TOPOLOGY
PROCEEDINGS
Volume 41, 2013
Pages 99–110

http://topology.auburn.edu/tp/

Almost completeness and the Effros Open Mapping Principle in normed groups

by

A. J. Ostaszewski

Electronically published on July 6, 2012

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 41 (2013)
Pages 99-110

E-Published on July 6, 2012

ALMOST COMPLETENESS AND THE EFFROS OPEN MAPPING PRINCIPLE IN NORMED GROUPS

A. J. OSTASZEWSKI

Abstract. We extend van Mill's version of the Effros Open Mapping Principle from analytic groups to almost complete normed groups.

1. Introduction

We extend to the class of almost complete normed groups (definitions below), a class which includes Polish (i.e. separable, completely metrizable) topological groups, a result that was originally proved by Effros [8] for Polish topological groups acting transitively and continuously on a non-meagre metric space. This has recently been improved by van Mill [27] to *analytic* metric groups with separately continuous, transitive action on a non-meagre metric space. Thus van Mill's variant includes meagre analytic groups acting transitively on a non-meagre metric space (for an example see [27, Remark 2]). In fact van Mill's argument applies more generally to analytic *normed* groups. Below we admit (separately) continuous actions by non-analytic normed groups, but at the price of the normed groups being non-meagre (and so Baire by virtue of almost completness). In this connection we note the result due to [16] and [11, Th. 2.3.6 p. 355] that a Baire analytic *topological group* is Polish.

A metric space is *almost complete* if it contains a dense absolute $\mathcal{G}_\delta$ (for a relaxation see Th. 2.3 below). The notion of 'almost completeness' is due to Frolík in [10] (but its name to Michael [17] – see also [1] and [4]). To state our version of the Effros Open Mapping Theorem we first recall the definition of normed groups and group actions.

2010 *Mathematics Subject Classification.* 26A03; 04A15; 02K20.

Key words and phrases. Almost completeness, Open Mapping Theorem, analytic sets, analytic Cantor theorem, non-commutative groups, group-norm, topological groups.

In the next section we recall the notion of analyticity and its connection with Cantor's Theorem; this will allow us to formulate a convergence criterion (Lemma 2.2) applicable under almost completeness (in lieu of the Cauchy criterion under completeness), a key tool alongside the notion of a (continuous) map that is 'irreducible in category' recalled at the end of that section.

Definition 1.1 (Normed groups)**.** 1. For T an *algebraic* group with neutral element $e = e_T$, say that $||\cdot|| : T \to \mathbb{R}_+$ is a *group-norm* ([4]) if the following properties hold:
(i) *Subadditivity* (Triangle inequality): $||st|| \leq ||s|| + ||t||$;
(ii) *Positivity*: $||t|| > 0$ for $t \neq e$ and $||e|| = 0$;
(iii) *Inversion* (Symmetry): $||t^{-1}|| = ||t||$.
Then $(T, ||.||)$ is called a *normed group.*
2. The group-norm generates a right and a left *norm-topology* via the right-invariant and left-invariant metrics $d_R^T(s,t) := ||st^{-1}||$ and $d_L^T(s,t) := ||s^{-1}t|| = d_R^T(s^{-1},t^{-1})$. In the right norm-topology the right shift $\rho_t(s) := st$ is a uniformly continuous homeomorphism, since $d_R(sy,ty) = d_R(s,t)$, so in particular the group is a right topological group; likewise in the left norm-topology the left shift $\lambda_s(t) = st$ is a uniformly continuous homeomorphism. Since $d_L^T(t,e) = d_L^T(e,t^{-1}) = d_R^T(e,t)$, convergence at e is identical under either topology, and generated by sets of the form $B^{-1}B$, for B open or closed balls centered at e. In the absence of a qualifier, the 'right' norm-topology is to be understood.
3. See Section 2 for a characterization of almost-complete normed groups.

Recall that a normed group G *acts continuously* on X if there is a continuous mapping $\varphi : G \times X \to X$ such that $\varphi(e,x) = x$ and $\varphi(gh,x) = \varphi(g,\varphi(h,x))$. In view of applications, it is convenient to regard G as having the topology generated by the left-invariant metric $d_L^G(g,h) := ||g^{-1}h||$. Thus $g : x \to \varphi(g,x)$ is a continuous self-map of X with a continuous inverse, and so is an autohomeomorphism, denoted $g(.)$. The action is *separately continuous* if $g : x \to \varphi(g,x)$ is continuous (so again an autohomeomorphism) and $\xi_x : g \to \varphi(g,x)$ is continuous. By a theorem of Bouziad ([5]), if the normed group G is Baire, as will be the case below, a separately continuous action is necessarily jointly continuous. The action is *transitive* if for any x, y in X there is $g \in G$ such that $g(x) = y$. The action of G on X is *weakly micro-transitive* if for each $x \in X$ and each neighbourhood (abbreviated henceforth to nhd) A of e_G the set

$$\mathrm{cl}(Ax) = \mathrm{cl}\{ax : a \in A\}$$

has x as an interior point (in X). We noted that the nhds of e_G under either norm topology are the same, so the weak microaction property is a norm property, rather than a topological property of G. The action is *micro-transitive* if for $x \in X$ and each nhd A of e_G the set

$$Ax = \{ax : a \in A\}$$

is a nhd of x. This is again a norm property. We refer to Ax as an *x orbit* (the A-orbit of x).

