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by

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ACTION ON HYPERSPACES

A. DI CONCILIO

Abstract. Let X be a Tychonoff space, $HomX$ the full group of self-homeomorphisms of X and CLX the hyperspace of all nonempty closed subsets of X. First, by constructing a not locally compact model of topologist's comb, we show that local compactness is not a necessary condition for $HomX$ equipped with the compact-open topology being a topological group acting continuously on X by the evaluation map $e : (f, x) \in HomX \times X \to f(x) \in X$. Then, generalizing the compact case, we give necessary and sufficient conditions for $HomX$ equipped with a set-open topology based on a Urysohn family being a topological group acting continuously on CLX by the evaluation map $E : (f, C) \in HomX \times CLX \to f(C) \in CLX$. Furthermore, when X is a uniform space, we give necessary and sufficient conditions for $HomX$ equipped with the topology of uniform convergence on a uniformly Urysohn family being a topological group acting continuously on CLX. Besides, we do the same in the proximal case.

1. Introduction

Let X be a Tychonoff space, $HomX$ the full group of self-homeomorphisms of X and $e : (f, x) \in HomX \times X \to f(x) \in X$ the evaluation map.

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Key words and phrases. Homeomorphism group, group topology, evaluation function, group action, uniformity, uniform topology, proximity, proximity function, $p-$isomorphism, strong inclusion, boundedness, local proximity space, Urysohn family, uniformly Urysohn family, topology of uniform convergence on members of a given family of subsets, proximal set-open topology.

We focus our investigation on topologies on $HomX$ and its subgroups which yield continuity of both the group operations, product and inverse function, and at the same time yield continuity of the evaluation map. It is well known that the compact-open topology, when taking up these two features, is the most eligible one. In local compactness the compact-open topology yields joint continuity of the product and of the evaluation map too, [3]. But, unfortunately in general the compact-open topology does not provide continuity of the inverse function. In [1] R. Arens proved that if further X is locally connected then the compact-open topology makes continuous the inverse map. In [7], J.J. Dijkstra proved that it is enough to require the property: *any point of X has a connected compact nhbd.* Again in local compactness, in the manuscript [10], unpublished in my knowledge, K.R. Wicks gave necessary and sufficient conditions for the compact-open topology being a group topology by using non-standard methods on one side and action on hyperspace on the other side, so inspiring this paper. But, under local compactness is Dijkstra's property a necessary condition for the compact-open topology being a group topology? And is local compactness a necessary condition for the compact-open topology being a group topology that more makes the evaluation map jointly continuous? In both cases we give a negative answer by using as counterexample first a model of locally compact topologist's comb, a typical space that is not locally connected, and then a non locally compact one. R.K. Wicks proved that $HomX$ equipped with the compact-open topology $\tau_{c.o}$ being a topological group is equivalent to joint continuity of the evaluation map $E : (f, C) \in HomX \times CLX \to f(C) \in CLX$ w.r.t. $\tau_{c.o}$ and the Fell hypertopology τ_F. Since for the compact-open topology three different formulations as set-open topology, as the topology of uniform convergence on compacta and also as proximal set-open topology can be displayed, three possible generalizations in topology, proximity and uniformity arise from those. After analyzing the compact case we improve and contemporaneously generalize the compact case in the topological, uniform and proximal frameworks by replacing the compact-open topology with a set-open topology based on a Urysohn family, with a topology of uniform convergence on a uniformly Urysohn family, with a proximal set-open topology relative to a proximity and a boundedness giving a local proximity space, respectively. Finally, we show that the *topologicality of $HomX$ is equivalent to topologicality of the evaluation map $E : (f, C) \in HomX \times CLX \to f(C) \in CLX$*, as in the Wicks compact case, in each generalized case.

2. Preliminaries

In order to give some useful background and for more exhaustive information, the definitions, the terminology and the results quoted below are drawn by [2,5,8,9,11].

- *Set-open topologies and Urysohn families*

Let α be a collection of non-empty closed sets in X, the *set-open topology of $HomX$ based on α*, usually denoted as $\tau_{\alpha.o}$, is that admitting as subbasic open sets the ones:

$$[A, W] := \{f \in HomX : f(A) \subset W\}$$

where A runs through α and W is open in X. The prototype of set-open topologies is the compact-open topology $\tau_{c.o}$ where α is the family of all non-empty compact subsets of X.

Now, let α be a family of non-empty closed subsets of X containing all singletons and, for simplicity, closed under finite unions. Then α is a *Urysohn family* if whenever $A \in \alpha$ and $C \in CLX$ are disjoint there exists $B \in \alpha$ such that $A \subset intB \subset B \subset X \setminus C$. A Urysohn family is also a local family, i.e. any point in X has arbitrarily small nhbds belonging to α, that when closed by inclusion contains all compact sets.

