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ČECH-STONE COMPACTIFICATIONS OF DISCRETE SPACES IN ZF AND SOME WEAK FORMS OF THE BOOLEAN PRIME IDEAL THEOREM

ERIC J. HALL AND KYRIAKOS KEREMEDIS

Abstract. For every infinite set X, we show that:
(i) In **ZF**, the Stone space $S(X)$ of the Boolean algebra of all subsets of X is homeomorphic with the Čech-Stone extension $\beta(X)$ of the discrete space $\mathbf{X}$.
(ii) In $\mathbf{ZF}+\mathbf{BPI}$ (the Boolean Prime Ideal theorem), $\mathbf{Id}(X)$ (X has an independent family of size $|\mathcal{P}(X)|$) $\leftrightarrow$ "$\forall X, \mathbf{2}^{\mathcal{P}(X)}$ is a continuous image of $S(X)$".
(iii) In $\mathbf{ZF}+\mathbf{Id}(X)$, "every filterbase of X is a subset of an ultrafilter of X" $\leftrightarrow$ "$\mathbf{2}^{\mathcal{P}(X)}$ is compact". However, in **ZF** alone, "every filterbase of X is a subset of an ultrafilter of X" $\not\rightarrow$ "$\mathbf{2}^{\mathcal{P}(X)}$ is compact".

1. Notation and terminology

Let $\mathbf{X} = (X, T)$ be a topological space.

$\mathbf{X}$ is said to be *compact* iff every open cover $\mathcal{U}$ of X has a finite subcover $\mathcal{V}$. Equivalently, $\mathbf{X}$ is compact iff every family $\mathcal{G}$ of closed subsets of X with the *finite intersection property*, fip for abbreviation, has a non-empty intersection.

$\mathbf{X}$ is said to be *ultrafilter compact* iff every ultrafilter $\mathcal{F}$ of X converges to some point $x \in X$.

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Given a set X, $\mathbf{2}^X$ denotes the Tychonoff product of the discrete space $\mathbf{2}$ ($2 = \{0,1\}$ is taken with the discrete topology) and,

$$\mathcal{B}(\mathbf{2}^X) = \{[p] : p \in Fn(X, 2, \omega)\},$$

where $Fn(X, 2, \omega)$ is the set of all finite partial functions from X into 2 and

$$[p] = \{f \in 2^{\mathbb{R}} : p \subset f\},$$

will denote the canonical (clopen) base for the product topology on $\mathbf{2}^X$.

If $X \neq \varnothing$ then $S(X)$ will denote the *Stone space* of the Boolean algebra of all subsets of X. i.e., the set of all ultrafilters on X together with the topology having as a base the collection of all (clopen) sets of the form

$$[Z] = \{\mathcal{F} \in S(X) : Z \in \mathcal{F}\}.$$

A family $\mathcal{F}$ of subsets of X is *independent* if for any two non-empty finite and disjoint subsets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{F}$ the set $\bigcap \mathcal{A} \cap (\bigcap \{B^c : B \in \mathcal{B}\}$ is infinite.

Let X be a non empty set endowed with the discrete topology. Let,

- χ_A denote the characteristic function of A,
- $\Delta_X = \{\chi_A : A \in \mathcal{P}(X)\}$,
- $C = [0,1]^X$ be the set of all functions from X to $[0,1]$,
- $\delta : \mathbf{X} \to \mathbf{2}^{\Delta_X}$ and $e : \mathbf{X} \to [\mathbf{0},\mathbf{1}]^C$ be the functions given by $\delta(x) = (\chi_A(x))_{A \in \mathcal{P}(X)}$ and $e(x) = (f(x))_{f \in C}$, where $[\mathbf{0},\mathbf{1}]^C$ denotes the Tychonoff product of C many copies of the subspace $[\mathbf{0},\mathbf{1}]$ of the real line $\mathbb{R}$,
- $\pi_{\Delta_X} : [\mathbf{0},\mathbf{1}]^C \to [\mathbf{0},\mathbf{1}]^{\Delta_X}$ be the projection of $[\mathbf{0},\mathbf{1}]^C$ onto $[\mathbf{0},\mathbf{1}]^{\Delta_X}$,
- $\mathcal{F}_x$ denote the principal ultrafilter of X generated by $x \in X$,
- $\mathcal{W}_x = \{\chi_A : A \in \mathcal{F}_x\}, x \in X$,
- for all $y \in 2^{\Delta_X}, \mathcal{F}(y)$ denotes the set $\{A \in \mathcal{P}(X) : h(\chi_A) = 1\}$.

Clearly,

- $\delta = \pi_{\Delta_X} \circ e$, and the image of $\pi_{\Delta_X} \circ e$ is contained in $\mathbf{2}^{\Delta_X}$.
- $\mathcal{F}(\chi_{\mathcal{W}_x}) = \{A \in \mathcal{P}(X) : \chi_{\mathcal{W}_x}(\chi_A) = 1\} = \{A \in \mathcal{P}(X) : x \in A\} = \mathcal{F}_x$,
- the collections Δ_X and C separate points from closed sets. Hence, the functions δ and e are embeddings, see Theorem 4.2 in [8] p. 220 for details.