Theorem 1.2 (Main Theorem)**.** *Let the normed group G have separately continuous and transitive action on X. If under either norm topology G is separable and almost complete (so Baire), and X is non-meagre, then the action of G is micro-transitive.*

A tribute to the importance of the Effros's result is the existence of several proofs with varying contexts, including [2], [12], [7], one attributed to Becker in [14] (based on the Kuratowski-Ulam, category analogue of the Fubini, theorem), as well as the already cited [27]. The last of these papers describes the historical development and applications to functional analysis and continuum theory; our interest arises with its application to topological regular variation (which draws on the 'crimping property' – for the definition and its derivation from the Effros Theorem see [4] Th.3.15, and for its applications see [3]). Our proof of the current version of the Effros Theorem blends the argument in [2] with that in [27], and relies on the convergence criterion of Lemma 2.2 below, valid in analytic spaces. It has recently emerged in [24] that the Effros Theorem is intimately connected with the notion of shift-compactness (for which see the survey [20], in this same volume), a fact that yields an altogether different, short proof of Th. 1.2, applicable beyond the separable context unlike the other proofs (by contradiction rather than directly, referring to one sequence rather than the two below); see also [18] for further background.

Remarks. 1. Working still under d_L^G and with G acting transitively, if for each fixed k the shift $\rho_k : g \to gk$ is a homeomorphism of G (having a continuous inverse $\rho_{k^{-1}}$), in particular if G is a topological group, then open-ness of any one map ξ_y implies open-ness of all the other maps ξ_x. Indeed, by transitivity write $ky = x$ for some $k \in G$; then $g(x) = g(k(y)) = gk(y)$, so that $\xi_x(g) = \xi_y \circ \rho_k(g)$, and then ξ_x is open (as a composition of two open maps).
If all the maps ξ_x are open, then for any fixed x and H an open nhd of e_G the set Hx is open, and so $x \in Hx$ is an open nhd of x, i.e. the action is microtransitive.
2. The theorem may be generalized in such a way that, in the case of a topological group, for T almost open (i.e. Baire non-meagre) in G the set $Tt^{-1}x$ is almost open for quasi all $t \in T$. For details see [18].

2. Analytic Cantor Theorem and convergent sequences

Recall that Cantor's Theorem on the intersection of a nested sequence of closed (or compact, as appropriate) sets has two formulations: (i) referring to vanishing diameters (in a complete-space setting), and (ii) to (countable) compactness. Our first aim in this section is to give a topological version that is in this same spirit but appropriate to an analytic, rather than complete or compact, context. For this we need first to recall that, in a metric space, a set is *analytic* if it is the continuous image of a Polish space P (as before, a separable topologically complete metrizable space), i.e. of the form $f(P)$ for f continuous and P Polish – see [13] for details.

Although our concern here is with metric spaces, there are several advantages in discussing analytic sets in the broader context of Hausdorff topological spaces, arising from explicitly exposing their underlying topological nature. The brief account below will suffice here – see [19] for a wider discussion.

For X a Hausdorff space write $\mathcal{K} = \mathcal{K}(X)$ for the family of compact subsets of a space X, and $\wp(X)$ for the power set. Following the the notation of [13], write I for $\mathbb{N}^{\mathbb{N}}$ endowed with the product topology (treating $\mathbb{N}$ as discrete), and with $i|n := (i_1, ..., i_n)$, for $i \in I$ and $n \in \mathbb{N}$, put $I(i|n) = \{j \in I : j|n = i|n\}$, a basic open nhd in I. For X a Hausdorff space a map $K : I \to \wp(X)$ is called *compact-valued* if $K(i)$ is compact for each $i \in I$, and *singleton-valued* if each $K(i)$ is a singleton. K is *upper semicontinuous* if, for each $i \in I$ and each open U in X with $K(i) \subseteq U$, there is a nhd $N = I(i|n)$ of i such that $K(j) \subseteq U$ for each j in N, i.e. $K(i|n) \subseteq U$ for some n, where we write $K(i|n) := K(I(i|n))$. A subset of X is $\mathcal{K}$-*analytic* if it is the image $K(I)$ under an upper-semicontinuous compact-valued map.