Proposition 2.1. *If α is a Urysohn family invariant under homeomorphisms and $HomX$ is equipped with $\tau_{\alpha.o}$ then the product function $\circ$: $(f, g) \in HomX \times HomX \to g \circ f \in HomX$ is jointly continuous or, equivalently, $HomX$ equipped with $\tau_{\alpha.o}$ is a topological semigroup. Moreover, $HomX$ equipped with $\tau_{\alpha.o}$ acts continuously on X via the evaluation map $e : (f, x) \in HomX \times X \to f(x) \in X$.*

Proof. We show first that the product $\circ$ is jointly continuous. In order to do so, we assume $g \circ f \in [A, W]$, with $A \in \alpha$ and W open in X. That is equivalent to $f(A) \subset g^{-1}(W)$. The betweenness property implies there exists $U \in \alpha$ such that $f(A) \subset intU \subset U \subset g^{-1}(W)$. Of course, $[A, intU], [U, W]$ are $\tau_{\alpha.o}$−nhbds of f and g respectively. Furthermore, if $h \in [A, intU]$ and $k \in [U, W]$ it happens that $k \circ h \in [A, W]$, so that the first result is acquired. Second, the evaluation map e is jointly continuous because α is a local family. □

For any family α of subsets of X, we will denote as $\alpha_{\mathcal{H}} := \{h(A) : A \in \alpha \text{ and } h \in HomX\}$ the minimal family invariant under homeomorphisms containing α.

Lemma 2.1. *If α is a Urysohn family, then $\alpha_{\mathcal{H}}$ is in turn a Urysohn family. So, by previous proposition, the set-open topology based on $\alpha_{\mathcal{H}}$, $\tau_{\alpha_{\mathcal{H}}.o}$, is a semigroup topology.*

Proposition 2.2. *If $\tau_{\alpha.o}$ is a right semigroup topology, then $\tau_{\alpha.o}$ coincides with $\tau_{\alpha_{\mathcal{H}}.o}$.*

Proof. Of course, $\tau_{\alpha_{\mathcal{H}}.o}$ is finer than $\tau_{\alpha.o}$. The vice versa is also true. Suppose $f \in [B, W]$ and $B = h(A),\ A \in \alpha,\ h \in HomX$. Continuity at f of the right translation $R_h : g \in HomX \to g \circ h \in HomX$ implies there exists a τ_α−nhbd U_f of f such that if $g \in U_f$ then $g \circ h \in [A, W]$, which is the same as $g \in [B, W]$. □

Corollary 2.1. *If $\tau_{\alpha.o}$ is a right semigroup topology, then it is also a semigroup topology.*

In looking for set-open topologies which make $HomX$ as a topological group acting continuously on X, accordingly with the previously proven facts, we have to restrict our investigation to Urysohn families invariant under homeomorphisms.

• *Uniform topologies and uniformly Urysohn families*

Let $(X, \mathcal{U})$ be a uniform space. Then the *uniform topology induced by $\mathcal{U}$ on $HomX$*, usually denoted as $\tau_{\mathcal{U}}$, has as subbasic open sets at any $f \in HomX$ the ones:

$$(U, f) := \{h \in HomX : (f(x), h(x)) \in U,\ \text{for each } x \in X\},$$

where U runs in $\mathcal{U}$.

Let α stand for a family of non-empty subsets of X. The *topology of uniform convergence on members of α derived from $\mathcal{U}$*, usually denoted as $\tau_{\alpha,\mathcal{U}}$, is that admitting as subbasic open sets at any $f \in HomX$ the following:

$$(A, U, f) := \{h \in HomX : (f(x), h(x)) \in U \text{ for each } x \in A\},$$

where U varies in $\mathcal{U}$ and A runs through α.

If for any $U \in \mathcal{U}$ and any $h \in HomX$ we put:

$$U_h := \{(x, y) \in X \times X : (h(x), h(y)) \in U\}$$

and then set: $\mathcal{S}_{\mathcal{H}} := \{U_h : U \in \mathcal{U},\ h \in HomX\}$, we produce a subbase for a uniformity $\mathcal{U}_{\mathcal{H}}$ on X, which is separated and totally bounded whenever $\mathcal{U}$ is so. Moreover, the uniformity $\mathcal{U}_{\mathcal{H}}$ is the least uniformity finer than $\mathcal{U}$ for which any self-homeomorphism of X is uniformly continuous.

A family α of non-empty closed subsets of X, containing all singletons and, for simplicity, closed under finite unions, is a *uniformly Urysohn family* provided that whenever $A \in \alpha$ and $C \in CLX$ are far (i.e. there exists a diagonal nhbd $U \in \mathcal{U}$ such that $U[A] \cap C = \emptyset$), there exist a diagonal nhbd $U \in \mathcal{U}$ and a member B of α so that $U[A] \subset B \subset U[B] \subset X \setminus C$.

