm{USC}(X)$ by $\downarrow h(f) = H(\downarrow f, \frac{\varepsilon}{4})$. Since $\overline{X_0} = X$ and X_m is compact, there exists a finite set $\{x_i : i \le k\} \subset X_0$ such that $\{B(x_i, \frac{\varepsilon}{4}) : i \le k\}$ covers X_m. Then we define $\downarrow\phi : \downarrow\mathrm{USC}_F(X) \to \downarrow\mathrm{USC}_F(X) \setminus \downarrow\mathrm{F}_n(X)$ and $\downarrow\psi : \downarrow\mathrm{F}_{n+1}(X) \to \downarrow\mathrm{F}_{n+1}(X) \setminus \downarrow\mathrm{F}_n(X)$ as follows:

$$\phi(f)(x) = \begin{cases} \sup\{h(f)(x) : x \in B(x_i, \frac{\varepsilon}{4})\} & x = x_i, 1 \le i \le k, \\ 0 & \text{otherwise;} \end{cases}$$

$$\psi(f)(x) = \begin{cases} \sup\{h(f)(x) : x \in B(x_i, \frac{\varepsilon}{4})\} & x = x_i, 1 \le i \le k, \\ \min\{f(x), \frac{1}{n+1}\} & \text{otherwise.} \end{cases}$$

It is trivial that $\rho'_H(\downarrow\phi, id_{\downarrow\mathrm{USC}_F(X)}) < \varepsilon$ and $\rho'_H(\downarrow\psi, id_{\downarrow\mathrm{USC}_F(X)}) < \varepsilon$. It can be verified that $\downarrow\phi$ and $\downarrow\psi$ are continuous. We get $\downarrow\mathrm{F}_n(X) \in Z(\downarrow\mathrm{USC}_F(X)) \cap Z(\downarrow\mathrm{F}_{n+1}(X))$. □

Let Y be a metric space and $A \subset Y$. We say that A is a *retract* of Y provided that there is a continuous function $r : Y \to A$ such that $r|_A = id_A$. Such a map r is referred to as a *retraction* from Y to A in the literature. The space Y is called an *absolute retract*, abbreviated to *AR*, provided Y is a retract of every metric space Z containing it as a closed subspace.

Definition 3.4. A *topological semilattice* is a topological space S equipped with a continuous operator $\vee : S \times S \to S$ which is idempotent, commutative and associative (i.e., $x \vee x = x$, $x \vee y = y \vee x$, $(x \vee y) \vee z = x \vee (y \vee z)$ for any $x, y, z \in S$). A topological semilattice S is called a *Lawson semilattice* if S admits an open basis consisting of subsemilattices.

Definition 3.5. Let Z be a space. A subset $Y \subset Z$ is *relatively* LC^0 in Z if for every $y \in Z$, each neighborhood U of y in Z contains a smaller neighborhood V of y such that any two points of $V \cap Y$ can be joined by a path in $U \cap Y$.

Lemma 3.6. *$\downarrow \mathrm{F}_n(X)$ is an AR and $\downarrow \mathrm{C}_n(X)$ is homotopy dense in $\downarrow\mathrm{F}_n(X)$.*

Proof. By Lemma 2.12(6), the operator $\downarrow\vee$ is continuous, idempotent, commutative and associative on $\downarrow F_n(X)$. Hence, $\downarrow F_n(X)$ is a semilattice. It is not hard to prove that if $\rho'_{\mathrm{H}}(\downarrow g, \downarrow f) < \varepsilon$ and $\rho'_{\mathrm{H}}(\downarrow h, \downarrow f) < \varepsilon$, then $\rho'_{\mathrm{H}}(\downarrow(g \vee h), \downarrow f) < \varepsilon$. So $\mathcal{U} = \{(B_{\rho'_{\mathrm{H}}}(\downarrow f, \varepsilon), \cup) : f \in \mathrm{USC(X)}\}$ is an open basis consisting of subsemilattices, which means that $\downarrow\mathrm{F}_n(X)$ is a Lawson semilattice.