The following result is implicit in a number of situations, and goes back to Frolík's characterization of completely regular Čech-complete spaces as $\mathcal{G}_\delta$ in some compactification ([10]; see [9] §3.9).

Theorem 2.1 (Theorem AC – Analytic Cantor Theorem, [19]). *Let X be a Hausdorff space, and let $A = K(I)$ be $\mathcal{K}$-analytic in X, where K is compact-valued and upper semicontinuous.*
Suppose that $F_n \subseteq X$ is a decreasing sequence of closed sets in X such that

$$F_n \cap K(i_1, ..., i_n) \neq \emptyset,$$

for some $i = (i_1, ...) \in I$ and each n. Then

$$K(i) \cap \bigcap_n F_n \neq \emptyset.$$

Equivalently, if there are open sets V_n in I with $\mathrm{cl}V_{n+1} \subseteq V_n$ *and* $\mathrm{diam}_I V_n \downarrow 0$ *such that $F_n \cap K(V_n) \neq \emptyset$, for each n, then*
(i) $\bigcap_n \mathrm{cl}V_n$ is a singleton, $\{i\}$ say,
(ii) $K(i) \cap \bigcap_n F_n \neq \emptyset$.

Proof. If not, then $\bigcap_n K(i) \cap F_n = \emptyset$ and so, by compactness, $K(i) \cap F_p = \emptyset$ for some p, i.e. $K(i) \subseteq X \backslash F_p$. So by upper semicontinuity $F_p \cap K(I(i_1, ..., i_n)) = \emptyset$ for some $n \geq p$, yielding the contradiction $F_n \cap K(I(i_1, ..., i_n)) = \emptyset$. □

We will make use of the following immediate corollary.

Lemma 2.2 (Convergence criterion)**.** *In a normed group for $r_n \searrow 0$ and $\alpha_n = a_n \cdot ... \cdot a_1$ with $\mathrm{cl}B_{r_{n+1}}(a_{n+1}) \subseteq B_{r_n}(e)a_n$, if $X = K(I)$ is an analytic subset and $K(i_1, ..., i_n) \cap B_{r_n}(\alpha_n) \neq \emptyset$ for some $i \in I$, then the sequence $\{\alpha_n\}$ is convergent.*

Proof. Indeed, $\alpha_n \to \alpha$, if $\{\alpha\} = K(i) \cap \bigcap_n F_n$ for $F_n = \mathrm{cl}(B_{r_n}(\alpha_n))$. □

The convergence criterion may be used to derive the following characterization of almost complete normed groups – for the proof see [21, Th. 2].

Theorem 2.3 (Characterization Theorem (Almost completeness))**.** *In a separable normed group X under d_R^X, the following are equivalent:*
(i) X is a non-meagre absolute-$\mathcal{G}_\delta$ modulo a meagre set (i.e. is almost complete);
(ii) X contains a non-meagre analytic subset;
(iii) X is non-meagre and almost analytic, i.e. non-meagre and analytic modulo a meagre set.

Definition 2.4. (cf. [9] Ex. 3.1.C)**.** Call a map $f : X \to Y$ *irreducible on X in the sense of category* if there is no proper closed $F \subseteq X$ such that modulo a meagre set $f(F)$ equals $f(X)$.
Equivalently: for non-empty open V in X the set $f(V)$ is non-meagre in Y. In particular, for f continuous, if W is open in Y and meets $f(X)$ then, as $V = f^{-1}(W)$ is non-empty and open in X, the set $W \cap f(V) = f(V)$ is non-meagre. For brevity, we shall say that a set S is *heavy* (resp. *heavy on W*) if $S \cap V$ (resp. $S \cap W \cap V$) is non-meagre for each open V meeting S (resp. meeting $S \cap W$). We follow [6] in using this term; [27, Prop. 2.2] calls sets that are dense and heavy 'fat'. Note that if S is heavy, then it is dense and heavy on the interior of $\mathrm{cl}S$.

The following result is the first step in [27, Prop. 2.2] and is inspired by a theorem of Levi [15]; more in fact is true – see [19]. We repeat the proof as it is short.

Lemma 2.5. *For a continuous surjective map $f : X \to Y$ with X separable, there is a closed set $X' \subseteq X$ such that the restriction map $f' := f|X'$ is irreducible on X' in the sense of category.*

Proof. Let $\mathcal{U}$ be the family of open sets U such that $f(U)$ is meagre; put $U := \bigcup \mathcal{U}$, $X' := X \backslash U$ (which is closed) and $f' := f|X'$. By separability there is a countable open family $\mathcal{V}$ with $U = \bigcup \mathcal{V}$; then $f(U) = \bigcup_{V \in \mathcal{V}} f(V)$, being a countable union of meagre sets, is meagre. Suppose that for some V open in X the set $f'(V \cap X')$ is meagre; as $V \backslash X' \subseteq U$, one has $f(V) \subseteq f'(V \cap X') \cup f(U)$, which is meagre. So $V \in \mathcal{U}$ and $V \subseteq U$, so that $V \cap X'$ is empty; so f' is irreducible. $\square$

3. Proof of the Effros Theorem

We first give the normed-group version of a key result; that will require a definition. In what follows we use letters from the beginning of the alphabet for (open) subsets in G and letters from the end for (open) subsets of X.