Proposition 2.3. *Let* $(X,\mathcal{U})$ *a uniform space. If* α *is a uniformly Urysohn family invariant under homeomorphisms, then* $HomX$ *equipped with* $\tau_{\alpha,\mathcal{U}}$ *is a right topological semigroup.*

Proof. For any given $h \in HomX$ the right translation R_h is continuous w.r.t. $\tau_{\alpha,\mathcal{U}}$ at any $f \in HomX$. In fact, any basic nhd $(A, U, f \circ h), A \in \alpha, U \in \mathcal{U}$ can be seen as the image through R_h of $(h(A), U, f)$. □

Proposition 2.4. *Let* $\mathcal{U}$ *a uniformity on* X*. If the topology* $\tau_{\alpha,\mathcal{U}}$ *of uniform convergence on* α *is a right semigroup topology, then it is also the topology of uniform convergence on* $\alpha_{\mathcal{H}}$ *induced by the same uniformity* $\mathcal{U}$*.*

Proof. It is easy to see that if $\tau_{\alpha,\mathcal{U}}$ makes continuous the right translations, then any net $\{f_\lambda\}$ of $HomX$ which uniformly converges to $f \in HomX$ on members of α, also uniformly converges to f on homeomorphic images of any member of α. □

Proposition 2.5. *If the topology* $\tau_{\alpha,\mathcal{U}}$ *of uniform convergence on* α *is a semigroup topology, then it is also the topology of uniform convergence on* $\alpha_{\mathcal{H}}$ *induced by the finer uniformity* $\mathcal{U}_{\mathcal{H}}$*.*

Proof. A net $\{f_\lambda\}$ of $HomX$ uniformly converges to $f \in HomX$ w.r.t. $\mathcal{U}_{\mathcal{H}}$ on members of $\alpha_{\mathcal{H}}$ if and only if for each $U \in \mathcal{U},\ h, k \in HomX,\ A \in \alpha$ any pair $(f_\lambda(k(x)), f(k(x)))$ belongs to U_h for each $x \in A$, eventually. That is equivalent to uniformly convergence to $h \circ f \circ k$ of the net $\{h \circ f_\lambda \circ k\}$ w.r.t. $\mathcal{U}$, that in turn is equivalent to the uniform convergence to f of the net $\{f_\lambda\}$ w.r.t. $\mathcal{U}$ on members of α. □

Corollary 2.2. *If* $(X,\mathcal{U})$ *is a uniform space for which any homeomorphism is uniformly continuous and* α *is a uniformly Urysohn family of* X*, invariant under homeomorphisms, then the topology of uniform convergence on* α *of* $HomX$ *makes the evaluation map* $e : (f, x) \in HomX \times X \to f(x) \in X$ *as a group action.*

Accordingly to the issues so far discussed, looking again for topologies of uniform convergence on members of some family, which make $HomX$ as a topological group acting continuously on X, we are conditioned to consider uniformly Urysohn families invariant under homeomorphisms and relative to uniformities for which any self-homeomorphism of X is a uniform isomorphism.

• *Proximal set-open topologies and boundedness*

Joining proximity with boundedness gives local proximity. A non-empty collection $\mathcal{B}$ of subsets of a set X is called a *boundedness* in X if and only if:
(a) $A \in \mathcal{B}$ and $B \subset A$ implies $B \in \mathcal{B}$ and (b) $A, B \in \mathcal{B}$ implies $A \cup B \in \mathcal{B}$. The elements of $\mathcal{B}$ are called *bounded sets.* A *local proximity space* $(X, \ll, \mathcal{B})$ consists of a set X, together with a strong inclusion $\ll$ on X and a boundedness $\mathcal{B}$ in X containing all singletons, which satisfy the following axioms:

1) $\emptyset \ll \emptyset$ 2) $A \ll B$ implies $A \subset B$
3) $A \ll B$ implies $X \setminus B \ll X \setminus A$ 4) $A \ll B$ and $A \ll C$ then $A \ll B \cap C$
5) $If A \in \mathcal{B}, C \subset X$ and $A \ll C$ then there exists some $B \in \mathcal{B}$ such that $A \ll B \ll C$.

The proximity δ associated with the above strong inclusion $\ll$ is defined by the formula: $A \bar{\delta} B$ if and only if $A \ll X \setminus B$. The notion of proximity and strong inclusion are interchangeable. The associated topological closure operator is given by putting as $Cl(A) := \{x \in X \ : \{x\} \ \delta \ A\}$.