For any $\downarrow f \in \downarrow\mathrm{F}_n(X)$ and $\varepsilon > 0$, we consider the set $B_{\rho'_{\mathrm{H}}}(\downarrow f, \varepsilon) \cap \downarrow\mathrm{C}_n(X)$. It is not hard to verify that this set is nonempty. For any $\downarrow g_1, \downarrow g_2 \in B_{\rho'_{\mathrm{H}}}(\downarrow f, \varepsilon) \cap \downarrow\mathrm{C}_n(X)$, let $g = g_1 \vee g_2$. Then $\downarrow g \in B_{\rho'_{\mathrm{H}}}(\downarrow f, \varepsilon) \cap \downarrow\mathrm{C}_n(X)$. Define $h : \mathbf{I} \to \mathrm{USC}(X)$ by

$$h(t) = \begin{cases} (1-2t)g_1 + 2tg & 0 \le t \le \frac{1}{2}, \\ (2t-1)g_2 + 2tg & \frac{1}{2} < t \le 1. \end{cases}$$

Trivially $\downarrow h$ is a path in $B_{\rho'_{\mathrm{H}}}(\downarrow f, \varepsilon)$ joining $\downarrow g_1$ and $\downarrow g_2$. Consequently, $\downarrow\mathrm{C}_n(X)$ is relatively LC^0 in $\downarrow\mathrm{F}_n(X)$. We can prove $\downarrow\mathrm{C}_n(X)$ is path connected in the same way.

Hence, we have showed that $\downarrow\mathrm{F}_n(X)$ is a Lawson semilattice, and $\downarrow\mathrm{C}_n(X)$ is path connected and $\downarrow\mathrm{C}_n(X)$ is relatively LC^0 in $\downarrow\mathrm{F}_n(X)$. It is not hard to verify that $\downarrow\mathrm{C}_n(X)$ is dense in $\downarrow\mathrm{F}_n(X)$. By [9, Theorem 5.1], $\downarrow\mathrm{F}_n(X)$ is an AR and $\downarrow\mathrm{C}_n(X)$ is homotopy dense in $\downarrow\mathrm{F}_n(X)$. □

We say a metric space (Y, d) has the *disjoint-cells property* provided that for any $\varepsilon > 0$, $n \in \omega$ and every continuous function $f : \mathbf{I}^n \times \{0,1\} \to Y$, there exists a continuous map $g : \mathbf{I}^n \times \{0,1\} \to Y$ such that $d(f,g) < \varepsilon$ and $g(\mathbf{I}^n \times \{0\}) \cap g(\mathbf{I}^n \times \{1\}) = \emptyset$. By [10, Corollary 7.8.4], we know that a compact AR with the disjoint-cells property is homeomorphic to Q.

Lemma 3.7. *$\downarrow\mathrm{F}_n(X) \approx Q$.*

Proof. By Lemmas 3.1, 3.3 and 3.6, we know that $\downarrow\mathrm{F}_n(X)$ is a compact AR. To complete the proof of this lemma, we only need to verify that $\downarrow\mathrm{F}_n(X)$ has the disjoint-cells property. For any $\varepsilon > 0$, we may choose two different isolated points $a_1, a_2 \in B(x_\infty, \frac{\varepsilon}{2})$ since $\overline{X_0} = X$. For any continuous function $\downarrow f : \mathbf{I}^k \times \{0,1\} \to \downarrow\mathrm{F}_n(X)$, define $g : \mathbf{I}^k \times \{0,1\} \to \mathrm{F}_n(X)$ as follows: for any $(q,t) \in \mathbf{I}^k \times \{0,1\}$, let

$$g(q,t)(x) = \begin{cases} \max\{\varepsilon, f(q,0)(a_1), f(q,0)(a_2)\} & t=0, x=a_1, \\ 0 & t=1, x=a_1, \\ 0 & t=0, x=a_2 \\ \max\{\varepsilon, f(q,1)(a_1), f(q,1)(a_2)\} & t=1, x=a_2, \\ f(q,t)(x) & \text{otherwise.} \end{cases}$$

Trivially, $\downarrow g : \mathbf{I}^k \times \{0,1\} \to \downarrow \mathrm{F}_n(X)$ is continuous and $\rho'_H(\downarrow g, \downarrow f) < \varepsilon$, $g(\mathbf{I}^k \times \{0\}) \cap g(\mathbf{I}^k \times \{1\}) = \emptyset$. Hence $\downarrow \mathrm{F}_n(X)$ has the disjoint-cells property. □

Definition 3.8. We say a tower $(T_i)_i$ in Q has the *deformation property* if there is a homotopy $H : Q \times \mathbf{I} \to Q$ such that $H_0 = id_Q$, and for each $t \in (0,1]$, there is an $m \in \omega$ such that $H(Q \times [t,1]) \subset T_m$.

