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SHIFT-COMPACTNESS IN ALMOST ANALYTIC SUBMETRIZABLE BAIRE GROUPS AND SPACES

A. J. OSTASZEWSKI

Abstract. We survey the harmonious interplay between the concepts of normed group, almost completeness, shift-compactness, and certain refinement topologies of a metrizable topology (the 'ground topology'), characterized by the existence of a weak base consisting of analytic sets in the ground topology. This leads to a generalized Gandy-Harrington Theorem. The property of shift-compactness leads to simplifications and unifications, e.g. to the Effros Open Mapping Principle.

1. Introduction and overview

This survey draws attention to a 'topological harmony of ideas' in the category of metrizable and submetrizable spaces, and is structured around three principal themes: *normed groups*; *shift-compactness*; *analyticity and almost-completeness.* A subspace A is analytic when it is the continuous image of a Polish space – an absolute property. Then, by a theorem of Lusin and Sierpiński (cf. [59, Th. 21.6]), A has the Baire property, 'has BP', in symbols $A \in \mathcal{B}a$ (or, is a 'Baire set', even 'is Baire' – if context permits); that is, A is almost open (open modulo a meagre set). Here we view this structurally via *Nikodym's theorem* that the Souslin operation preserves the BP (cf. [55, §2.9]): analytic sets are Souslin-$\mathcal{F}$, for $\mathcal{F}$ the closed sets, and a closed set differs from its interior by a nowhere dense set. Having BP makes analytic sets almost absolute-$\mathcal{G}_\delta$ and so in our group context 'almost complete', if non-meagre (and then a 'Baire space', cf. Th. 7.3).

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Key words and phrases. Almost completeness, analytic sets, analytic Baire theorem, analytic Cantor theorem, automatic continuity, density topology, fine topology, Gandy-Harrington topology, group-norm, measure-category duality, Nikodym's Theorem, non-commutative groups, shift-compactness, Subgroup Theorem, submetrizable, group-oscillation function.

A non-meagre almost absolute-$\mathcal{G}_\delta$ space is non-meagre *almost analytic* (cf. Th. 7.4). The term 'almost complete' is due to Frolík, and its name to Michael (see [44], [1], [72]). The structure of analytic sets often **directly** provides enough completeness to obviate re-metrization of their absolute-$\mathcal{G}_\delta$ 'cores' (e.g. via Prop. 7.2, the Convergence Criterion). In our context in an almost-complete space, 'Baire set' and 'Baire space' are almost synonyms: for B non-meagre, $B \in \mathcal{B}a$ iff B is a Baire space iff B is almost-complete (cf. Th. 7.4). We define normed groups in §2, indicating a 'canonical' example, and place them historically. In §3 we define shift-compactness (a term borrowed from the probability context of convolution semi-groups of measures, due to Parthasarathy [85]), and explain why this is a notion of compactness; in §4 we sketch a proof of a (primal) version of the shift-compactness theorem (Prop. 4.4) based on a strong separation property (another is Th. 7.5 based on analyticity). The theorem was inspired by the following old result in real analysis, first studied by Kestelman [61], later by Borwein-Ditor [25] with a first generalization due to Harry I. Miller (e.g. [73]), and rediscovered by Trautner [98]. Below a property holds for almost all (resp. *quasi-all*) t if it holds for all t off a null (meagre) set.

Theorem 1.1 (Kestelman-Borwein-Ditor Theorem, KBD)**.** *Let $\{z_n\} \to 0$ be a null sequence in $\mathbb{R}$. If T is a measurable/Baire subset of $\mathbb{R}$, then for generically all (= almost all/quasi-all) $t \in T$ there is an infinite set $\mathbb{M}_t$ such that*

$$\text{(sub)} \qquad \{t + z_m : m \in \mathbb{M}_t\} \subseteq T.$$

We show how to prove the Banach-Steinhaus Theorem (§6.1) directly from shift-compactness. We give an easy proof of the Steinhaus-Picard-Pettis Theorem (§6.2) in a simple topological-group context, and later (§9.2) identify a topological generalization. Next in §7 we introduce the ABC of analyticity: the analytic Baire and analytic Cantor Theorems, and note a useful characterization of almost-completeness; our viewpoint is informed by a recent generalization (in [78]) of the Gandy-Harrington Theorem as unifying the "Baire recognition" literature, cf. [1], [50] (and perhaps bridging between forcing and fine-topology methods). We identify a dual shift-compactness theorem (Th. 9.2) and by reference to the density topology recapture the measure case of KBD.