Furthermore, for all $A \in \mathcal{P}(X)$,

(1)

$$\chi_{\mathcal{W}_x}(\chi_A) = \begin{cases} 1 \text{ if } \chi_A \in \mathcal{W}_x \\ 0 \text{ if } \chi_A \notin \mathcal{W}_x \end{cases} = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases} = \chi_A(x) = \delta(x)(\chi_A)$$

and consequently,

(2) $$\delta(X) = \{\chi_{\mathcal{W}_x} : x \in X\}.$$

Definition 1. Let X be a non empty set endowed with the discrete topology. $\gamma(X) = \overline{\delta(X)}$ is called the *Stone extension* of $\mathbf{X}$ and $\beta(X) = \overline{e(X)}$ is called the *Čech-Stone extension* of $\mathbf{X}$.

Let $\mathbf{Y}$ be an ultrafilter compact T_2 space and $f : X \to Y$ be a function. Clearly, for every $\mathcal{F} \in S(X)$, $\{f(F) : F \in \mathcal{F}\}$ is an ultrafilter of $f(X)$. Since closed subspaces of ultrafilter compact spaces are clearly ultrafilter compact, it follows that the ultrafilter $\mathcal{H}$ of $\overline{f(X)}$ which is generated by $\{f(F) : F \in \mathcal{F}\}$ converges to a unique point $y_{\mathcal{F}} \in \overline{f(X)}$. It is straightforward to verify that

$$\{y_F\} = \bigcap\{\overline{f(F)} : F \in \mathcal{F}\}. \tag{3}$$

Definition 2. Let X be a non empty set, $\mathbf{Y}$ an ultrafilter compact T_2 space and $f : X \to Y$ be a function. The function $\bar{f} : S(X) \to \mathbf{Y}$ given by $\bar{f}(\mathcal{F}) = y_{\mathcal{F}}$ where, for every $\mathcal{F} \in S(X), y_{\mathcal{F}}$ is given by (3), is called the *Stone extension* of f.

Clearly, if $\mathcal{F} = \mathcal{F}_x$ then $\bar{f}(\mathcal{F}_x) = f(x)$ and this justifies the term "$\bar{f}$ is an extension of f" in case we identify $\mathcal{F}_x$ with x.

Let X be an infinite set.

(1) $\mathbf{C}(X)$: *The Tychonoff product* $\mathbf{2}^{\mathcal{P}(X)}$ *is compact.*
(2) $\mathbf{BPI}(X)$: *Every filterbase of* X *is included in an ultrafilter of* X.
(3) $\mathbf{BPI}$: $(\forall Y)\mathbf{BPI}(Y)$.
(4) $\mathbf{Id}(X)$: X *has an independent family of size* $|\mathcal{P}(X)|$.
(5) $\mathbf{AC}$: *Every family of non-empty sets has a choice function.*

2. Introduction and some preliminary results

Let $\mathbf{X}$ be a topological space and $\mathcal{C}$ be a base for the closed subsets of $\mathbf{X}$ (every closed subset of $\mathbf{X}$ can be expressed as an intersection of members of $\mathcal{C}$). Clearly, $\mathbf{X}$ is compact iff for every $\mathcal{G} \subset \mathcal{C}$ with the fip, $\bigcap \mathcal{G} \neq \varnothing$. In particular, if $\mathbf{X} = S(X)$ and $\mathcal{C} = \{[A] : A \in \mathcal{P}(X)\}$ then we have the following well known result:

Proposition 3. *"$S(X)$ is compact" iff* $\mathbf{BPI}(X)$.

Proof. ($\rightarrow$) Assume $S(X)$ is compact and let $\mathcal{H}$ be a filter of X. Clearly, $\{[H] : H \in \mathcal{H}\} \subset \mathcal{C}$ has the fip and, by our hypothesis, $\bigcap\{[H] : H \in \mathcal{H}\} \neq \varnothing$. It is easy to see that every $\mathcal{F} \in \bigcap\{[H] : H \in \mathcal{H}\}$ is an ultrafilter of X extending $\mathcal{H}$.

($\leftarrow$) Fix a family $\{[A_i] : i \in I\} \subset \mathcal{C}$ with the fip. Clearly, $\{A_i : i \in I\}$ has the fip and by $\mathbf{BPI}(X)$, $\{A_i : i \in I\}$ is included in an ultrafilter $\mathcal{F}$. Clearly, $\mathcal{F} \in \bigcap\{[A_i] : i \in I\} \neq \varnothing$ and consequently $S(X)$ is compact as required. □

Let X be an infinite set endowed with the discrete topology. In **ZFC** it is known that $S(X)$ is homeomorphic with the Čech-Stone extension $\beta(X)$ of $\mathbf{X}$ and the latter extension is compact (see e.g. [9] Prop 1.42). In **ZF** however, $S(X)$ and $\beta(X)$ need not be compact. So, the question which arises at this point is:

Question 1. Are $S(X)$ and $\beta(X)$ homeomorphic in **ZF**?