When X carries a proximity δ and α is a network in X, the *proximal set-open topology relative to* α *and* δ, by acronym $PSOT_{\alpha.\delta}$, or $PSOT_\delta$ when α is the network $CL(X)$ of all non-empty closed subsets of X, is that admitting as subbasic open sets the ones:

$$[A, W]_\delta := \{ \ f \in HomX \ : \ f(A) \ll_\delta W \ \}$$

where A runs through α and W is an open subset of X and where $\ll$ is the strong inclusion associated with δ.

A collection α of subsets of a space X is a *network* provided that for any point x in X and any open set U containing x there is a member A of α such that $x \in A \subset U$. Again, the compact-open topology, when α is the family of all non-empty compact sets of X, is the prototype, since any compact set contained in an open set is also strongly contained whatever is the proximity.

Whenever α is a closed, hereditarily closed network in X and δ an EF-proximity on Y, then, $PSOT_{\alpha,\delta}$ is the topology of uniform convergence on α w.r.t. the unique totally bounded uniformity $\mathcal{U}^*(\delta)$ compatible with δ, [4]. This recasting takes up the opportunity of reformulating topologies of uniform convergence on members of a network, when the range space carries a proximity,[4]. In association with a local proximity space $(X, \ll, \mathcal{B})$ can be introduced the Alexandroff proximity δ_A, which is an EF-proximity, while usually δ is not, by setting : For any two non-empty subsets of X, $A \overline{\delta_A} B$ if and only if $A \bar{\delta} B$ and one of them is bounded. It can be shown that:

Theorem 2.1. *Let $(X, \mathcal{B}, \ll)$ be a local proximity space and δ the proximity associated with $\ll$. The proximal set-open topology $PSOT_{\mathcal{B},\delta}$ agrees with the topology of uniform convergence on members of $\mathcal{B}$ w.r.t. the unique totally bounded uniformity associated with the Alexandroff proximity δ_A.*

As a consequence of Corollary 2.2 and of the previous theorem we have:

Theorem 2.2. *Let $(X, \mathcal{B}, \ll)$ be a local proximity space and δ the proximity associated with $\ll$. If any self-homeomorphism of X preserves proximity and boundedness, then the proximal set-open topology $PSOT_{\mathcal{B}.\delta}$ makes the evaluation map e as a group action.*

• *Homeomorphisms and hypertopologies*

Let α be a subfamily of CLX. The *hit and miss (hyper)topology associated with* α, τ_α, is generated by the join of the hit topology τ^- generated by the hit sets $A^- := \{E \in CLX : E \cap A \neq \emptyset\}$, where A is open in X, with the miss topology τ_α^+ generated by the miss sets A^+, so defined: $A^+ := \{E \in CLX : \ E \subset A\}$, where A is an open set of X whose complement is a member of α. The Vietoris topology, when $\alpha = CLX$, is the archetype and the Fell topology, when α is the collection of all non-empty compact sets, is the prototype.

When X is endowed with a uniformity $\mathcal{U}$ (with a proximity δ) the notion of *far-miss topology associated with* α, $\tau_\alpha^{++}(\mathcal{U})$ $(\tau_\alpha^{++}(\delta)$) , can be naturally displayed by replacing the miss sets with far-miss sets defined as:

$$A^{++} := \{ \ E \in CLX : U[E] \subset A, \text{ for some } U \in \mathcal{U} \ (\ E \ll_\delta \ A) \ \},$$

where again A is an open subset of X whose complement is a member of α. Subsequently, the *hit and far-miss topology associated with* α can be defined in the same fashion.

Since homeomorphic images of closed sets are closed, any $f \in HomX$ naturally extends to CLX as $C \in CLX \to f(C) \in CLX$. Then, of course, it is natural to consider the evaluation map $E : (f, C) \in HomX \times CLX \to f(C) \in CLX$.

Proposition 2.6. *Let $\alpha \subset CLX$. If α is invariant under homeomorphisms, then any self-homeomorphism of X continuously extends to CLX equipped with τ_α^+ or also equipped with τ_α.*

Proof. If we denote, for simplicity, also the natural extension of f to CLX by f we can easily show that $f^{-1}[(X \setminus A)^+] = (X \setminus f^{-1}(A))^+$. When A runs through α, $(X \setminus A)^+$generates τ_α^+ and at the same time $(X \setminus f^{-1}(A))^+$ is in τ_α^+. Next, since it is simply checked that $f^{-1}(A^-) = (f^{-1}(A))^-$ for each subset $A \in X$, the final result follows. □

Proposition 2.7. *Let $(X,\mathcal{U})$ a uniform space for which any self-homeomorphism of X is a uniform isomorphism and $\alpha \subset CLX$. If α is invariant under homeomorphisms then any self-homeomorphism of X continuously extends to CLX equipped with the far-miss topology $\tau_\alpha^{++}(\mathcal{U})$ or also equipped with the hit and far-miss topology, the join of $\tau_\alpha^{++}(\mathcal{U})$ with τ^-.*