Lemma 3.9. *$(\downarrow \mathrm{F}_n(X))_n$ is a tower with the deformation property in $\downarrow\mathrm{USC}_F(X) \approx Q$.*

Proof. Trivially $(\downarrow\mathrm{F}_n(X))_n$ is a tower in $\downarrow\mathrm{USC}_F(X)$. Define $H : \downarrow\mathrm{USC}_F(X) \times \mathbf{I} \to \downarrow\mathrm{USC}_F(X)$ by

$$H(\downarrow f, t) = \downarrow f \cup (X \times [0,t]),\ \ f \in \mathrm{USC}(X), t \in \mathbf{I}.$$

Then H is a homotopy, $H_0 = id_Q$, and for each $t \in (0,1]$, there is an $m \in \omega$ such that $H(Q \times [t,1]) \subset F_m(X)$. □

Lemma 3.10. *$\bigcup_{n=1}^{\infty} \downarrow\mathrm{F}_n(X)$ is a capset of $\downarrow\mathrm{USC}_F(X)$.*

Proof. Let $\varepsilon \in (0,1)$, $n \in \omega$ and $A \in Z(\downarrow\mathrm{USC}_F(X))$. By [11, Corollary 5.3.8], there exists $\gamma > 0$ such that every homeomorphism between two Z-sets in $\downarrow\mathrm{USC}_F(X)$ that moves the points less than γ can be extended to a homeomorphism of $\downarrow\mathrm{USC}_F(X)$ moving the points less than ε. Let $t = \min\{\frac{1}{2n}, \frac{\gamma}{5}\}$. Choose $m \in \omega$ such that $\frac{1}{m} < t$.

By [11, Theorem 4.1.1], there exists $\delta > 0$ such that, for every separable metrizable space Y, any two δ-close continuous maps are $\frac{\gamma}{8}$-homotopic. By Lemma 3.6, there is a retraction $r : \downarrow\mathrm{USC}_F(X) \to \downarrow\mathrm{F}_n(X)$.

Put

$$B = \{f \in \mathrm{USC}(X) : \rho'_{\mathrm{H}}(\downarrow f, r(\downarrow f)) \leq \frac{\delta}{2}\},$$

$$C = \{f \in \mathrm{USC}(X) : \rho'_{\mathrm{H}}(\downarrow f, r(\downarrow f)) \geq \frac{\delta}{2}\}.$$

Then $\downarrow B$ is a closed neighborhood of $\downarrow\mathrm{F}_n(X)$. Since $id_{\downarrow B}$ and $r|_{\downarrow B}$ are δ-close, there exists a homotopy $\Psi : \downarrow B \times \mathbf{I} \to \downarrow\mathrm{USC}_F(X)$ which is $\frac{\gamma}{8}$-limited and connects $id_{\downarrow B}$ and $r|_{\downarrow B}$. By [11, Theorem 4.1.3], there exists another $\frac{\gamma}{8}$-limited homotopy $\Phi : \downarrow\mathrm{USC}_F(X) \times \mathbf{I} \to \downarrow\mathrm{USC}_F(X)$ such that $\Phi_0 = id_{\downarrow\mathrm{USC}_F(X)}$ and $\Phi|_{(\downarrow B \times \mathbf{I})} = \Psi$.

Since $\downarrow\mathrm{F}_n(X)$ and $\downarrow C$ are disjoint closed subsets of $\downarrow\mathrm{USC}_F(X)$, there exists a continuous map $\alpha : \downarrow\mathrm{USC}_F(X) \to [0,t]$ such that $\alpha(\downarrow\mathrm{F}_n(X)) = \{0\}$ and $\alpha(\downarrow C) = \{t\}$. Define $\varphi : \downarrow\mathrm{USC}_F(X) \to \downarrow\mathrm{F}_m$ by $\varphi(\downarrow f) = \Phi_1(\downarrow f) \cup (X \times [0, \alpha(\downarrow f)])$. It is easy to see φ is continuous and $\varphi|_{\downarrow\mathrm{F}_n(X)} = id_{\downarrow\mathrm{F}_n(X)}$. Moreover,

$$\rho'_{\mathrm{H}}(\varphi(\downarrow f), \downarrow f) \leq \rho'_{\mathrm{H}}(\varphi(\downarrow f), \Phi_1(\downarrow f)) + \rho'_{\mathrm{H}}(\Phi_1(\downarrow f), \downarrow f) \leq \frac{\gamma}{5} + \frac{\gamma}{4} < \frac{\gamma}{2}$$

for every $f \in \mathrm{USC}(X)$.