As an application we give two versions of the Steinhaus Subgroup Theorem and derive from it a theorem of Loy [69] and Hoffmann-Jørgensen [54] concerning analytic topological groups (that when non-meagre they are Polish). We contrast this with a related result that under comparable circumstances a 'semi-analytic' normed group is Polish (§8).

The presentation is based on a series of papers, many with Nick Bingham, and one with Harry Miller, cited in the bibliography. It is a pleasure to thank them both for their intellectual stimulus. I am grateful also to Roman Pol and to Henryk Toruńczyk for some key conversations. There have been two other sources of inspiration, of which I am conscious despite the long germination time; I thank Anatole Beck, for introducing me to topological dynamics (flows on planes), and Ralph Kopperman, for his advocacy of a bitopological viewpoint. I am also grateful for the invitation of the organizers of the Kielce 25$^{\text{th}}$ Summer Topology Conference not only to give an overview of this corpus of work, but also to place on record an abridged, but updated, version of the July 2010 talk. A longer website version offers further commentary and applications, e.g. to additive combinatorics – extensions of Ruziewicz [91] (which anticipated the van der Waerden theorem of 1927, for bibliography and details see [20]) – though this is not the first time such a connection has been made, cf. [43] and [53].

2. Normed groups

Motivated by normed *vector* spaces and topological regular variation (see [6] for the case of $\mathbb{R}$), where asymptotics (actually 'divergence') may be defined via $||x|| := d^X(x, e_X) \to \infty$ (with d^X a metric on X), we have:

Definition 2.1. (a) For T an *algebraic* group with neutral element e, say that $||\cdot|| : T \to \mathbb{R}_+$ is a *group-norm* ([21]) if the following properties hold:
(i) *Subadditivity* (Triangle inequality): $||st|| \leq ||s|| + ||t||$;
(ii) *Positivity*: $||t|| > 0$ for $t \neq e$ and $||e|| = 0$;
(iii) *Inversion* (Symmetry): $||t^{-1}|| = ||t||$.
Then $(T, ||.||)$ is called a *normed-group.*
(b) The group-norm generates a right and a left *norm topology* via the right-invariant and left-invariant metrics $d^T_R(s,t) := ||st^{-1}||$ and $d^T_L(s,t) := ||s^{-1}t|| = d^T_R(s^{-1}, t^{-1})$. In the right norm topology the right shift $\rho_t(s) := st$ is a uniformly continuous homeomorphism, $\rho_t \in \mathcal{H}_{unif}(X)$, since $d_R(sy, ty) = d_R(s,t)$; likewise in the left norm topology for the left shift. Since $d^T_L(t,e) = d^T_L(e, t^{-1}) = d^T_R(e,t)$, convergence at e is identical under either topology. In the absence of a qualifier, the 'right' norm topology is to be understood.
(c) Under the d_R topology, $B_r(x) = \{t : d_R(t,x) < r\} = B_r(e_T)x$.
(d) If d^X is a one-sidedly invariant metric, then $||x|| := d^X(x, e_X)$ is a norm.
(e) Under either norm topology, there is continuity of operations at e. At further distances the topology may force the group operations to be increasingly 'less' continuous.
(f) The *symmetrization metric* $d^T_S := \max\{d^T_R, d^T_L\}$ is also of interest below, and again in §8.

The *Birkhoff-Kakutani Theorem* ([52] Th. 8.3; cf. [90] Th. 1.24, albeit in a topological-vector-space setting) asserts that a metrizable topological group has an equivalent right-invariant metric. Inspection of Kakutani's proof yields the following sharpening.