In addition to Proposition 3, other characterizations of $\mathbf{BPI}(X)$ in terms of compactness are known for certain special cases:

Theorem 4. *(i) ([6]) $\mathbf{BPI}(\omega)$ iff $\mathbf{2}^{\mathcal{P}(\omega)}$ is compact.*
(ii) ([5]) $\mathbf{BPI}(\mathbb{R})$ iff $\mathbf{2}^{\mathcal{P}(\mathbb{R})}$ is compact.

In view of Theorem 4, one may ask:

Question 2. Is it true in **ZF** that for every infinite set X, $\mathbf{BPI}(X)$ iff $\mathbf{C}(X)$?

The research in this paper is motivated by questions 1 and 2.

Regarding Question 1 we will establish in Theorems 14 and 15 that $S(X) \simeq \beta(X) \simeq \gamma(X)$ in **ZF**.

With respect to Question 2 we will show in Theorems 18 and 20 that in $\mathbf{ZF} + \mathbf{Id}(X)$, $\mathbf{BPI}(X)$ iff $\mathbf{C}(X)$ but in general in **ZF**, $\mathbf{BPI}(X)$ does not imply $\mathbf{C}(X)$.

The last result in this section is listed here for future reference.

Theorem 5. *([7, Proposition 3]) (**ZF**) If $|X| = |Y|$, then $\mathbf{2}^X \simeq \mathbf{2}^Y$. i.e., the Tychonoff products $\mathbf{2}^X$ and $\mathbf{2}^Y$ are topologically homeomorphic.*

3. Compactifications

The main results about Stone spaces and their proofs are in the following section, mostly independently of this section. This section provides further context and comparison between what is provable in **ZFC** versus what is provable in **ZF**.

Let $\mathbf{T}$ be a Tychonoff space (meaning Hausdorff and completely regular). Recall that the Čech-Stone compactification of $\mathbf{T}$ can be characterized as follows: $\mathbf{B}$ is a Čech-Stone compactification of $\mathbf{T}$ if and only if (i) There is an embedding $j\colon \mathbf{T} \to \mathbf{B}$ whose image is dense in $\mathbf{B}$, (ii) $\mathbf{B}$ is compact Hausdorff, and (iii) for every continuous $f\colon \mathbf{T} \to \mathbf{K}$ where $\mathbf{K}$ is compact Hausdorff, there exists a unique continuous $F\colon \mathbf{B} \to \mathbf{K}$ which extends f in the sense that $F \circ j = f$. It is well-known that in **ZFC** (or even $\mathbf{ZF} + \mathbf{BPI}$) it can be proven that every Tychonoff space has a Čech-Stone compactification. Furthermore, using the universal property (iii) (and some lemmas about dense subsets of Hausdorff spaces), well-known arguments show that the Čech-Stone compactification of a given **T**, if it exists, is unique in the sense that there exists a homeomorphism between the compactifications which fixes all elements of **T**. (See, e.g., [10] 19.6–19.10.)

If **T** is an infinite discrete space, then it cannot be proved in **ZF** that $\beta(T)$ (as defined in Section 1) is the Čech-Stone compactification of **T**. However, by changing the definition somewhat, we can find a universal property for $\beta(T)$ which holds in **ZF**.

Definition 6. A *cube* is a product of copies of the space $[\mathbf{0},\mathbf{1}]$. A space is *cube-compact* if it is homeomorphic to a closed subspace of a cube.

Clearly $\beta(X)$, $\gamma(X)$, and $\mathbf{2}^{\Delta_X}$ as defined in Section 1 are cube-compact.

Proposition 7. *Every cube-compact space is ultrafilter compact.*

Proof. It is straightforward that closed subspaces of ultrafilter compact spaces are ultrafilter compact, so it suffices now to show that cubes are ultrafilter compact. (The following argument can be generalized to show that every product of ultrafilter compact spaces is ultrafilter compact.)

Let $\mathcal{F}$ be an ultrafilter of $[\mathbf{0},\mathbf{1}]^Y$. Clearly, for every $x \in Y, \mathcal{F}^x = \{\pi_x(F) : F \in \mathcal{F}\}$ is an ultrafilter of the compact space $[\mathbf{0},\mathbf{1}]$ (resp. $\mathbf{2}$). Hence, $\mathcal{F}^x$ converges to a unique point, say $f(x) \in [0,1]$ (resp. $f(x) \in \{0,1\}$). It is straightforward to see that every neighborhood V of $(f(x))_{x\in Y}$ is a member of $\mathcal{F}$. Hence, $\mathcal{F}$ converges to $(f(x))_{x\in Y}$ and $[\mathbf{0},\mathbf{1}]^Y$ is ultrafilter compact as required. □

For Hausdorff spaces, the notions of compactness and cube-compactness are equivalent in **ZFC**. However, in **ZF** they are not equivalent, and neither notion is stronger than the other. BPI is equivalent to the statement "every cube is compact"; since BPI is not provable in **ZF** it is consistent that there are cube-compact Hausdorff spaces (for example, cubes) which are not compact. For the consistency with **ZF** of a compact Hausdorff space that cannot be embedded in a cube, see Example 2.4 in [1].