By [11, Theorem 5.3.11] and Lemma 3.7 of this paper, there exists a Z-embedding $\psi : \downarrow\mathrm{USC}_F(X) \to \downarrow\mathrm{F}_m$ such that $\psi|_{\downarrow\mathrm{F}_n(X)} = id_{\downarrow\mathrm{F}_n(X)}$ and $\rho'_H(\varphi, \psi) < \frac{\gamma}{2}$. So $\rho'_H(\psi, id_{\downarrow\mathrm{USC}_F(X)}) < \gamma$ and $\psi|_{\downarrow\mathrm{F}_n(X)\cup A}$ is a homeomorphism between Z-sets in $\downarrow\mathrm{USC}_F(X)$. Hence, there exists a homeomorphism $h : \downarrow\mathrm{USC}_F(X) \to \downarrow\mathrm{USC}_F(X)$ extending $\psi|_{\downarrow\mathrm{F}_n(X)\cup A}$ and $\rho'_H(h, id_{\downarrow\mathrm{USC}_F(X)}) < \varepsilon$. Consequently, $\bigcup_{n=1}^{\infty} \downarrow\mathrm{F}_n(X)$ is a capset of $\downarrow\mathrm{USC}_F(X)$. □

Now we are in a position to give our key lemma.

Lemma 3.11. $(\downarrow\mathrm{F}_n(X), \downarrow\mathrm{C}_n(X))$ *is strongly* $(\mathcal{M}_0, \mathcal{F}_{\sigma\delta})$*-universal for any* $n \in \omega$.

Proof. Let C, K be an $\mathrm{F}_{\sigma\delta}$-subset and a closed subset of a compact metrizable space Y, respectively. And let $\Phi : Y \to \mathrm{F}_n(X)$ be a map such that $\downarrow\Phi : Y \to \downarrow\mathrm{F}_n(X)$ is continuous and $\downarrow\Phi|_K : K \to \downarrow\mathrm{F}_n(X)$ is a Z-embedding. By [2, Lemma 1.1] and Lemma 3.7, we may assume that $\downarrow\Phi(K) \cap \downarrow\Phi(Y \setminus K) = \emptyset$. For every $\varepsilon \in (0,1)$, we will define a map $\Psi : Y \to \mathrm{F}_n(X)$ such that $\downarrow\Psi : Y \to \downarrow\mathrm{F}_n(X)$ is a Z-embedding, $\Psi|_K = \Phi|_K$, $\Psi^{-1}(\mathrm{C}_n(X)) \setminus K = C \setminus K$ and $\rho'_H(\downarrow\Psi(y), \downarrow\Phi(y)) < \varepsilon$ for each $y \in Y$.

Let $\delta : Y \to [0,1)$ be a map defined by

$$\delta(y) = \frac{1}{5}\min\{\varepsilon, \rho'_H(\downarrow\Phi(y), \downarrow\Phi(K))\}.$$

Then δ is continuous and $\delta(y) = 0$ if and only if $y \in K$. For every $k \in \omega$, let

$$Y_k = \{y \in Y : 2^{-k} \leq \delta(y) \leq 2^{-k+1}\}.$$

Then Y_k is compact and $\bigcup_{k=1}^{\infty} Y_k = Y \setminus K$.

We consider the following two cases:

Case A. The subspace X' of X contains a non-isolated point.

We will use the same method as in [16, Proposition 1] to construct a map $\Psi : Y \to \mathrm{USC}(X)$ satisfying the above conditions. Hence we omit some details.

By our assumption, $x_\infty \in X$ is a non-isolated point in the subspace X'. Without loss of generality, we may assume that there exists a sequence $(x_m)_{m=0}^\infty$ in X' such that $d(x_m, x_\infty) = 2^{-m}$ for every m. It follows from [11, Proposition 4.1.7] and Lemma 3.6 that there exists a homotopy $H : \downarrow\mathrm{F}_n(X) \times \mathbf{I} \to \downarrow\mathrm{F}_n(X)$ such that

$$H_0 = \mathrm{id}_{\downarrow\mathrm{F}_n(X)},\ \ H_t(\downarrow\mathrm{F}_n(X)) \subset \downarrow\mathrm{C}_n(X) \text{ and } \rho'_H(H_t(\downarrow f), \downarrow f) \leq t$$

for each $f \in \mathrm{F}_n(X)$ and each $t \in (0,1]$. For each $y \in Y$ and $t \in \mathbf{I}$, let

$$\begin{aligned} \downarrow h(y) &= H(\downarrow\Phi(y), \delta(y)), \text{and} \\ M(y,t) &= \sup\{h(y)(x) : d(x, x_\infty) < t\}. \end{aligned}$$