Proof. Denote again as f both $f \in HomX$ and its natural extension to CLX. It is enough to show that $f \in HomX$, $A \in \alpha$ yield $f^{-1}[(X\diagdown A)^{++}] = (X\diagdown f^{-1}(A))^{++}$, or, equivalently, since any $f \in HomX$ is a uniform isomorphism, if $C \in CLX$, $A \in \alpha$ and $U[f(C)] \subset X\diagdown A$ for some diagonal nhbd $U \in \mathcal{U}$, then there exists $V \in \mathcal{U}$ such that $V[C] \subset X\diagdown f^{-1}(A)$, the same as $C \in (X\diagdown f^{-1}(A))^{++} \in \tau_\alpha^{++}(\mathcal{U})$. Uniform continuity of f allows us to select a diagonal nhbd $V \in \mathcal{U}$ so that if $(x, y) \in V$ then $(f(x), f(y)) \in U$. This gets $f(V[C]) \subset U[f(C)] \subset X\diagdown A$ which in turn implies $V[C] \subset X\diagdown f^{-1}(A)$, that means exactly $C \in (X\diagdown f^{-1}(A))^{++}$. So, the first result follows. Accordingly with $f^{-1}(A^-) = (f^{-1}(A))^-$ for each subset A of X, the final result is acquired. □

Now we examine the proximal case. Arguments similar to the previous ones work in showing that:

Proposition 2.8. *Let $(X, \ll, \mathcal{B})$ be a local proximity space and δ the proximity associated with $\ll$. Then any self-homeomorphism of X which preserves proximity and boundedness continuously extends to CLX equipped with $\tau_{\mathcal{B}}^{++}(\delta)$.*

Accordingly with the previous results $HomX$ embeds as a subgroup in $HomCLX$ when α is Urysohn and CLX is equipped with τ^-, τ_α^+ or τ_α, the same when X carries a uniformity $\mathcal{U}$, α is uniformly Urysohn and CLX is equipped with $\tau^-, \tau_\alpha^{++}(\mathcal{U})$ or their join and also when X carries a proximity δ, α is a boundedness compatible with δ and CLX is endowed with $\tau^-, \tau_\alpha^{++}(\delta)$ or their join.

3. The compact-open topology as a group topology

When the compact-open topology $\tau_{c.o}$ on the full group of self-homeomorphism $HomX$ of a Tychonoff X space is a group topology? Local compactness of X gets $\tau_{c.o}$ being the topology of continuous convergence which in turn provides joint continuity of both the product function and evaluation map $e : (f, x) \in HomX \times X \to f(x) \in X$. Remind the notions of continuous convergence $\overrightarrow{c.c}$ and $g-$convergence $\overrightarrow{g}$:

$$\{f_\lambda\}_{\lambda\in\Lambda} \overrightarrow{c.c}\ f \text{ iff } \{x_\mu\}_{\mu\in M} \to x \Rightarrow \{f_\lambda(x_\mu)\}_{(\lambda,\mu)\in\Lambda\times M} \to f(x),$$

for each net $\{x_\mu\}$in X.

$$\{f_\lambda\}_{\lambda\in\Lambda} \overrightarrow{g}\ f \text{ iff } \{f_\lambda\}_{\lambda\in\Lambda} \overrightarrow{c.c}\ f \text{ and } \{f_\lambda^{-1}\}_{\lambda\in\Lambda} \overrightarrow{c.c}\ f^{-1}.$$

In local compactness $\tau_{c.o}$ being a group topology is equivalent to $\tau_{c.o}-$continuity of the inverse map at the identity function of X or also, equivalently, $\tau_{c.o}-$convergence coinciding with the $g-$convergence.

In local compactness and when becoming a group topology, $\tau_{c.o}$ has to be the uniform topology associated with the Alexandroff uniformity: *Two non-empty sets A and B are far if and only if their closures do not intersect and one of them is compact.* In [7], J.J. Dijkstra proved that *if any point in X admits a connected compact nhbd, then $\tau_{c.o}$ is a group topology.* But, under local compactness is Dijkstra's property a necessary condition for the compact-open topology being a group topology? And is local compactness a necessary condition for the compact-open topology being a group topology that more makes the evaluation map jointly continuous? In both cases we give a negative answer by using as counterexample first a model of locally compact topologist's comb, a typical space that is not locally connected, and then a non locally compact one.