Theorem 2.2 (Birkhoff-Kakutani Normability Theorem, [24], [58])**.** *A first-countable right topological group X is a normed group iff inversion and multiplication are continuous at the identity.*

Some history. Early use of group-norms occurs in work of A. D. Michal and his collaborators and was in providing a canonical setting for differential calculus (starting in the 1940s); a noteworthy example is the implicit function theorem by Bartle (1955). In name the group-norm resurfaces in 1950 in a paper of Pettis [86] in the course of his classic closed-graph theorem (in connection with Banach's closed-graph theorem and the Banach-Kuratowski category dichotomy for groups). The notion reappears in the group context first in 1961 by Dudley [34], who added a further condition to derive an automatic continuity result, and then in 1963 under the name 'length function', motivated by word length, in the work of R. C. Lyndon on Nielsen's Subgroup Theorem. Gromov theory has a normed group context. See the Introduction of [21] for a wider discussion.

Key (and principal) example. For a metric space (X, d^X) consider $Auth(X)$, the *algebraic* group (i.e. not equipped with a topology) of homeomorphisms $h : X \to X$ (under composition) with identity $e_X(x) = id_X(x) = x$. A candidate metric, when finite, is the *supremum metric*

$$\hat{d}(h, h') := \sup_x d^X(h(x), h'(x)). \tag{sup}$$

One thus restricts attention to the subgroup $\mathcal{H}(X)$ comprising those $h(\cdot)$ such that $\sup_x d^X(h(x), e_X(x)) < \infty$; these are the 'bounded elements' (compare with h being *limited* by an open cover of X, as in [101, p.5422]). We note the following properties.

1. $\hat{d}$ is right invariant, so may be denoted $d_R^{\mathcal{H}}$:

$$\hat{d}(hg, h'g) = \sup_x d^X(h(g(x)), h'(g(x))) = \sup_y d^X(h(y), h'(y)) = \hat{d}(h, h').$$

2. Hence the equation $||h|| = ||h||_{\mathcal{H}} := \hat{d}(h, e_X)$ defines a norm on $\mathcal{H}(X)$ with

$$d_R^{\mathcal{H}}(g, h) = ||g^{-1}h|| \text{ and } d_L^{\mathcal{H}}(g, h) = ||gh^{-1}||.$$

3. The symmetrized topology provides a natural refinement topology, via the symmetrized metric

$$d_S^{\mathcal{H}}(g, h) = \max\{d_R^{\mathcal{H}}, d_L^{\mathcal{H}}\} = \max\{\hat{d}(g, h), \hat{d}(g^{-1}, h^{-1})\},$$

which is complete provided (X, d^X) is complete. When (X, d^X) is compact, $d_L^{\mathcal{H}}$ is equivalent to $d_R^{\mathcal{H}}$ (see e.g. [99]). Thus:

4. The norm topology is topologically complete if X is compact.

5. Since $d_R^{\mathcal{H}} \leq d_S^{\mathcal{H}}$ the symmetrized topology is indeed a finer topology, and while the latter need not be invariant, one may pursue a bi-topological study (cf. §8).

6. Baire's theorem holds for $\hat{d}$ iff the group is non-meagre under $\hat{d}$ (see [54] Prop. 2.2.3 and §3.2).

7. The norm topology provides continuity when $\mathcal{H}(X)$ acts on X via the evaluation $\varphi : (h, x) \to h(x)$. *The action* φ *is continuous as a map from* $(\mathcal{H}(X), \hat{d}) \times (X, d^X) \to (X, d^X)$ (see [36] XII.8.3, p. 271), which permits a development of topological dynamics (cf. [77]). In particular each point evaluation $\varphi_x : h \to h(x)$ is continuous.

8. Our final property, Th.2.5, yields the viewpoint that the group $\mathcal{H}_u(X)$ below is a topological dual of X (a theme pursued by [21]):

Definition 2.3. Say that h is *bi-uniformly continuous* if both h and h^{-1} are uniformly continuous wrt d^X and write

$$\mathcal{H}_u(X) = \{h \in \mathcal{H}_{unif}(X) : h^{-1} \in \mathcal{H}_{unif}\} \subseteq \mathcal{H}(X).$$

Theorem 2.4 (Dieudonné [33], cf. [21, Th. 3.13])**.** $\mathcal{H}_u(X)$ *is complete under* $\hat{d}$*, provided* X *is complete under* d^X*.*

2.1. **Normed versus topological: Equivalence Theorem.**

Theorem 2.5 (Equivalence Theorem, [21])**.** *A normed group under either norm topology is a topological group iff the* d_R^X *topology is equivalent to the* d_L^X *topology. Furthermore, either of the following is equivalent to this condition:*
(i) each conjugacy $\gamma_t(x) := txt^{-1}$ *is continuous at* e *in norm,*
(ii) inversion is continuous in either d_R^X *or* d_L^X*.*

This motivates the following definition to which we refer in §8.