Then $h(y) \in \mathrm{C}_n(X)$ for each $y \in Y \setminus K$ and $\downarrow h|_{Y\setminus K} : Y \setminus K \to \downarrow\mathrm{C}_n(X)$ is continuous. Moreover, $\rho'_H(\downarrow h(y), \downarrow\Phi(y)) \leq \delta(y)$ for every $y \in Y$. Note that for a fixed $t_0 \in (0,1]$, $M(y,t_0) : Y \setminus K \to \mathbf{I}$ is continuous and, for a fixed $y_0 \in Y \setminus K$, $M(y_0, t) : \mathbf{I} \to \mathbf{I}$ is increasing. It follows from Lebesgue's Dominated Convergence Theorem that

$$s(y) = \frac{1}{\delta(y)} \int_{\delta(y)}^{2\delta(y)} M(y,t)\, dt$$

is continuous on $Y \setminus K$ and

$$M(y, \delta(y)) \leq s(y) \leq M(y, 2\delta(y)) \qquad \text{for } y \in Y \setminus K. \tag{3.1}$$

By Lemma 2.9, there exists an embedding $g : Y \to Q_u$ such that $g^{-1}(c_1) = C$.

Let

$$S_0 = \{x \in X : d(x, x_\infty) \geq 1\} \quad \text{and}$$

$$S_m = \{x \in X : 2^{-m} \leq d(x, x_\infty) \leq 2^{-m+1}\} \quad \text{for } m \in \omega.$$

Then $x_{m-1}, x_m \in S_m$. Thus $S_m \cap S_{m'} \neq \emptyset$ if and only if $|m - m'| \leq 1$. And $\bigcup_{m=0}^\infty S_m = X \setminus \{x_\infty\}$. Define continuous maps $\varphi : Y_k \to \mathbf{I}$ by $\varphi(y) = 2 - 2^k \delta(y)$ and $\phi_m : X \to \mathbf{I}$ by

$$\phi_m(x) = \begin{cases} 0 & \text{if } d(x, x_\infty) \leq 2^{-m}, \\ 2^m(d(x,x_\infty) - 2^{-m}) & \text{if } 2^{-m} \leq d(x, x_\infty) \leq 2^{-m+1}, \\ 1 & \text{if } d(x, x_\infty) \geq 2^{-m+1} \end{cases}$$

for each $m = 1, 2, \cdots$. Then $\phi_m(x_m) = 0$ and $\phi_m(x_{m-1}) = 1$ for every every $m = 1, 2, \cdots$. Now we define a sequence $(f_m : Y_k \to \mathrm{C}(X))_m$ as follows:

$$
\begin{aligned}
&f_1(y)(x) = h(y)(x),\\
&f_2(y)(x) = (1-\varphi(y))s(y) + \varphi(y)h(y)(x),\\
&f_3(y)(x) = h(y)(x)\varphi(y),\\
&f_4(y)(x) = s(y),\\
&f_5(y)(x) = 0,\\
&f_6(y)(x) = (1-\varphi(y))\delta(y) + \varphi(y)s(y),\\
&f_7(y)(x) = (1-\varphi(y))\delta(y)g(y)(1),\\
&f_m(y)(x) = \delta(y), && \text{if } m \text{ is even and } m \geq 8,\\
&f_m(y)(x) = \delta(y)((1-\varphi(y))g(y)(\tfrac{m+1}{2}-3)+\\
&\qquad\qquad \varphi(y)g(y)(\tfrac{m+1}{2}-4)), && \text{if } m \text{ is odd and } m \geq 9.
\end{aligned}
$$

For every $i \in \omega$, define a map $g_i : Y_k \to \mathrm{C}(X)$ by

$$g_i(y)(x) = \phi_{2k+i}(x)f_i(y)(x) + (1-\phi_{2k+i}(x))f_{i+1}(y)(x).$$

Using the above maps we define a map $\Psi_k : Y_k \to \mathrm{USC}(X)$ as follows:

$$\Psi_k(y)(x) = \begin{cases} f_1(y)(x) = h(y)(x) & \text{if } x \in \bigcup_{i=0}^{2k} S_i,\\ g_i(y)(x) & \text{if } x \in S_{2k+i},\\ \delta(y) & \text{if } x = x_\infty. \end{cases}$$

Fact 1: For every $y \in Y_k$, $\Psi_k(y)$ is well-defined and continuous on $X \setminus \{x_\infty\}$. And it is upper semi-continuous at x_∞. Moreover, it is continuous at x_∞ if and only if $\lim_{n\to\infty} g(y)(n) = 1$ if and only if $y \in C$. Therefore, for every $y \in Y_k$, $\Psi_k(y) \in \mathrm{C}(X)$ if and only if $y \in C$.

Proof of this fact is exactly the same as one of [16, Fact 1 of Proposition 1] and hence we omit it.

Fact 2: $\Psi_k(y) \in \mathrm{F}_n(X)$ for every $y \in Y_k$.