Theorem 3.1. *The Dijkstra property is not a necessary condition for $\tau_{c.o}$ being a group topology on $HomX$.*

Proof. Let X be the union of the horizontal line segments $T_n = \{(x, 1/n) : 0 < x < 1\}$, where n is a positive integer, with the unit open interval on the $x-$axis $T_0 = \{(x, 0) : 0 < x < 1\}$. As subspace of the Euclidean plane the space X is locally compact but not locally connected. In fact, any point on the tooth of level 0 has no connected nhbd. So, the space X does not satisfy Dijkstra's condition, but anyway $\tau_{c.o}$ makes $HomX$ as a topological group. The proof is based on the following peculiar facts. a) Any homeomorphism between open intervals of the real line is a monotone function relatively to the usual order. b) If a net $\{f_\lambda\}$, $f_\lambda \in HomX$ for each λ, $\tau_{c.o}-$converges to the identity function of X, then for any fixed integer n it happens that $f_\lambda(T_n) = T_n$ eventually and, in particular, $f_\lambda(T_0) = T_0$, for each λ. c) If a net $\{f_\lambda\}$, $f_\lambda \in HomX$ for each λ, $\tau_{c.o}-$converges to the identity function of X, then there exists an index λ_0 such that for any $\lambda > \lambda_o$ and for each integer n the restrictions $f_\lambda : T_n \to f_\lambda(T_n)$ are monotone increasing. Now, if we denote by ∞ the ideal point of the one-point compactification of X it will be enough to show that if a net $\{f_\lambda\}$ in $HomX$ $\tau_{c.o}-$converges to the identity function of X and a net $\{x_\mu\}$ in X converges to ∞, then the net $\{f_\lambda(x_\mu)\}$ converges to ∞ as well. The trace on X of a basic nhbd of ∞ is of the type $L_1 \cup L_2$ with:

$$L_1 = \{(x, 1/n) : 0 < x < \epsilon,\ n \in N^+\} \ \cup\ \{(x, 0) : 0 < x < \epsilon\}$$

and

$$L_2 = \{(x, 1/n) : 1-\epsilon < x < 1,\ n \in N^+\} \ \cup\ \{(x, 0) : 1-\epsilon < x < 1\}$$

where $0 < \epsilon < 1$. Of course, x_μ is in $L_1 \cup L_2$ eventually. But, we can prove that also $\{f_\lambda(x_\mu)\}$ is in $L_1 \cup L_2$ eventually. Let K_1, K_2 two vertical line segments contained in L_1, L_2 respectively. Since they are both compact, and $\{f_\lambda\}$ $\tau_{c.o}-$ converges to the identity, then $f_\lambda(K_1) \subset L_1$ and $f_\lambda(K_2) \subset L_2$ eventually. So, if x_μ is in $L_1 \cap T_n$, that means x_μ is eventually greater than the unique point in $K_1 \cap T_n$, then $f_\lambda(x_\mu)$ is eventually greater than the unique point in $f_\lambda(K_1 \cap T_n)$. Since each $f_\lambda(K_1 \cap T_n)$ stays inside L_1, consequently $f_\lambda(x_\mu)$ stays in L_1 too. The same happens for L_2. □

Theorem 3.2. *Local compactness is not a necessary condition for $\tau_{c.o}$ being a group topology acting continuously on $HomX$.*

Proof. We work now on a topologist's comb that is not locally compact. Let $Y = X \cup \{(0,0), (1,0)\}$ the space obtained by adding to X, the previous model of topologist's comb, the extreme points of the tooth T_0. As subspace of the Euclidean plane the space Y is not locally compact at both the new added points. But it is rim-compact: any of its points admits arbitrarily small nhbs with compact boundary. In fact, the boundary of the intersection of the space Y with any square with vertical sides and center at an extreme point of T_0 consists of a convergent sequence jointly with its limit. Since the space Y is metrizable, the $\tau_{c.o}-$convergence agrees with the continuous convergence. So, it is enough to show that if $\{f_\lambda\}$ $\tau_{c.o}-$converges to the identity function f and $\{x_\mu\}$ is a net in X which does not accumulate, then $\{f_\lambda(x_\mu)\}$ does not converge in X. Any net in X has to accumulate in the Freudenthal compactification FY of Y, which is the closure of Y in the Euclidean plane. Any ideal point of FY is extreme point of a tooth distinct from T_0. For that there is a subnet of $\{f_\lambda(x_\mu)\}$ which converges to an ideal point. And the result follows. □

4. Beyond local compactness: main results

In [10], R.K. Wicks proved that $HomX$ equipped with the compact-open topology, $\tau_{c.o}$, being a topological group is equivalent to joint continuity of the evaluation map $E : (f, C) \in HomX \times CLX \to f(C) \in CLX$ w.r.t. $\tau_{c.o}$ and the Fell hypertopology τ_F. Since for the compact-open topology three different formulations as set-open topology, as the topology of uniform convergence on compacta and also as proximal set-open topology can be displayed, three possible generalizations in topology, uniformity and proximity arise from those. Having analyzed the properties of the compact-open topology in section 3, now we focus on generalizations in the topological, uniform and proximal frameworks by substituing on one side the compact-open topology with a set-open topology based on a Urysohn family, with a topology of uniform convergence on a uniformly Urysohn family, with a proximal set-open topology relative to a proximity and a boundedness giving a local proximity space and on the other side the Fell topology with a hit and miss topology in the topological setting, with a hit and far-miss topology in the uniform and proximity setting.