Definition 2.6 (cf. [53] Defn. 2.4)**.** A point z *lies in the topological centre* $Z_\Gamma(X)$ of a normed group X, if the conjugacy $\gamma_z(\cdot)$ is norm-continuous at e.

We note the immediate corollary: *an abelian group equipped with a group norm is topological under the norm topology.* We mention other consequences of the Equivalence Theorem, all asserting that a slight amount of regularity in the relationship between the left and right norm topologies, often as below in the presence of separability and some topological completeness – such as implied by analyticity of X under d_R^X, draws the two into coincidence. Straightforward instances (see [81]) are:

(i) if the graph of the self-homeomorphism $x \to x^{-1}$ is analytic;
(ii) if all conjugacies $\gamma_t(x) = txt^{-1}$ are Baire under d_R^X;
(iii) if X is locally compact and all conjugacies are Haar-measurable;
(iv) if the norm is such that $\kappa_n||x|| \leq ||x^n||$ for all n, x and some $\kappa_n \to \infty$ ([21], Th. 3.39, where the normed group is then called *Darboux-normed*).
The result (ii) is connected with the Cauchy dichotomy governing automatic continuuity of homomorphisms. For more subtle connections (especially the group's oscillation function, $\omega(t) := \lim_{\delta\searrow 0} \sup_{||z||\leq\delta} ||\gamma_t(z)||$) see [21]. For a recent discussion of when a group is topological see [30].

3. Shift-compactness

In a normed topological group G, say that a set A is *(properly) right-shift compact* if, for any sequence of points a_n in A (resp. in G), there are a point t and a subsequence $\{a_n : n \in \mathbb{M}_t\}$ such that $a_n t$ lies entirely in A and converges through $\mathbb{M}_t$ to a point $a_0 t$ in A; similarly for *left-shift compact* (cf. [22], for the real line case). Evidently, finite Cartesian products of shift-compact sets are shift-compact. Taken in the later context of e.g. Prop 3.5, demonstrating that shift-compactness is a strengthening of the Baire theorem, this productivity is an improvement over the Baire space property, which may fail to be productive - see [40], [87], and [102]. Thus a right-shift compact set A is precompact. (If the subsequence $a_m t$ converges to $a_0 t$, for m in $\mathbb{M}_t$, then likewise a_m converges to a_0, for m in $\mathbb{M}_t$.) Say that a set is *strongly right-shift compact* (or right-shift compact for arbitrarily small shifts) if the conditions just given hold and in addition the point t may be selected with $||t||$ arbitrarily small.

Remark 3.1. The compactness terminology is justified on two counts.
Firstly, suppose $T \subseteq \mathbb{R}$ is as in KBD, and a_n is a bounded sequence of points in T. Assume without loss of generality that $a_n \to a_0$; then $z_n := a_n - a_0 \to 0$. Now for some $t \in T$, $t + z_m \in T$ for m in some infinite set $\mathbb{M}_t$. Take $s := t - a_0$; then

$$s + a_n = (t - a_0) + a_n = t + z_n \in T$$

and $s + a_m$ converges through $\mathbb{M}_t$ to $s + a_0 = t \in T$. Thus after translation a subsequence of a_n converges to a point of T.
Secondly, this shift-compactness implies an (open) finite sub-covering theorem covering after shifts (also after small shifts), as follows.

Definition 3.2. (a) For N a nhd (=neighbourhood) of e_G say that $\mathcal{D}$:= $\{D_1, ..., D_h\}$ *shift-covers A (resp. N-strongly shift-covers* A, or is an N-*strong shifted-cover* of A), if for some $d_1, ..., d_h$ in G (in N),

$$(D_1 d_1) \cup ... \cup (D_h d_h) \supseteq A.$$

(b) Say that A is *compactly shift-covered* (resp. compactly strongly shift-covered, or *compactly shift-covered with arbitrarily small shifts),* if for every open cover $\mathcal{U}$ of A (and for each nhd N of e_G) there is a finite subfamily $\mathcal{D}$ which shift-covers A (resp. N-strongly shift-covers A).