For the next result note that, since $d_L^G(g, h) = d_R^G(g^{-1}, h^{-1})$, the map $g \to g^{-1}$ from (G, d_L^G) to (G, d_R^G) is an isometry. So if G is separable in either norm topology, then it is separable in the other; likewise with almost completeness.

Theorem 3.1 (Effros' Theorem – weak micro-transitive form)**.** *Let the normed group G act on X transitively. If G is separable under either norm topology and X is non-meagre, then the action of G is weakly micro-transitive.*

Proof. Here it is convenient to work under d_L^G so that each left shift $\lambda_g(h) = gh$ is a (uniformly) continuous bijection, as $d_L^G(gh', gh) = d_L^G(h', h)$, likewise its inverse $\lambda_{g^{-1}}$ and so λ_g is a homeomorphism. So if H is an open nhd of e_G, then gH is open in the left norm-topology. As G is second-countable there are elements g_n in G such that $\{g_n H : n \in \omega\}$ covers G. Fix $x \in X$. Now G acts transitively on X, so $\{g_n Hx : n \in \omega\}$ covers X. As X is non-meagre, for some n the set $\mathrm{cl}(g_n Hx)$ has non-empty interior. That is, for some non-empty open set W in X the set $g_n Hx$ is dense in W. Then Hx is dense in the open set $U := g_n^{-1}(W)$; indeed for any open $V \subseteq g_n^{-1}(W)$, the set $g_n(V)$ is open (since $g_n : X \to X$ is a homeomorphism) and, being contained in W, meets $g_n Hx$. So V meets Hx. Thus $\emptyset \neq U \subseteq \mathrm{int}(\mathrm{cl}(Hx))$.
As Hx is dense in U, for some $h \in H$, the point hx is in U, i.e. in $\mathrm{int}(\mathrm{cl}(Hx))$. So $x \in h^{-1}\mathrm{int}(\mathrm{cl}(Hx)) = \mathrm{int}(\mathrm{cl}(h^{-1}Hx)) \subseteq \mathrm{int}(\mathrm{cl}(H^{-1}Hx))$, since h is a homeomorphism. But sets of the form $H^{-1}H$ with H nhds of e_G form a basis for the open nhds of e_G, so x is in the interior of $\mathrm{cl}(Ax)$ for any nhd A of e_G. $\square$

Theorem 3.2 (Effros' Theorem from weak micro-transitivity)**.** *If the normed group G is almost complete under the right norm-topology and the separately continuous action of G on X is weakly micro-transitive, then the action of G is micro-transitive.*

Proof. Let $d = d^X$ be any metric on X and for a fixed $x \in X$ denote by $\xi := \xi_x : G \to X$ the evaluation map $\xi_x(g) = g(x)$, which is continuous. Fix x in X, and let $H_0 = K_0 = B_\varepsilon(e)$ be any open ball about $e = e_G$. By the Characterization Theorem (Th. 2.3), the group G, being almost complete, is almost analytic, i.e., modulo a meagre set, is the continuous image of a Polish space, P say under continuous f say. As G is separable metrizable, Lemma 2.5 applies and, for some closed $P_0 \subseteq P$ (so again a Polish space), we may as context permits dual use of the letter G write $H_0 = K_0 = G(P_0) \cup N$ with N meagre and $G(.)$ an irreducible continuous map on P_0, i.e. with the property that $G(A)$ is heavy for each non-empty open subset A of P_0. Without loss of generality, $G_0 = G(P_0)$ is dense. (Otherwise expand N to a meagre $\mathcal{F}_\sigma$. Then $G_0 \backslash N$ is a $\mathcal{G}_\delta$ and comeagre; then, G being a Baire space, $G_0 \backslash N$ is dense.) We put $Q_0 = P_0$ and without loss of generality assume that $\mathrm{diam}(P_0) = 1$.
Pick U_0 open with $x \in U_0 \subseteq \mathrm{cl}(H_0 x)$. Let $y \in U_0$; we will show that $y = gx$ for some $g \in H_0^{-1} H_0$.
We work inductively. We begin with the (rather long) first step in the induction. We then set out the general inductive step (which follows the same pattern). What follows is a 'back and forth' argument, performed within successively smaller sub-orbits of the orbit $H_0 x$ of x and sub-orbits of the orbit $K_0 y$ with the intention of showing that the limiting sub-orbits meet (i.e. $hx = ky$ for some $h \in H_0$ and $k \in K_0$, so that for $g = k^{-1} h \in H_0^{-1} H_0$ one has $y = gx$).
Put $x_0 = x$ and $y_0 = y$. We thus have