We now list our main results starting from the topological case.

Theorem 4.1. *Whenever α is a Urysohn family invariant under homeomorphisms, then the following properties are equivalent:*

1) *$HomX$ equipped with $\tau_{\alpha.o}$ is a topological group;*

2) *The inverse map $f \in HomX \to f^{-1} \in HomX$ is $\tau_{\alpha.o}$-continuous at the identity function;*

3) *$E_C : f \in (HomX, \tau_{\alpha.o}) \to f(C) \in (CLX, \tau_\alpha^+)$ is continuous for any $C \in CLX$;*

4) *$E : (f, C) \in HomX \times CLX \to f(C) \in CLX$ is jointly continuous w.r.t. $\tau_{\alpha.o}$ and τ_α^+;*

5) *$E : (f, C) \in HomX \times CLX \to f(C) \in CLX$ is jointly continuous w.r.t. $\tau_{\alpha.o}$ and τ_α.*

Proof. Observe before that $f \in [C, X \setminus A]$ if and only if $f^{-1} \in [A, X \setminus C]$. Next, since proposition 2.1 guarantees to $Hom(X)$ a topological semigroup structure, as previously remarked 1) and 2) are equivalent. 1) $\Rightarrow$ 3). For any $C \in CLX$, $A \in \alpha$ and $f \in HomX$, $f(C) \in (X \setminus A)^+$ is equivalent to $f \in [C, X \setminus A]$. But, as observed, $[C, X \setminus A]$, as image by the inverse map in $HomX$ of the $\tau_{\alpha.o}$ subbasic open set $[A, X \setminus C]$, is in turn a $\tau_{\alpha.o}$ open. Moreover, $E_C^{-1}[(X \setminus A)^+]$ is the same as $[C, X \setminus A]$. From that continuity of E_C follows. 3) $\Rightarrow$ 4): separate continuity gets joint continuity. Again, assume that $C \in CLX, A \in \alpha$, $f \in HomX$, and $f(C) \in (X \setminus A)^+$. Since α is a Urysohn family invariant under homeomorphisms, we can find $B \in \alpha$ so that $f(C) \subset X \setminus f(B) \subset X \setminus f(intB) \subset X \setminus A$. Furthermore, continuity of the map $E_{X \setminus intB}$ guarantees that $E_{X \setminus intB}^{-1}[(X \setminus A)^+]$ is a $\tau_{\alpha.o}$-nhbd of f. Also, $(X \setminus B)^+$ is a τ_α^+-nhbd of C. Thus, whenever $g \in E_{X \setminus intB}^{-1}[(X \setminus A)^+]$ and $K \in (X \setminus B)^+$, it happens that $g(K) \subset g(X \setminus B) \subset g(X \setminus intB) \subset X \setminus A$. The same as $g(K) \in (X \setminus A)^+$. And the result follows. 4) $\Rightarrow$ 5). Keep in mind that τ_α is the join of τ_α^+ with τ^- and assume that $C \in CLX$,

W is an open set of X and $f(C) \in W^-$. Then, we can select a point $x \in C$, a member $U \in \alpha$ so that $f(x) \in f(intU) \subset f(U) \subset W$. Accordingly, $[U, W]$ is a $\tau_{\alpha.o}$-nhbd of f and $(intU)^-$ is a τ^--nhbd of C. So, whenever $g \in [U, W]$ and $K \in (intU)^-$ it happens that $g(K \cap U)$ is a non empty subset of $g(K) \cap W$, that yields $g(K) \in W^-$. Finally, 5) $\Rightarrow$ 1). It is enough to show that the inverse image of any $\tau_{\alpha.o}$ subbasic open set $[A, X \setminus C]$, where $A \in \alpha$, $C \in CLX$, is in $\tau_{\alpha.o}$. But, it is really so, since, as already seen, $[A, X \setminus C]^{-1}$ that agrees with $[C, X \setminus A]$ can be expressed equally well as $E_C^{-1}[(X \setminus A)^+]$. □

Secondly, we approach the uniform case.