Correspondingly one has a strong and weak compactness result with almost identical proof.

Theorem 3.3 (Strong Compactness Theorem – modulo shift, cf. [21])**.** *Let A be a (strongly) right-shift compact subset of a separable normed topological group G. Then A is compactly (strongly) shift-covered, i.e. for any norm-open cover $\mathcal{U}$ of A (and any nhd N of e_G), there is a finite subset $\mathcal{V}$ of $\mathcal{U}$, and for each member of $\mathcal{V}$ a translator (resp. a translator in N) such that the corresponding translates of $\mathcal{V}$ cover A.*

3.1. **Basis of generic behaviour.** Imagine a construction with aim a set of points $F(T)$ associated with the set T. For example:

$$t \in F(T) := \bigcap\nolimits_n \bigcup\nolimits_{m>n} (T - z_m) \text{ iff } t + z_n \in T \text{ infinitely often.}$$

The following result formalizes that if one can get at least one point of $F(T)$ in T itself, then 'most' points of T will be in $F(T)$. This explains the 'almost all/quasi all' aspect of the KBD.

Theorem 3.4 (Generic Dichotomy (Completeness) Principle, see [20])**.** *For $F : \mathcal{B}a \to \mathcal{B}a$ monotonic, if $W \cap F(W) \neq \varnothing$ for all non-meagre $W \in \mathcal{G}_\delta$, then, for each non-meagre $T \in \mathcal{B}a$, $T \cap F(T)$ is quasi all of T. That is, either*
(i) there is a non-meagre $S \in \mathcal{G}_\delta$ with $S \cap F(S) = \varnothing$, or,
(ii) for every non-meagre $T \in \mathcal{B}a$, $T \cap F(T)$ is quasi-all of T.

3.2. **Shift-compact spaces.** For a subgroup $\mathcal{G} \subseteq Auth(X)$ say that X is *$\mathcal{G}$-shift-compact* if for any convergent sequence $x_n \to x_0$, any open subset $U \subseteq X$ and $T \in \mathcal{B}a(X)$ co-meagre in U, there is $g \in \mathcal{G}$ such that $g(x_n) \in T \cap U$ along a subsequence. We abbreviate '$\mathcal{H}(X)$-shift-compact' to *shift-compact.* In such a space, any Baire non-meagre set is locally co-meagre (co-meagre on open sets) in view of the following: *For any subgroup $\mathcal{G} \subseteq \mathcal{H}(X)$, if X is $\mathcal{G}$-shift compact, then X is Baire.* This has proof identical with [101] Prop 3.1 (1), which proof we adapt to establish:

Proposition 3.5. *For any subgroup $\mathcal{G} \subseteq \mathcal{H}(X)$, if $\mathcal{G}$ acts transitively on a non-meagre X, then X is a Baire space.*

Proof. Suppose otherwise; then X contains a non-empty meagre open set. By Banach's localization principle, or Category Theorem ([84] Ch. 16, [55] p. 42, [60] Th. 6.35, or [66] §10.III under the name Union theorem), the union of all such sets is a largest open meagre set M, and is non-empty.

Being meagre, M has non-empty complement. For $x_0 \in X\backslash M$ and any non-empty open U pick $u \in U$ and $g \in G$ such that $g(x_0) = u$. As g is continuous, $g^{-1}(U)$ is a nhd of x_0, so is non-meagre, since every nhd of x_0 is non-meagre (cf. Definition 7.6 below). But g is a homeomorphism, so $U = g(g^{-1}(U))$ is non-meagre. (For a generalization see [54, Prop. 2.2.3].) So X is a Baire space, as every non-empty open set is non-meagre. □

4. Separation properties and shift-compactness

The definition below, inspired by recent work of van Mill, allows the interpretation of shift-compactness via a separation property, which is closely related to that considered in [101]. The underlying theme is homeomorphic shifting of points, sequences and eventually nowhere dense compact sets into disjointness.