$$x_0 \in U_0 \subseteq \mathrm{cl}(H_0 x_0) \text{ with } y_0 \in U_0,$$

where the diameter of U_0 is less than 1 without loss of generality.
We first work within the y orbit: by weak microaction pick open V_0 with diameter less than $1 = 2^0$ such that $y_0 \in V_0 \subseteq \mathrm{cl}(K_0 y_0)$. Thus

$$x_0 \in U_0 \subseteq \mathrm{cl}(H_0 x_0) \text{ with } y_0 \in U_0 \text{ and } y_0 \in V_0 \subseteq \mathrm{cl}(K_0 y_0). \qquad \text{(ind-0)}$$

Combining the information about y_0, we have $y_0 \in U_0 \cap V_0$, so that $U_0 \cap V_0$ is non-empty open; furthermore $U_0 \cap V_0 \subseteq U_0 \subseteq \mathrm{cl}(H_0 x)$. So the x orbit $H_0 x$ in particular meets $U_0 \cap V_0$, i.e. there is $h_1' \in H_0$ such that

$$x_1' := h_1' x_0 \in U_0 \cap V_0 \subseteq V_0. \qquad (1')$$

As $x_1' = h_1' x_0 \in U_0 \cap V_0$, we have $h_1' \in A_0' := \xi^{-1}(U_0 \cap V_0) \cap H_0$, so this open set is non-empty. (Recall that ξ is continuous.) As $G_0 = G(P_0)$ is dense and heavy, $A'_0 \cap G_0$ is non-empty, and so by continuity $G^{-1}(A'_0)$ is a non-empty open subset of P_0. By Lemma 2.5 there is thus a closed subset $P_1 \subseteq P_0$, with diam(P_1) <diam$(P_0)/2 = 2^{-1}$, such that $G(P_1) \subseteq A'_0$ and $G(P_1)$ is heavy. So for some non-empty open $A_0 \subseteq A'_0$ the set $G(P_1)$ is dense and heavy on A_0. For $h_1 \in A_0 \subseteq A'_0 \subseteq \xi^{-1}(U_0 \cap V_0)$ we have $\xi(h_1) = h_1(x) \in U_0 \cap V_0$, so

$$x_1 := h_1 x_0 \in U_0 \cap V_0 \subseteq V_0. \tag{1}$$

Now there exists a ball H_1 about e_G of diameter at most $\varepsilon/2$ such that $H_1 h_1 \subseteq A_0 \subseteq H_0$, and $G_1 := G(P_1)$ is dense and heavy on $H_1 h_1$.
Now we work in the orbit of $h_1 x_0$: by weak microaction, for some U_1 open and of diameter less than 2^{-1} in X,

$$x_1 = h_1 x_0 \in U_1 \subseteq \mathrm{cl}(H_1 h_1 x_0). \tag{2$'$}$$

By (1) and (2$'$) and (ind-0), $x_1 \in U_1 \cap V_0 \subseteq V_0 \subseteq$cl$(K_0 y_0)$. So here too the orbit $K_0 y_0$ meets $U_1 \cap V_0$ and so there is $k'_1 \in K_0$ such that

$$y'_1 := k'_1 y_0 \in U_1 \cap V_0.$$

Write $\tilde{\xi} := \xi_y$. Since $k'_1 \in \tilde{A}'_0 := \tilde{\xi}^{-1}(U_1 \cap V_0) \cap K_0$, this open set is non-empty. As G_0 is dense and heavy in G, $\tilde{A}'_0 \cap G_0$ is non-empty and so $G^{-1}(\tilde{A}'_0)$ is a non-empty subset of Q_0. There is thus a closed subset Q_1, with diam(Q_1) <diam$(Q_0)/2 = 2^{-1}$, such that $G(Q_1) \subseteq \tilde{A}'_0$ and $G(Q_1)$ is heavy. So for some non-empty open $\tilde{A}_0 \subseteq \tilde{A}'_0$ the set $G(Q_1)$ is dense and heavy on $\tilde{A}_0$. For $k_1 \in \tilde{A}_0 \subseteq \tilde{A}'_0 \subseteq \tilde{\xi}^{-1}(U_1 \cap V_0)$ we have $\tilde{\xi}(k_1) = k_1(y) \in U_1 \cap V_0$, so

$$y_1 := k_1 y_0 \in U_1 \cap V_0 \subseteq V_0. \tag{2}$$

Now there exists a ball K_1 about e_G of diameter less than $\varepsilon 2^{-1}$ such that $K_1 k_1 \subseteq \tilde{A}_0 \subseteq K_0$, and $\tilde{G}_1 := G(Q_1)$ is dense and heavy on $K_1 k_1$.
Working again in the y_1 orbit: by weak microaction, for some V_1 open with diameter less than 2^{-1}