For every $y \in Y_k$, there exists $z \in X_n \cap X'$ such that $h(y)(z) \geq \frac{1}{n}$ since $h(y) \in \mathrm{C}_n(X)$.

Case a: $z \in \bigcup_{i=0}^{2k} S_i$. Then $\Psi_k(y)(z) = h(y)(z) \geq \frac{1}{n}$. Hence $\Psi_k(y) \in \mathrm{F}_n(X)$.

Case b: $z \in X \setminus \bigcup_{i=0}^{2k} S_i$. Then $d(z, x_\infty) < 2^{-2k}$. It follows from (3.1) that

$$\begin{aligned} s(y) \geq M(y, \delta(y)) \geq M(y, 2^{-k}) = \sup\{h(y)(x) : d(x, x_\infty) < 2^{-k}\}\\ \geq h(y)(z) \geq \frac{1}{n}. \end{aligned}$$

Hence, by $x_{2k+3} \in S_{2k+4}$,

$$\begin{aligned} \Psi_k(y)(x_{2k+3}) = \phi_{2k+4}(x_{2k+3})f_4(y)(x_{2k+3}) +\\ (1-\phi_{2k+4}(x_{2k+3}))f_5(y)(x_{2k+3}) = s(y) \geq \frac{1}{n}. \end{aligned}$$

Since $x_{2k+3} \in X_n \cap X'$, $\Psi_k(y) \in \mathrm{F}_n(X)$.

Fact 3: $\downarrow\Psi_k : Y_k \to \downarrow\mathrm{USC}(X)$ is continuous for every k.

Note that g_1 is a weighted mean of f_1 and f_2 and hence it is also a weighted mean of the continuous map $s : Y_k \to \mathbf{I}$ and the map h. Note that $s : Y_k \times X \to \mathbf{I}$, where $s(y,x) = s(y)$ for $(y,x) \in Y_k \times X$, and $\downarrow h : Y_k \to \mathrm{C}_F(X)$ are continuous. By Lemma 2.12(3), we have that $\downarrow g_1 : Y_k \to\downarrow \mathrm{C}_F(X)$ is continuous. Similarly, $\downarrow g_i : Y_k \to\downarrow \mathrm{C}_F(X)$ is continuous for every $i \geq 2$. Let U_i be the interior of S_i for $i = 0, 1, \cdots$. It follows from Lemma 2.12(5) that $\downarrow h_{\overline{U_i}}$ and $\downarrow(g_i)_{\overline{U_{2k+i}}}$ are continuous for all $i \in \omega$. Note that for every $l \in \omega$,

$$\bigcup_{i=0}^{l-1} S_i \subset \bigcup_{i=0}^{l} \overline{U_i} \subset \bigcup_{i=0}^{l} S_i.$$

By Lemma 2.12(4)(5), $\downarrow(\Psi_k)_{\bigcup_{i=0}^{l}\overline{U_i}}$ is continuous for every $l \in \omega$. Moreover, we note that $\Psi_k(y)(x_\infty) = \delta(y)$ is continuous from Y_k to $\mathbf{I}$ and $\Psi_k(y)(x) \leq \Psi_k(y)(x_\infty)$ for every $y \in Y_k$ and $x \in \bigcup_{i=2k+8}^{\infty} S_i$. Using those facts, we may show that $\downarrow\Psi_k : Y_k \to\downarrow\mathrm{USC}_F(X)$ is continuous.