Definitions. 1. Say that a subgroup $G \subseteq \mathcal{H}(X)$ *separates points and closed nowhere dense sets* in $(X, \mathcal{T}_X)$ if for each $p \in X$ and F closed and nowhere dense in $\mathcal{T}_X$ there is in each nhd of the identity e_G an element $g \in G$ such that $g(p) \notin F$. Here we assume G is given either a norm topology, or some refinement of it.

2. Say that the separation of p from F, as in (1) above, is *strong* if each nhd of e_G contains a non-empty open set H such that $h(p) \notin F$ for every $h \in H$.

Equivalently (when the group is right-topological), in each open nhd U of e_G there is $g \in U$ and an open nhd V of e_G such that $Vg \subseteq U$ and $Vg(p)$ is disjoint from F.

3. Denote by $Tr(\mathbb{R}^d)$ the group of c-translations $x \to x + c$ in $\mathbb{R}^d$. Under the sup-norm, as in (sup) above, this group is isometric with $\mathbb{R}^d$. Thus any refinement of the Euclidean topology can be used as a topology also on $Tr(\mathbb{R}^d)$: see Th. 9.5. Particularly useful refinements are provided by *density topologies*, as they permit measure properties to be handled topologically. Recall that density open sets are measurable sets W all of whose members are *density points*, that is $1 = \lim_{\varepsilon\to 0} |W \cap B_\varepsilon(w)|/|B_\varepsilon(w)|$ for every $w \in W$. Here $|\cdot|$ denotes Lebesgue measure and $B_\varepsilon(w)$ is the open ball of radius ε. For other density topologies in $\mathbb{R}^d$ (e.g. using density bases other than these balls) in particular, and refinement topologies in general see [70]; for the locally compact case see [21] for (metric) topological groups, and [81] for normed groups. We recall that in $\mathcal{D}$, the density topology on the line, a set A is Baire iff A is measurable; A is meagre in $\mathcal{D}$ iff A is null (has measure zero); so $\mathcal{D}$ is a Baire space (for all these see e.g. [59, 17.47]). The following result relies on a Lemma which we cite in §6.3.

Lemma 4.1 (Separation Lemma)**.** *If G is a separable normed group, acting separately continuously and transitively on a non-meagre space X, then for any point x and F closed nowhere dense the set $W_{x,F} := \{\alpha : \alpha(x) \notin F\}$ is dense open. In particular, G separates points from nowhere dense closed sets.*

Proof. $W_{x,F} = \varphi_x^{-1}(X\backslash F)$ is open, as φ_x is assumed continuous. By Lemma 6.8 below, for $\emptyset \neq U \subseteq G$ open, Ux is non-meagre, and so $Ux\backslash F$ is non-empty, as F is meagre. So $u(x) \notin F$ for some $u \in U$. □

Proposition 4.2 (Finitary Strong Separation, generalizing the linear shifts of [74, Prop. 2])**.** *Suppose a subgroup G of $\mathcal{H}(X)$ strongly separates points from closed nowhere dense sets of X. In X let U be open, $u_i \in U$ for $i \leq n$ and F closed and nowhere dense. Then in G, for each $\varepsilon > 0$, in $B_\varepsilon(e)$ there is a non-empty open set V of homeomorphisms η such that $\eta(u_i) \in U$ and $\eta(u_i) \notin F$ for each $i \leq n$.*

Proof. Let $\varepsilon > 0$. By assumption $\delta := \min\{\varepsilon, \min_i\{d(u_i, X\backslash U)\}\}/(n+1) > 0$. Let $B_0 := B_\delta(e)$. By induction on $i \leq n$, we select $\tau_1, ..., \tau_n$ and open nhds $B_1, ..., B_n$ of e such that for $\eta_i := \tau_i \circ ... \circ \tau_1$

(i) $\tau_i \in B_{i-1}$, $B_i\tau_i \subseteq B_{i-1}$,
(ii) $\tau\eta_i(u_j) \in U\backslash F$ for $\tau \in B_i$ for $j \leq i-1$, and
(iii) $\tau\eta_i(u_j) \in U$ for $\tau \in B_i$ for $j \leq n$.