$$y_1 \in V_1 \subseteq \mathrm{cl}(K_1 y_1).$$

This completes the first step in the induction, as we now have closed sets P_1, Q_1 in P, open nhds H_1, K_1 of e_G of diameter less than $\varepsilon 2^{-1}$, points h_1, k_1 in G, points x_1, y_1 in X, and open sets U_1, V_1 in X with diameter less than 2^{-1} such that

$$x_1 \in U_1 \subseteq \mathrm{cl}(H_1 x_1) \text{ with } y_1 \in U_1 \text{ and}$$

$$y_1 \in V_1 \subseteq \mathrm{cl}(K_1 y_1), \tag{ind-1}$$

and

$$\begin{aligned} G_1 &:= G(P_1) \text{ is dense and heavy on } H_1h_1, \\ \tilde{G}_1 &:= G(Q_1) \text{ is dense and heavy on } K_1k_1. \end{aligned}$$

In general suppose that we now have closed sets P_{n-1}, Q_{n-1} in P, open nhds H_{n-1}, K_{n-1} of the identity in G of diameter less than $\varepsilon 2^{-(n-1)}$, points $h_1, ..., h_{n-1}$ and $k_1, ..., k_{n-1}$ in G, points $x_1, .., x_{n-1}$ and $y_1, ..., y_{n-1}$ in X with $x_i = h_i x_{i-1}$ and $y_i = k_i y_{i-1}$ for $i \leq n-1$, and open sets U_{n-1}, V_{n-1} in X with diameter less than $2^{-(n-1)}$ such that

$$\begin{aligned} &x_{n-1} \in U_{n-1} \subseteq \mathrm{cl}(H_{n-1}x_{n-1}) \text{ with } y_{n-1} \in U_{n-1} \text{ and} \\ &\quad y_{n-1} \in V_{n-1} \subseteq \mathrm{cl}(K_{n-1}y_{n-1}). \qquad \text{(ind-}(n-1)) \end{aligned}$$

and

$$\begin{aligned} G_{n-1} &:= G(P_{n-1}) \text{ is dense and heavy on } H_{n-1}\eta_{n-1}, \\ \text{where } \eta_{n-1} &:= h_{n-1} \cdot ... \cdot h_1, \\ \tilde{G}_{n-1} &:= G(Q_{n-1}) \text{ is dense and heavy on } K_{n-1}\kappa_{n-1}, \\ \text{where } \kappa_{n-1} &:= k_{n-1} \cdot ... \cdot k_1, \end{aligned}$$

with $\mathrm{diam}(P_{n-1}) < 2^{-(n-1)}$ and likewise $\mathrm{diam}(Q_{n-1}) < 2^{-(n-1)}$.
Then $y_{n-1} \in U_{n-1} \cap V_{n-1} \subseteq U_{n-1} \subseteq \mathrm{cl}(H_{n-1}x_{n-1})$, so as above there is $h_n' \in H_{n-1}$ such that

$$x_n' := h_n' x_{n-1} \in U_{n-1} \cap V_{n-1} \subseteq V_{n-1}. \qquad (1' : n)$$

Write $\xi_{n-1} := \xi_{x(n-1)}$. As $x_n' = h_n' x_{n-1} \in U_{n-1} \cap V_{n-1}$ and $x_n' = h_n' \eta_{n-1} x_0$ with $h_n' \in H_{n-1}$, we have $h_n' \eta_{n-1} \in A_{n-1}' := \xi_{n-1}^{-1}(U_{n-1} \cap V_{n-1}) \cap H_{n-1}\eta_{n-1}$, and this open set is non-empty (as ξ_{n-1} is continuous). As $G_{n-1} = G(P_{n-1})$ is dense and heavy on A_{n-1}' we have that $A_{n-1}' \cap G_{n-1}$ is non-empty and so $G^{-1}(A_{n-1}')$ is a non-empty subset of P_{n-1}. There is thus a closed subset P_n, with $\mathrm{diam}(P_n) < \mathrm{diam}(P_{n-1})/2 < 2^{-n}$, such that $G(P_n) \subseteq A_{n-1}'$ and $G(P_n)$ is heavy. So for some non-empty open $A_{n-1} \subseteq A_{n-1}'$ the set $G(P_n)$ is dense and heavy on A_{n-1}. For $h_n \in A_{n-1} \subseteq A_{n-1}' \subseteq \xi_{n-1}^{-1}(U_{n-1} \cap V_{n-1})$ we have $\xi_{n-1}(h_n) = h_n(x_{n-1}) = h_n \eta_{n-1} \in U_{n-1} \cap V_{n-1}$, so

$$x_n := h_n x_{n-1} \in U_{n-1} \cap V_{n-1} \subseteq V_{n-1}. \qquad (1 : n)$$