Definition 8. For a Tychonoff space **T**, a *Čech-Stone cube-compactification* of **T** is a space **B** such that (i) there is an embedding $j\colon \mathbf{T} \to \mathbf{B}$ whose image is dense in **B**, (ii) **B** is cube-compact and Hausdorff, and (iii) for every continuous $f\colon \mathbf{T} \to \mathbf{K}$ where **K** is cube-compact and Hausdorff, there exists a unique continuous $F\colon \mathbf{B} \to \mathbf{K}$ which extends f in the sense that $F \circ j = f$.

Remark 9. The argument mentioned above for the uniqueness of the Čech-Stone compactification of **T** can be easily adapted to show uniqueness of the Čech-Stone cube-compactification of **T**.

Some of the results of the next section can be summarized as follows:

Theorem 10. *For a discrete space* $\mathbf{X}$*, the Stone space* $S(X)$ *is a Čech-Stone cube-compactification.*

Proof. The embedding $\mathbf{X} \to S(X)$ obtained by mapping each $x \in X$ to the fixed ultrafilter $\mathcal{F}_x \in S(X)$ certainly has a dense image in $S(X)$, since every basic open set of $S(X)$ contains fixed ultrafilters. So part (i) of Definition 8 is satisfied.

It will follow from Theorem 14 below that $S(X)$ is cube-compact (part (ii) of Definition 8), using the observation that $\gamma(X)$ is clearly cube-compact.

Finally, it will follow from Theorem 13 below (along with Proposition 7) that $S(X)$ has the universal property (iii) of Definition 8. □

After Theorem 10 is established, the result that $S(X) \simeq \beta(X)$ could be proven by showing that $\beta(X)$ is also a Čech-Stone cube-compactification (see Remark 9). This may be done by modifying usual **ZFC** arguments (e.g. [10] 19.5) showing that $\beta(X)$ is a Čech-Stone compactification. (More generally, this approach shows that a Čech-Stone cube-compactification exists for every Tychonoff space, not just discrete spaces). We will not follow this approach here; instead we give a more direct proof that $S(X) \simeq \beta(X)$ in Theorem 15.

4. Main results

Proposition 11. *Let* X *be a non empty set,* $\mathbf{Y}$ *an ultrafilter compact* T_2 *space and* $f : X \to Y$ *be a function. Then:*
(i) For every $\mathcal{F} \in S(X)$, $\{\bar{f}(\mathcal{F})\} = \bigcap\{\bar{f}([F]) : F \in \mathcal{F}\}$*, where* $\bar{f}$ *is the Stone extension of* f*.*
(ii) For every $A \in \mathcal{P}(X), \overline{\bar{f}([A])} = \overline{f(A)}$*. In particular, for* $A = X$, $\overline{\bar{f}(S(X))} = \overline{f(X)}$*.*

Proof. (i) By (3) we have,

$$\{\bar{f}(\mathcal{F})\} = \bigcap\{\overline{f(F)} : F \in \mathcal{F}\}. \tag{4}$$

We claim that

$$\text{for every } A \in \mathcal{P}(X), \bar{f}([A]) \subseteq \overline{f(A)}. \tag{5}$$

We have:

$y \in \bar{f}([A]) \leftrightarrow$ there exists $\mathcal{H} \in [A]$ with $\bar{f}(\mathcal{H}) = y \to \{y\} = \bigcap\{\overline{f(H)} : H \in \mathcal{H}\}$.

Since $A \in \mathcal{H}$ we see that $y \in \overline{f(A)}$ and consequently $\bar{f}([A]) \subseteq \overline{f(A)}$ as required.

Since for every $F \in \mathcal{F}, \bar{f}(\mathcal{F}) \in \bar{f}([F])$, we see, in view of (4) and (5) that:

$$\{\bar{f}(\mathcal{F})\} \subseteq \bigcap\{\bar{f}([F]) : F \in \mathcal{F}\} \subseteq \bigcap\{\overline{f(F)} : F \in \mathcal{F}\} = \{\bar{f}(\mathcal{F})\} \tag{6}$$

finishing the proof of (i).

(ii) This follows at once from (5) and the observation that for every $A \in \mathcal{P}(X), f(A) \subseteq \bar{f}([A])$. □

Our next result in this section shows that for every $y \in \gamma(X)\backslash\delta(X)$, $\mathcal{F}(y)$ is a free ultrafilter of X.