It will follow that $\tau\eta_n(u_i) \in U\backslash F$ for all $i \leq n$ and each $\tau \in B_n$.

Choose $\tau_1 \in B_0 = B_\delta(e)$ and B_1 an open nhd of e such that $B_1\tau_1 \subseteq B_\delta(e)$ so that $\tau\tau_1(x_1) \in U\backslash F$ for each $\tau \in B_1$. For each such τ and each i one has $\tau\tau_1(u_i) \in U$, since $||\tau\tau_1|| \leq ||\tau|| + ||\tau_1|| < 2\delta \leq \varepsilon$.

Now choose τ_2 in B_1 and B_2 a nhd of e such that $B_2\tau_2 \subseteq B_1$ so that $\tau\eta_2(u_2) \in U\backslash F$ for each $\tau \in B_2$. For any such τ and each i one has $\tau\eta_2(u_i) \in U$ as $||\tau_1|| + ||\tau_2|| + ||\tau|| < 3\delta \leq \varepsilon$ and $\tau\eta_2(u_1) \in U\backslash F$ as $\tau\eta_2 \in B_2\tau_2\tau_1 \subseteq B_1\tau_1$.

Proceed similarly for any $i < n$, by selecting τ_i in B_{i-1} and B_i a nhd of e such that $B_i\tau_i \subseteq B_{i-1}$ so that $\tau\tau_i\eta_{i-1}(u_i) \in U\backslash F$ for each $\tau \in B_i$.

For any such τ and each $j < i$ one has $\tau\eta_i(u_j) \in U$ as $||\tau_1|| + ||\tau_2|| + ... + ||\tau_i|| + ||\tau|| < (i+1)\delta \leq n\delta < \varepsilon$ and $\tau\tau_i\eta_{i-1}(u_1) \in U\backslash F$ as $\tau\tau_i \in B_{i-1}$. Likewise for each $j < i$ one has $\tau\tau_i\eta_{i-1}(u_j) \in U$ as $\tau\tau_i \in B_{i-1}$. This completes the inductive step from $i-1$ to i.

Taking $V = B_n$, one has for $\tau_n' \in B_n$ that the shift $\eta := \tau_n'\eta_n$ has $||\eta|| < \min_i\{\varepsilon, d(u_i, X\backslash U)\}$, so $\eta(u_i) \in U$ and $\eta(u_i) \notin F$, as asserted. □

One may re-interpret the preceding result as saying that any finite number of points may be shifted locally into the complement of a closed nowhere dense set; in the semigroup setting a set into which any finite set may be left-shifted was termed by Mitchell [75] *left thick*; for further connections see [32]. We now improve the last result from finite sets to convergent sequences, a matter we return to in the final remark of this section. The analogous result is obvious in the case of the compact-open topology.

Lemma 4.3 ([83])**.** *For $K = \{x_n : n = 0, 1, 2, ..\}$ with $x_n \to x_0$ the set $W_{K,F} := \{\alpha : \alpha(K) \notin F\} = \{\alpha : \alpha(K) \subseteq X\backslash F\}$ is dense open in the norm topology.*

Proof. As to open-ness, for $u \in W_{K,F}$ one has $u(K) \subseteq X\backslash F$, so $\varepsilon := \min_{k\in K}\{d(u(k), F)\} > 0$, as K is compact. Then $Bu \subseteq W_{K,F}$ for $B = B_\varepsilon(e)$.
As to density, fix u and write $u_n := u(x_n)$. By Lemma 4.1 we may assume $u(x_0) \notin F$. Now for some $\varepsilon > 0$ and integer N one has $B_\varepsilon(u(x_0)) \subseteq X\backslash F$ and $u(x_n) \in B_{\varepsilon/2}(u(x_0))$ for $n > N$. As in Prop. 4.2 find η with $||\eta|| < \varepsilon/2$ such that $\eta(u_i) \notin F$ for $i \leq N$. But for $n > N$ one has $\eta(u(x_n)) \in B_{\varepsilon/2}(u(x_n)) \subseteq B_\varepsilon(u(x_0)) \subseteq X\backslash F$. Thus for all n one has $\eta(u(x_n)) \notin F$, as required. □

A first generalization of KBD follows. Here shift-compactness is a consequence of separation properties. (For a refinement of the argument below see [83]). A dual version in §9 turns out to be broader, but has other underpinnings.