In fact, let $y_0 \in Y_k$. Assume that U is an open set in $X \times \mathbf{I}$ and $\downarrow\Psi_k(y_0) \cap U \neq \emptyset$. Then there exist $x \in X$ and $t \in \mathbf{I}$ such that $(x,t) \in U$ and $t \leq \Psi_k(y_0)(x)$. If $x = x_\infty$, then, using the continuity of δ, it is easy to verify that there exists a neighborhood V of y_0 in Y_k such that $\downarrow\Psi_k(y) \cap U \neq \emptyset$ for $y \in V$. If $x \neq x_\infty$, then there exists $l \in \omega$ such that $x \in \bigcup_{i=0}^{l} U_i$. Since $\downarrow(\Psi_k)_{\bigcup_{i=0}^{l}\overline{U_i}}$ is continuous, there exists a neighborhood V of y_0 in Y_k such that $\downarrow\Psi_k(y)_{\bigcup_{i=0}^{l}\overline{U_i}} \cap U \neq \emptyset$ for $y \in V$. It implies $\downarrow\Psi_k(y) \cap U \neq \emptyset$ for $y \in V$. Therefore, $(\downarrow\Psi_k)^{-1}(U^-)$ is open in Y_k. Moreover, for every compact A in $X \times \mathbf{I}$, if $\downarrow\Psi_k(y_0) \cap A = \emptyset$, then there exists a neighborhood V_1 of y_0 and $\delta \in (0, 2^{-2k-8})$ such that $(\overline{B(x_\infty,\delta)} \times [0,\delta(y)]) \cap A = \emptyset$ for every $y \in V_1$. It follows from $\Psi_k(y)(x) \leq \delta(y)$ for $x \in \overline{B(x_\infty,\delta)}$ that $\downarrow(\Psi_k(y))_{\overline{B(x_\infty,\delta)}} \cap A = \emptyset$ for $y \in V_1$. Choose $l \in \omega$ such that $X \setminus B(x_\infty,\delta) \subset \bigcup_{i=0}^{l}\overline{U_i}$. Since $\downarrow(\Psi_k)_{\bigcup_{i=0}^{l}\overline{U_i}}$ is continuous, there exists a neighborhood V_2 of y_0 in Y_k such that $\downarrow\Psi_k(y)_{\bigcup_{i=0}^{l}\overline{U_i}} \cap A = \emptyset$ for $y \in V_2$. Then $\downarrow\Psi_k(y) \cap A = \emptyset$ for every $y \in V_1 \cap V_2$. It follows that $(\downarrow\Psi_k)^{-1}((X \times \mathbf{I} \setminus A)^+)$ is open in Y_k.

We may show that $\Psi_k(y) = \Psi_{k+1}(y)$ for every k and every $y \in Y_k \cap Y_{k+1}$. Thus the following map is well-defined:

$$\Psi(y) = \begin{cases} \Phi(y) = h(y) & \text{if } y \in K, \\ \Psi_k(y) & \text{if } y \in Y_k. \end{cases}$$

Then $\Psi : Y \to \mathrm{USC}(X)$ is as required, see proof of [16, Proposition 1].

Case B. The subspace X' contains no non-isolated point.

Then $X = \bigoplus_{i\in A} S_i \oplus B$, where S_i is a convergent sequence with some limit for every $i \in A$ and B is countable discrete space and $1 \le |A| \le \omega$, $0 \le |B| \le \omega$ and at least one of A and B is infinite. Thus

$$(\mathrm{F}_n(X), \mathrm{C}_n(X)) \approx (\mathrm{F}_n(\bigoplus_{i\in A} S_i) \times \mathbf{I}^B, \mathrm{C}_n(\bigoplus_{i\in A} S_i) \times \mathbf{I}^B).$$

Using [3, Lemma 6.4], it is not hard to verify that, for a pair (Y, Z) of spaces, $(Y \times \mathbf{I}^B, Z \times \mathbf{I}^B)$ is strongly $(\mathcal{M}_0, \mathcal{F}_{\sigma\delta})$-universal if so is (Y, Z). Therefore, we may assume that $B = \emptyset$. Furthermore, without loss of generality, we may also assume that

$$X = \{(i, \frac{1}{2^j}) \in \mathbb{R}^2 : j \in \omega \cup \{\infty\}, i \in \omega\}$$

which is a subspace of the metric space $\mathbb{R}^2$, where $\frac{1}{2^\infty} = 0$. We may think that $x_\infty = (1, 0)$. In [14, Case B of Lemma 9], authors constructed a required map $\Psi : Y \to F_n(X)$ in the case $X = \bigoplus_{i\in A} S_i$ for every finite set A. We will use the same method as employed in [14, Case B of Lemma 9] to construct a required map $\Psi : Y \to F_n(X)$ and hence we omit all checks. Define $M_i : Y \times S_i \setminus \{(i, 0)\} \to \mathbf{I}$ by, for any $y \in Y$, and $j < \infty, M_i(y, (i, \frac{1}{2^j})) = \max\{\Phi(y)(i, \frac{1}{2^k}) : k \ge j\}$. Then M_i is continuous for every i.

For $y \in Y_k$, define $\Psi_k(y) \in F_n(X)$ as follows, for any $(i, \frac{1}{2^j}) \in X$,

$$\Psi_k(y)(i, \frac{1}{2^j}) = \begin{cases} \min\{\Phi(y)(i, \frac{1}{2^j}) + \frac{\delta(y)}{2}, 1\} & j \le 2k \\ \min\{\frac{\delta(y)}{2}(1-t) + \Phi(y)(i, \frac{1}{2^j})t + \frac{\delta(y)}{2}, 1\} & j = 2k+1 \\ \min\{\Phi(y)(i, \frac{1}{2^j})t + \frac{\delta(y)}{2}, 1\} & j = 2k+2 \\ \delta(c) & j = 2k+3 \\ \frac{\delta(y)}{2}t & j = 2k+4 \\ B_i(y)(1-t) + \delta(y)t & j = 2k+5 \\ 0 & j = 2k+6 \\ B_i(y) & j = 2k+7 \\ B_i(y)g(y)(1)(1-t) & j = 2k+8 \\ B_i(y)[g(y)(m+1)(1-t) + g(y)(m)t] & j > 2k+8, \\ & \quad j \text{ is even} \\ B_i(y) & \text{others,} \end{cases}$$