Proposition 12. *Let X be an infinite set. Then, for every $y \in \gamma(X)\backslash\delta(X)$, $\mathcal{F}(y)(= \{A \in \mathcal{P}(X) : y(\chi_A) = 1\})$ is a free ultrafilter of X.*

Proof. First we show that $\mathcal{F}(y)$ is a filter of X.

(a) $\varnothing \notin \mathcal{F}(y)$. Assume on the contrary and let $\varnothing \in \mathcal{F}(y)$. Then, $V = [\{(\chi_\varnothing, 1)\}]$ is a neighborhood of y. Hence, $V \cap \delta(X) \neq \varnothing$. Fix $\chi_{\mathcal{W}_x} \in V \cap \delta(X)$. It follows that $\chi_{\mathcal{W}_x}(\chi_\varnothing) = 1$. Therefore, $\chi_\varnothing \in \mathcal{W}_x$ and $x \in \varnothing$. Contradiction!

(b) $X \in \mathcal{F}(y)$. Assume, aiming for a contradiction, that $X \notin \mathcal{F}(y)$. Hence, $y(\chi_X) = 0$. Fix $\chi_{\mathcal{W}_x} \in \delta(X) \cap [\{(\chi_X, 0)\}]$. Then, $\chi_X \notin \mathcal{W}_x$ and $x \notin X$. Contradiction!

(c) Assume that $A, B \in \mathcal{F}(y)$. We show that $A \cap B \in \mathcal{F}(y)$. Assume on the contrary that $A \cap B \notin \mathcal{F}(y)$. Then, $V = [\{(\chi_A, 1), (\chi_B, 1), (\chi_{A\cap B}, 0)\}]$ is a neighborhood of y. Fix $\chi_{\mathcal{W}_x} \in V \cap \delta(X)$. Then, $\chi_A, \chi_B \in \mathcal{W}_x$ and $\chi_{A\cap B} \notin \mathcal{W}_x$. Therefore, $x \in A, x \in B$ but $x \notin A \cap B$. Contradiction!

(d) Assume that $A \in \mathcal{F}(y)$ and $A \subseteq B \in \mathcal{P}(X)$. If $y(\chi_B) = 0$ then $V = [\{(\chi_A, 1), (\chi_B, 0)\}]$ is a neighborhood of y. Fix $\chi_{\mathcal{W}_x} \in V \cap \delta(X)$. Clearly, $\chi_A \in \mathcal{W}_x$ and $\chi_B \notin \mathcal{W}_x$. Thus, $x \in A$ and $x \notin B$. Contradiction!

From (a) - (d) it follows that $\mathcal{F}(y)$ is a filter of X.

Next we show that $\mathcal{F}(y)$ is maximal. Let $A \in \mathcal{P}(X)$. We show that either $A \in \mathcal{F}(y)$ or $A^c \in \mathcal{F}(y)$. Assume on the contrary and let $A, A^c \notin \mathcal{F}(y)$. Fix $\chi_{\mathcal{W}_x} \in \delta(X) \cap [\{(\chi_A, 0), (\chi_{A^c}, 0)\}]$. It follows that $x \notin A$ and $x \notin A^c$. Contradiction!

Since for all $x \in X$, $y \neq \chi_{\mathcal{W}_x}$, it follows that $\mathcal{F}(y) \neq \mathcal{F}(\chi_{\mathcal{W}_x}) = \mathcal{F}_x$ for all $x \in X$. Hence, $\mathcal{F}(y)$ is free finishing the proof of the proposition. □

We show next that the Stone extension $\bar{f} : S(X) \to \mathbf{Y}$ of any function $f : X \to Y$ is continuous in case $\mathbf{Y}$ is ultrafilter compact and T_3.

Theorem 13. *Let X be a non empty set, $\mathbf{Y}$ an ultrafilter compact T_3 space and $f : X \to Y$ be a function. Then:*

(a) The Stone extension $\bar{f}$ of f is continuous and unique.
(b) **BPI**(X) *implies $\bar{f}$ is closed and onto $\overline{f(X)}$.*

Proof. (a) Let $\mathcal{W}$ be a filter of $S(X)$ converging to $\mathcal{F} \in S(X)$. This means that for every $F \in \mathcal{F}, [F] \in \mathcal{W}$. Hence,

$$\{\bar{f}([F]) : F \in \mathcal{F}\} \subseteq \{\bar{f}(W) : W \in \mathcal{W}\} = \mathcal{G}.$$

By (5), $\bar{f}([F]) \subseteq \overline{f(F)}$ for every $F \in \mathcal{F}$. Thus, $\overline{f(F)} \in \mathcal{G}$ for every $F \in \mathcal{F}$. We show that $\mathcal{G}$ converges to $\bar{f}(\mathcal{F}) = y_{\mathcal{F}}$. To see this, fix V a closed neighborhood of y_F in $\overline{f(X)}$. By (6), $\bigcap\{\overline{f(F)} : F \in \mathcal{F}\} = \{y_{\mathcal{F}}\}$. Hence, $V \cap f(X)$ meets non trivially every member of the ultrafilter $\{f(F) : F \in \mathcal{F}\}$ of $f(X)$ and consequently $V \cap f(X) = f(F)$ for some $F \in \mathcal{F}$. Hence, $\overline{V \cap f(X)} = \overline{f(F)} \subseteq V$ and $V \in \mathcal{G}$ as required. Thus, $\bar{f}$ is continuous as required.