Now there exists a ball H_n about e_G of diameter at most $\varepsilon 2^{-n}$ such that $H_n h_n \eta_{n-1} \subseteq A_{n-1} \subseteq H_{n-1}\eta_{n-1}$, and one has that $G(P_n)$ dense and heavy on $H_n \eta_n$ for $\eta_n = h_n \cdot \eta_{n-1}$.
Next we work in the orbit of $h_n x_{n-1}$: by weak microaction, for some U_n open and of diameter less than 2^{-n} in X,

$$x_n = h_n x_{n-1} \in U_n \subseteq \mathrm{cl}(H_n h_n x_{n-1}). \qquad (2' : n)$$

By (1:n), (2$'$:n) and (ind-($n-1$)), $x_n \in U_n \cap V_{n-1} \subseteq V_{n-1} \subseteq$cl$(K_{n-1}y_{n-1})$. So here too the orbit $K_{n-1}y_{n-1}$ meets $U_n \cap V_{n-1}$ and so there is $k'_n \in K_{n-1}$ such that

$$y'_n := k'_n y_{n-1} \in U_n \cap V_{n-1}.$$

Write $\tilde{\xi}_{n-1} := \xi_{y(n-1)}$. Since $k'_n \in \tilde{A}'_{n-1} := \tilde{\xi}^{-1}_{n-1}(U_n \cap V_{n-1}) \cap K_{n-1}\kappa_{n-1}$, this open set is non-empty. As $\tilde{G}_{n-1}$ is dense and heavy in $K_{n-1}\kappa_{n-1}$ by the inductive hypothesis, $\tilde{A}'_{n-1} \cap \tilde{G}_{n-1}$ is non-empty and so $G^{-1}(\tilde{A}'_{n-1})$ is a non-empty subset of Q_{n-1}. There is thus a closed subset Q_n, with diam(Q_n) <diam$(Q_{n-1})/2 < 2^{-n}$, such that $G(Q_n) \subseteq \tilde{A}'_{n-1}$ and $G(Q_n)$ is heavy. So for some non-empty open $\tilde{A}_{n-1} \subseteq \tilde{A}'_{n-1}$ the set $G(Q_n)$ is dense and heavy on $\tilde{A}'_{n-1}$. For $k_n \in \tilde{A}_{n-1} \subseteq \tilde{A}'_{n-1} \subseteq \tilde{\xi}^{-1}_{n-1}(U_n \cap V_{n-1})$ we have $\tilde{\xi}_{n-1}(k_n) = k_n(y_{n-1}) \in U_n \cap V_{n-1}$, so

$$y_n := k_n y_{n-1} \in U_n \cap V_{n-1} \subseteq V_{n-1}. \qquad (2:n)$$

Now there exists a ball K_n about e_G of diameter less than $\varepsilon 2^{-n}$ such that $K_n\kappa_n \subseteq \tilde{A}_{n-1} \subseteq K_{n-1}\kappa_{n-1}$, and $\tilde{G}_n := G(Q_n)$ dense and heavy on $K_n\kappa_n$ for $\kappa_n = k_n \cdot \kappa_{n-1}$.

Working again in the y_n orbit: by weak microaction, for some V_n open with diameter less than 2^{-n}

$$y_n \in V_n \subseteq \text{cl}(K_n y_n).$$

This completes the general induction step, as we now have subsets P_n, Q_n in P, sets H_n, K_n nhds of the identity in G, points x_n, y_n and sets U_n, V_n in X such that $x_n \in U_n$ with $y_n \in U_n$ and $y_n \in V_n \subseteq$cl$(K_n y_n)$.

By Lemma 2.2, the products $\eta_n = h_n h_{n-1}...h_1$ and $\kappa_n = k_n k_{n-1}...k_1$ are convergent sequences, with limit say h and k resp. Thus $h \in$cl(H_0) and $k \in$cl(K_0) =cl(H_0), which are closed balls centered at e_G. Thus $k^{-1}h \in$cl$(H_0)^{-1}$cl(H_0). But sets of the form $B^{-1}B$ with B a closed ball around e_G are a base for the topology at e_G, so $k^{-1}h$ is as small as we wish.

For fixed x the map $g \to gx$ is continuous and $h_n h_{n-1}...h_1 \to h$, so $x_n = h_n h_{n-1}...h_1 x \to hx$, and likewise $y_n = k_n k_{n-1}...k_1 y \to ky$. (For instance, if $G \subseteq Auth(X)$ has the supremum metric derived from d^X, then $d^X(h_n h_{n-1}...h_1 x, hx) \leq \hat{d}(h_n h_{n-1}...h_1, h) \to 0$.)

But x_n and y_n have a common limit (since $x_n, y_n \in U_n$ and d^X-diam$(U_n) \to 0$), so $hx = ky$. Thus $y = k^{-1}hx$, as promised. $\square$

Theorems 3.1 and 3.2 now yield the Main Theorem (Th. 1.2).