Theorem 4.2. *Let $(X, \mathcal{U})$ be a uniform space for which any self-homeomorphism of X is a uniform isomorphism and α a uniformly Urysohn family invariant under homeomorphisms. Then the following are equivalent:*

1) *$HomX$ equipped with the topology of uniform convergence on α, $\tau_{\alpha,\mathcal{U}}$, is a topological group;*

2) *$f \in HomX \to f^{-1} \in HomX$ is $\tau_{\alpha,\mathcal{U}}$-continuous at the identity function of X;*

3) *$E : (f, C) \in HomX \times CLX \to f(C) \in CLX$ is jointly continuous w.r.t. $\tau_{\alpha,\mathcal{U}}$ and the hit and far-miss topology join of $\tau_\alpha^{++}(\mathcal{U})$ with τ^-.*

Proof. For simplicity we assume that all diagonal nhbds at work are open and symmetric. From Proposition 2.3 and a previous remark 1) and 2) are equivalent. 1) $\Rightarrow$ 3). Let $f(C) \in (X \setminus A)^{++}$ or equivalently let $U[f(C)] \subset X \setminus A$ with $U \in \mathcal{U}, f \in HomX$, $C \in CLX$ and $A \in \alpha$. Since α is uniformly Urysohn and invariant under homeomorphisms, then we can find a diagonal nhbd $V \in \mathcal{U}$ and $B \in \alpha$ so that $V[f(C)] \subset X \setminus B \subset V[X \setminus B] \subset X \setminus A$. Next, uniform continuity of f allows us to choose a diagonal nhbd $V_1 \in \mathcal{U}$ so that if $(x, y) \in V_1$ then $(f(x), f(y)) \in V$ so having $f(V_1[C]) \subset V[f(C)] \subset X \setminus B$ and then $V_1[C] \subset X \setminus f^{-1}(B)$, the same as $C \in (X \setminus f^{-1}(B))^{++}$. Furthermore, $V[X \setminus B] \subset X \setminus A$ yields $V[A] \subset intB$. Thus, since the images by a uniform isomorphism of two far sets have to be far, there does exist a diagonal nhbd $V_2 \in \mathcal{U}$ so that $V_2[f^{-1}(A)] \subset f^{-1}(intB)$. Now, choose $W \in \mathcal{U}$, $W \circ W \subset V_2$ and, as natural, consider the $\tau_{\alpha,\mathcal{U}}-$nhbd (A, W, f^{-1}) of f^{-1}. Whenever $h \in (A, W, f^{-1})$, then $W[h(A)] \subset W \circ W[f^{-1}(A)] \subset V_2[f^{-1}(A] \subset f^{-1}(intB)$. Consequently, for some $W_1 \in \mathcal{U}$ it happens that $W_1[h^{-1}(X \setminus f^{-1}(intB))] \subset X \setminus A$. But, by hypothesis $(A, W, f^{-1})^{-1}$ has to be a $\tau_{\alpha,\mathcal{U}}-$nhbd of f. Accordingly, whenever g belongs to $(A, W, f^{-1})^{-1}$ and K to $(X \setminus f^{-1}(B))^{++}$, then for some $D \in \mathcal{U}$, $D[g(K)] \subset D[g(X \setminus f^{-1}(B))] \subset D[g(X \setminus f^{-1}(intB))] \subset X \setminus A$, the same as $g(K) \in (X \setminus A)^{++}$. Finally, given $f \in HomX, C \in CLX$, O an open set in X and $f(C) \in O^-$, select $x \in C, V \in \mathcal{U}, A \in \alpha$ so that $f(x) \in intf(A) \subset V[f(A)] \subset O$. It follows that whenever g belongs to (A, V, f) and K belongs to $(intA)^-$, then $g(K \cap intA)$ is a non empty subset of $V[f(A)]$ that in turn is contained in O, the same as $g(K) \in O^-$. So, the result is definitively acquired. □

Finally, we conclude with the proximal case by adding:

Theorem 4.3. *Let $(X, \mathcal{B}, \delta)$ be a local proximity space. If any self-homeomorphism of X preserves proximity and boundedness, then the following properties are equivalent:*

1) $HomX$ equipped with $PSOT_{\mathcal{B},\delta}$ is a topological group;

2) $f \in HomX \ \rightarrow \ f^{-1}$ is $PSOT_{\mathcal{B},\delta}$-continuous at the identity function of X;

3) $E : (f, C) \in HomX \times CLX \quad \rightarrow \quad f(C) \in CLX$ is jointly continuous w.r.t. $PSOT_{\mathcal{B},\delta}$ and $\tau_{\mathcal{B}}^{++}(\delta)$;

4)$PSOT_{\mathcal{B},\delta}$ agrees with the uniform topology associated with the Alexandroff uniformity.

Proof. The proof is similar to that one in Theorem 4.2 considering that the symmetry property of δ jointly with the preservation of the strong inclusion yields:

$$f \in [C, W]_\delta \ \text{ if and only if } f^{-1} \in [X \setminus W, X \setminus C]_\delta,$$

for each $f \in HomX$. For the equivalence of 4) with 1) through 3) see theorem 2.1. □

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