To see that $\bar{f}$ is unique let $g : S(X) \rightarrow \overline{f(X)}$ be a continuous extension of f. Since $\bar{f}, g$ are continuous functions, it follows that $K = \{\mathcal{F} \in S(X) : \bar{f}(\mathcal{F}) = g(\mathcal{F})\}$ is a closed subspace of $S(X)$. Since $X \subseteq K$ and $\overline{X} = S(X)$, we see that $K = S(X)$. Thus, $\bar{f} = g$ as required.

(b) By Proposition 3, **BPI**(X) implies $S(X)$ is compact. Thus, by the continuity of $\bar{f}$, $\bar{f}$ maps closed, hence compact, subsets of $S(X)$, to compact, hence closed subsets of $\overline{f(X)}$. Hence, $\bar{f}$ is a closed map.

By Proposition 11 and the closedness of $\bar{f}$ we have $\bar{f}(S(X)) = \overline{\bar{f}(S(X))} = \overline{f(X)}$ finishing the proof of the theorem. □

Theorem 14. *Let X be an infinite set. Then, $S(X) \simeq \gamma(X)$.*

Proof. Clearly, the family $\Gamma_X = \{\chi_{[A]} : A \in \mathcal{P}(X)\}$ separates points from closed sets in $S(X)$. Therefore, by Theorem 4.2 in [8] p. 220, the mapping $G : S(X) \rightarrow \mathbf{2}^{\Gamma_X}$, $G(\mathcal{F}) = (\chi_{[A]}(\mathcal{F}))_{A \in \mathcal{P}(X)}$ is an embedding. Since the mapping $H : \mathbf{2}^{\Gamma_X} \rightarrow \mathbf{2}^{\Delta_X}$ given by $H(f)(\chi_A) = f(\chi_{[A]}), A \in \mathcal{P}(X)$ is clearly a homeomorphism, it follows that $T = H \circ G : S(X) \rightarrow \mathbf{2}^{\Delta_X}$ is an embedding. Since

$$G(\mathcal{F})(\chi_{[A]}) = \chi_{[A]}(\mathcal{F}) = \begin{cases} 1 \text{ if } A \in \mathcal{F} \\ 0 \text{ if } A \notin \mathcal{F} \end{cases} = \chi_{\mathcal{F}}(A),$$

we see that, $T(\mathcal{F})(\chi_A) = H(G(\mathcal{F})(\chi_A) = G(\mathcal{F})(\chi_{[A]}) = \chi_{\mathcal{F}}(A), \mathcal{F} \in S(X)$. In particular, for every $x \in X$,

$$T(\mathcal{F}_x)(\chi_A) = \chi_{\mathcal{F}_x}(A) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases} = \chi_A(x) = \delta(x)(\chi_A).$$

Hence, $T|X = \delta$ and by Proposition 7 and Theorem 13 $T = \bar{\delta}$. By Proposition 12, for every $y \in \gamma(X) = \overline{\{\delta(x) : x \in X\}}, \mathcal{F}(y)$ is an ultrafilter of X. Since $\bar{\delta}(\mathcal{F}(y)) = T(\mathcal{F}(y))$ and for every $A \in \mathcal{P}(X)$,

$$T(\mathcal{F}(y))(\chi_A) = \chi_{\mathcal{F}(y)}(A) = \begin{cases} 1 \text{ if } A \in \mathcal{F}(y) \\ 0 \text{ if } A \notin \mathcal{F}(y) \end{cases} = \begin{cases} 1 \text{ if } y(\chi_A) = 1 \\ 0 \text{ if } y(\chi_A) = 0 \end{cases} = y(A),$$

we see that $\bar{\delta}(\mathcal{F}(y)) = y$ and consequently $\bar{\delta}$ is onto $\gamma(X)$. Hence, $\bar{\delta} : S(X) \to \gamma(X)$ is a homeomorphism as required. □

Theorem 15. *Let X be an infinite set. Then:*
(i) For every $A \subseteq X, \delta(A) = \pi_{\Delta_X}(e(A))$.
(ii) For every $\mathcal{F} \in S(X), \bar{\delta}(\mathcal{F}) = \pi_{\Delta_X}(\bar{e}(\mathcal{F}))$.
(iii) For every $z \in \gamma(X)$, there exists a unique $y \in \beta(X)$ with $z = \pi_{\Delta_X}(y)$. In particular, the restriction $\pi_{\Delta_X}|\beta(X) : \beta(X) \to \gamma(X)$ is a homeomorphism.