Acknowledgement. I am grateful to the Referee for a careful reading of the paper and useful comments regarding exposition, and much indebted to Henryk Toruńczyk for bringing to my notice Ancel's work and his own. A study of [2] has led to a broadening of perspectives on the

Effros Theorem both here and in the companion papers cited after Th. 1.2, also beyond – in topological regular variation, the original motivation, an area developed jointly with Nick Bingham, whose mathematical taste has thus left a mark here also.

References

[1] J. M. Aarts and D. J. Lutzer, *Completeness properties designed for recognizing Baire spaces.* Dissertationes Math., **116** (1974), 48p.

[2] F. D. Ancel, *An alternative proof and applications of a theorem of E. G. Effros*, Michigan Math. J. **34** (1987), no. 1, 39–55.

[3] N. H. Bingham and A. J. Ostaszewski, *Topological regular variation: I. Slow variation*, Top. App. **157** (2010), 1999–2013.

[4] N. H. Bingham and A. J. Ostaszewski, *Normed versus topological groups: dichotomy and duality*, Dissertationes Math., **472** (2010), 138 pp.

[5] A. Bouziad, *Continuity of separately continuous group actions in p-spaces*, Topology & its App. **71** (1966), 119–124.

[6] J. C. Bradford and C. Goffman, *Metric spaces in which Blumberg's theorem holds*, Proc. Amer. Math. Soc. **11** (1960), 667–670.

[7] J. J. Charatonik and T. Maćkowiak, *Around Effros' theorem*, Trans. Amer. Math. Soc. **298** (1986), no. 2, 579–602.

[8] E. G. Effros, *Transformation groups and C^*-algebras.* Ann. of Math. (2) **81** (1965), 38–55.

[9] R. Engelking, *General topology*, Heldermann Verlag, 1989.

[10] Z. Frolík, *Generalizations of the $\mathcal{G}_\delta$-property of complete metric spaces*, Czechoslovak Math. J **10** (85) (1960), 359–379.

[11] J. Hoffmann-Jørgensen, *Automatic continuity*, Section 3 of [25].

[12] A. Hohti, *Another alternative proof of Effros' theorem*, Top. Proc. **12** (1987), 295–298.

[13] J. Jayne and C. A. Rogers, *Analytic sets,* Part 1 of [26].

[14] A. S. Kechris, *Topology and descriptive set theory*, Top. Appl. **58** (1994), no. 3, 195–222.

[15] S. Levi, *On Baire cosmic spaces*, General topology and its relations to modern analysis and algebra, V (Prague, 1981), 450–454, Heldermann, Berlin, 1983.

[16] R. J. Loy, *Multilinear mappings and Banach algebras.* J. London Math. Soc. (2) **14** (1976), no. 3, 423–429.

[17] E. Michael, *Almost complete spaces, hypercomplete spaces and related mapping theorems*, Top. Appl. **41** (1991), no. 1-2, 113–130.

[18] H. I. Miller and A. J. Ostaszewski, *Group-action and Shift-compactness*, Journal of Mathematical Analysis and Applications **392** (2012), 23–39.

[19] A. J. Ostaszewski, *Analytically heavy spaces: analytic Baire and analytic Cantor theorems*, Top. Appl., **158** (2011), 253–275.

[20] A. J. Ostaszewski, *Shift-compactness in almost analytic submetrizable Baire groups and spaces*, invited survey article, Topology Proc., this issue.

[21] A.J. Ostaszewski, *Beyond Lebesgue and Baire: III. Steinhaus' Theorem and its descendants,* preprint.

[22] A. J. Ostaszewski, *The semi-Polish Theorem: one-sided vs joint continuity in groups*, preprint.

[23] A. J. Ostaszewski, *Analytic Baire spaces*, Fund. Math. **217** (2012), 189–210.

[24] A. J. Ostaszewski, *On the Effros Open Mapping Principle*, preprint.

[25] F. Topsøe, J. Hoffmann-Jørgensen, *Analytic spaces and their applications*, Part 3 of [26].

[26] C. A. Rogers, J. Jayne, C. Dellacherie, F. Topsøe, J. Hoffmann-Jørgensen, D. A. Martin, A. S. Kechris, A. H. Stone, *Analytic sets,* Academic Press, 1980.

[27] J. van Mill, *A note on the Effros Theorem*, Amer. Math. Monthly **111** (2004), no. 9, 801–806.

[28] J. van Mill, *Analytic groups and pushing small sets apart,* Trans. Amer. Math. Soc. **361** (2009), no. 10, 5417–5434.

Mathematics Department, London School of Economics, Houghton Street, London WC2A 2AE

E-mail address: a.j.ostaszewski@lse.ac.uk