where $t = 2 - 2^k\delta(y)$, $m = 2^{-1}j - k - 4$, and $B_i(y) = M_i(y, (i, \frac{1}{2^{2k-1}}))$ $(1-t) + M_i(y, (i, \frac{1}{2^{2k+1}}))t$.

Observe that $\Phi(y) \in F_n(X)$ and $B_i(y) \ge \Phi(y)(i, 0)$ for every $y \in Y \setminus K$ and $i \in \omega$. Thus, for every $y \in Y \setminus K$, we have $\Psi_k(y)(i, 0) \ge \frac{1}{n}$ for some $i \in \omega$, that is, $\Psi_k(y) \in F_n(X)$. If $y \in Y_k \cap Y_{k-1}$ for some $k \in \omega$ then $\delta(c) = \frac{1}{2^{k-1}}$ and an easy check shows that $\Psi_k(c) = \Psi_{k-1}(c)$.

We may define $\Psi : Y \to F_n(X)$ by $\Psi(c) = \Psi_k(c)$ for $c \in Y_k$ and $\Psi(c) = \Phi(c)$ if $c \in K$. Then Ψ is as required. □

Proof of Theorem 1.1. Consider the following triple of spaces.

$$(\downarrow\mathrm{USC}_F(X), \downarrow\mathrm{USC}_F(X) \setminus \downarrow\mathrm{C}_0(X), \downarrow\mathrm{C}_F(X) \setminus \downarrow\mathrm{C}_0(X)).$$

(1) By Lemma 3.1, $\downarrow\mathrm{USC}_F(X) \approx Q$, ,

(2) $\downarrow\mathrm{USC}_F(X) \setminus \downarrow\mathrm{C}_0(X) = \bigcup_{n=1}^{\infty} \downarrow\mathrm{F}_n(X)$ is the union of a tower $(\downarrow\mathrm{F}_n(X))_n$ in $\downarrow\mathrm{USC}_F(X)$ such that

(a) $\downarrow\mathrm{F}_n(X) \in Z(\downarrow\mathrm{USC}_F(X)) \cap Z(\downarrow\mathrm{F}_{n+1}(X))$ by Lemma 3.3;

(b) $(\downarrow\mathrm{F}_n(X), \downarrow\mathrm{F}_n(X) \cap (\downarrow\mathrm{C}(X) \setminus \mathrm{C}_0(X))) = (\downarrow\mathrm{F}_n(X), \downarrow\mathrm{C}_n(X))$ is strongly $(\mathcal{M}_0, \mathcal{F}_{\sigma\delta})$-universal for every $n \in \omega$, by Lemma 3.11;

(c) $\downarrow\mathrm{USC}_F(X) \setminus \downarrow\mathrm{C}_0(X)$ is a capset for $\downarrow\mathrm{USC}_F(X)$, by Lemma 3.10.

Consequently, by Theorem 2.11, we get

$$(\downarrow\mathrm{USC}_F(X), \downarrow\mathrm{USC}_F(X) \setminus \downarrow\mathrm{C}_0(X), \downarrow\mathrm{C}_F(X) \setminus \downarrow\mathrm{C}_0(X)) \approx (Q, \Sigma, c_0).$$

Thus,

$$\begin{aligned} & (\downarrow\mathrm{USC}_F(X), \downarrow\mathrm{C}_F(X)) \\ = \; & (\downarrow\mathrm{USC}_F(X), (\downarrow\mathrm{C}_F(X) \setminus \downarrow\mathrm{C}_0(X)) \cup \downarrow\mathrm{C}_0(X)) \\ = \; & (\downarrow\mathrm{USC}_F(X), (\downarrow\mathrm{C}_F(X) \setminus \downarrow\mathrm{C}_0(X)) \cup (\downarrow\mathrm{USC}_F(X) \setminus (\downarrow\mathrm{USC}_F(X) \setminus \downarrow\mathrm{C}_0(X)))) \\ \approx \; & (Q, c_0 \cup (Q \setminus \Sigma)). \end{aligned}$$

We complete the proof of Theorem 1.1. □

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