Proof. (i) This follows at once from the observation for every $x \in X$, $\delta(x) = \pi_{\Delta_X}(e(x))$.

(ii) Fix $\mathcal{F} \in S(X)$. We have $\bar{e}(\mathcal{F})$ is the unique limit point $y_{\mathcal{F}}$ of the ultrafilter $\{e(F) : F \in \mathcal{F}\}$ of $e(X)$ in $\beta(X)$ and $\bar{\delta}(\mathcal{F})$ is the unique limit point $z_{\mathcal{F}}$ of the ultrafilter $\{\delta(F) = \pi_{\Delta_X}(e(F)) : F \in \mathcal{F}\}$ of $\delta(X)$ in $\gamma(X)$. Since $z_{\mathcal{F}} \subseteq y_{\mathcal{F}}$, the conclusion follows from the fact that $\bar{\delta}(\mathcal{F}) = z_{\mathcal{F}} = \pi_{\Delta_X}(y_{\mathcal{F}}) = \pi_{\Delta_X}(\bar{e}(\mathcal{F}))$.

(iii) Fix $z \in \gamma(X)$. By Proposition 12 $\mathcal{F}(z)$ is a free ultrafilter of X and by the proof of Theorem 14, $\bar{\delta}(\mathcal{F}(z)) = z$. Since, by part (ii), $\bar{\delta}(\mathcal{F}(z)) = \pi_{\Delta_X}(\bar{e}(\mathcal{F}(z)))$ we see that for $y = \bar{e}(\mathcal{F}(z))$ we have $z = \pi_{\Delta_X}(y)$.

Assume on the contrary that there exists $w \in \beta(X), w \neq y$ with $z = \pi_{\Delta_X}(w)$. Fix $f \in C = [0,1]^X$ with $y(f) \neq w(f)$. Clearly, $V = \pi_f^{-1}(y(f))$ and $U = \pi_f^{-1}(w(f))$ are disjoint neighborhoods of y and w respectively in $[\mathbf{0},\mathbf{1}]^C$. Since $\mathcal{F}(z)$ is an ultrafilter of X, it follows at once that $e^{-1}(V) \in \mathcal{F}(z)$ and $e^{-1}(U) \in \mathcal{F}(z)$. Contradiction! Thus, y is unique as required finishing the proof of the proposition.

The second assertion follows from (iii) and the fact that projections are continuous and open. □

Next we get, as a corollary to Theorems 14 and 15, a list of characterizations of $\mathbf{BPI}(X)$, as well as an answer to Question 1.

Corollary 16. *Let X be an infinite set. The following are equivalent:*
(i) $\mathbf{BPI}(X)$.
(ii) $S(X)$ is compact.
(iii) $\gamma(X)$ is compact.
(iv) $\beta(X)$ is compact.
In particular, $\mathbf{BPI}$ iff "for every infinite set X, $\mathbf{C}(X)$".

Theorem 17. $\mathbf{BPI}(X)$ *implies "every ultrafilter compact* T_3 *space* $\mathbf{Y}$ *with a dense subset* D *of size* $\leq |X|$ *is compact".*
In particular, for every infinite set J*, if* $\mathbf{2}^J$ *(resp.* $[\mathbf{0},\mathbf{1}]^J$*) has a dense set* D *of size* $\leq |X|$ *then* $\mathbf{BPI}(X)$ *implies* $\mathbf{2}^J$ *(resp.* $[\mathbf{0},\mathbf{1}]^J$*) is compact.*

Proof. Fix X an infinite set and let $\mathbf{Y}$ be an ultrafilter compact T_3 space having a dense subset D with $|D| \leq |X|$. Fix $f : X \to D$ an onto function. By Theorem 13 the Stone extension $\bar{f} : S(X) \to \mathbf{Y}$ of f is continuous and onto $\overline{f(X)} = Y$. By $\mathbf{BPI}(X)$ and Corollary 16 $S(X)$ is compact. Hence, by the continuity of $\bar{f}$, $\mathbf{Y}$ is compact as required. □

Proof of Theorem 4. Follow the proof of Theorem 17 and use the fact that "*the product* $\mathbf{2}^{\mathcal{P}(\omega)}$ *has a dense set of size* $\aleph_0$" and "*the product* $\mathbf{2}^{\mathcal{P}(\mathbb{R})}$ *has a dense set of size* $|\mathbb{R}|$" hold true in $\mathbf{ZF}$.

Next, we show that in $\mathbf{ZF}+\mathbf{Id}(X)$, $\mathbf{BPI}(X)$ and $\mathbf{C}(X)$ are equivalent.

Theorem 18. *(i)* $\mathbf{Id}(X)$ *implies "*$\mathbf{BPI}(X)$ *iff* $\mathbf{C}(X)$*".*
In particular, $\mathbf{BPI}(X)$ *and* $\mathbf{Id}(X)$ *together imply "*$\mathbf{2}^{\mathcal{P}(X)}$ *is a continuous image of* $S(X)$*".*
(ii) "$\mathbf{2}^{\mathcal{P}(X)}$ *is a continuous image of* $S(X)$*" implies* $\mathbf{Id}(X)$*. Hence, under* $\mathbf{BPI}$*, the Fichtenholz-Kantorovich-Hausdorff theorem, i.e., the proposition: "For every infinite set* X*,* $\mathbf{Id}(X)$*" is equivalent to the statement: "*$\forall X, \mathbf{2}^{\mathcal{P}(X)}$ *is a continuous image of* $S(X)$*".*

Proof. (i) $\mathbf{BPI}(X) \to \mathbf{C}(X)$. Fix X an infinite set and let, by $\mathbf{Id}(X)$, $\mathcal{A}$ be an independent family of X of size $|\mathcal{P}(X)|$. We will show that $\mathbf{2}^{\mathcal{P}(X)}$ is compact. By Proposition 5, $\mathbf{2}^{\mathcal{A}}$ and $\mathbf{2}^{\mathcal{P}(X)}$ are homeomorphic. So, it suffices to show that $\mathbf{2}^{\mathcal{A}}$ is compact. Since, by Proposition 7, $\mathbf{2}^{\mathcal{A}}$ is ultrafilter compact, it suffices in view of Theorem 17, to show that $\mathbf{2}^{\mathcal{A}}$ has a dense subset of size $\leq |X|$. This has been established in [3] where it is shown that $\overline{\{(\chi_A(x))_{A\in\mathcal{A}} : x \in X\}} = \mathbf{2}^{\mathcal{A}}$.

$\mathbf{C}(X) \to \mathbf{BPI}(X)$. In view of Corollary 16, the fact that $\gamma(X)$ is a closed subspace of $\mathbf{2}^{\mathcal{P}(X)}$ and our hypothesis, it follows that $\mathbf{BPI}(X)$ holds true.

The second assertion of (i) is straightforward and it follows from the proof of $\mathbf{BPI}(X) \to \mathbf{C}(X)$.

(ii) To see this, fix $H : S(X) \to \mathbf{2}^{\mathcal{P}(X)}$ a continuous onto function. Clearly, $D = H(X)$ is dense in $\mathbf{2}^{\mathcal{P}(X)}$. It is well known and easy to see that

$$\mathcal{A} = \{A_x = \{d \in D : d(x) = 1\} : x \in \mathcal{P}(X)\}$$

is an independent family of D of size $|\mathcal{P}(X)|$. It is a routine work to verify that $\mathcal{A}' = \{H^{-1}(A_x) : x \in \mathcal{P}(X)\}$ is the required independent family of X. □

Remark 19. We would like to point out here that if $H : S(X) \to \mathbf{2}^{\mathcal{P}(X)}$ a continuous onto function, then certainly $D = H(X)$ is dense in $\mathbf{2}^{\mathcal{P}(X)}$ but it is not known if $|D| = |X|$. In [3] it has been shown that $\mathbf{Id}(X)$ implies *"the product $\mathbf{2}^{\mathcal{P}(X)}$ has a dense set of size $|X|$"*. Hence, in the light of the latter implication and Theorem 18 (ii), we see that "$\mathbf{2}^{\mathcal{P}(X)}$ is a continuous image of $S(X)$" implies *"the product $\mathbf{2}^{\mathcal{P}(X)}$ has a dense set of size $|X|$"*.

Next, we answer Question 2.

Theorem 20. *In* **ZF***,* $\mathbf{BPI}(X)$ *does not imply* $\mathbf{C}(X)$*.*

Proof. We recall that in [11] it has been established that there exists a **ZF** model $\mathcal{M}$ containing an unbounded amorphous set Y (every subset of Y is finite or cofinite and for every $n \in \mathbb{N}$, there is a (necessarily infinite) partition Π of Y into finite sets such that infinitely many members of Π have size greater than n. Clearly, $\mathbf{BPI}(Y)$ holds true in $\mathcal{M}$.

We show next that the product $\mathbf{2}^Y$, hence $\mathbf{2}^{\mathcal{P}(Y)}$ also, fails to be compact in $\mathcal{M}$. Assume on the contrary that $\mathbf{2}^Y$ is compact in $\mathcal{M}$. Let for every $\pi \in \Pi$,

$$G_\pi = \{f \in \mathbf{2}^Y : |f^{-1}(1) \cap \pi| = 1\}.$$

It is easy to see that $\mathcal{G} = \{G_\pi : \pi \in \Pi\}$ is a family of closed subsets of $\mathbf{2}^Y$ with the fip. Hence, $\bigcap \mathcal{G} \neq \varnothing$. Fix $g \in \bigcap \mathcal{G}$ and let $G = g^{-1}(1)$. Clearly, $\{G, Y \backslash G\}$ is a partition of Y into two infinite sets. Contradiction! Thus, $\mathbf{C}(Y)$ fails in $\mathcal{M}$. □

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