Proposition 4.4 ([83])**.** *For metric X, non-meagre $T \in \mathcal{B}a(X)$, and G a separable normed group, Baire in its right norm topology (e.g. almost-complete, and so non-meagre, in the norm topology), acting separately continuously and transitively on X : for every convergent sequence x_n with limit x there are $\tau \in G$ and an integer N such that $\tau x \in T$ and*

$$\{\tau(x_n) : n > N\} \subseteq T.$$

Proof. Write $T := M \cup U\backslash \bigcup_n F_n$ with U open, M meagre and each F_n closed and nowhere dense in X. Let $u_0 \in T \cap U$. By transitivity there is $\sigma \in G$ with $\sigma x_0 = u_0$. Put $u_n := \sigma x_n$. Then $u_n \to u_0$. Take $K = \{u_n : n = 0, 1, 2, ..\}$. As G is Baire the set $\{\alpha : \alpha(u_0) \in U\} \cap C$, where $C := \bigcap_n \{\alpha : \alpha(K) \notin F_n\}$ is a dense $\mathcal{G}_\delta$, is non-empty. For α in the above set we have: $\alpha(u_0) \in U\backslash \bigcup_n F_n$. Now $\alpha(u_n) \to \alpha(u_0)$, by continuity of α, and U is open. So for some N we have for $n > N$ that $\alpha(u_n) \in U$. Since $\{\alpha(u_m) : m = 1, 2, ..\} \in X\backslash \bigcup_n F_n$, we have for $m > N$ that $\alpha(u_m) \in U\backslash \bigcup_n F_n \subseteq T$.
Finally put $\tau := \alpha\sigma$; then $\tau(x_0) = \alpha\sigma(x_0) \in T$ and $\{\tau(x_n) : n > N\} \subseteq T$. □

Remark 4.5. **(Generalizing shift-compactness via 'shifting into disjointness'.)** A metric space (X, d^X) is *strongly locally homogeneous* if there is a base consisting of open sets U with the property that for any $x, y \in U$ there is $h \in \mathcal{H}(X)$ taking x to y, equal to the identity outside U. (Denoting d^X-diameters also by d^X, evidently $||h|| < d^X(U)$.) In such a space any compact nowhere dense set has a disjoint homeomorphic image under a homeomorphism in $\mathcal{H}(X)$. Thus a compact nowhere dense set may be shifted into disjointness from itself. Indeed one might consider shift-compactness in the broader context of k-spaces, since closure of a set may be tested by reference to whether the trace on nowhere dense compact sets is closed (cf. [63], and note also [93] in regard to smallness of σ-compact sets). Then a complete group which shifts a zero-dimensional compact nowhere dense set into disjointness from another nowhere dense set exhibits a more general form of shift-compactness - compare [95] and [2]. (As to complications in regard to zero-dimensional subspaces, see e.g. [31].)

5. Properties of normed groups

Unless otherwise stated, the default norm topology is the right norm topology.

Theorem 5.1 (Squared Pettis Theorem, [21], Th. 5.8)**.** *Let X be an almost-complete normed group and $A \subseteq X$ a non-meagre Baire set (under the right norm topology). Then e_X is interior to $(AA^{-1})^2$.*

Squaring and higher powers of AA^{-1} were studied by R. Henstock [51] and E. Følner [42], and more recently by Rosendal and Solecki [88].

When X is a locally compact normed group there exists an invariant Haar measure on X ([81]), and so a Haar-measure variant of Theorem KBD holds, as does also a measure version of the Pettis Theorem (in fact without any squaring).

Theorem 5.2 (Baire Homomorphism Theorem, [21], Th. 11.11, and [81])**.** *Let X and Y be normed groups analytic in the right-norm topology with X non-meagre. If $f : X \to Y$ is a Baire homomorphism (i.e. a homomorphism with preimages of open sets being Baire sets), then f is continuous.*

5.1. **Application: Subgroup Theorem.** There are two well-known dichotomies (here 'small or large', rather than 'nice or nasty', as later) which assert that a Baire subset is *either meagre or clopen*. From our current perspective theses dichotomies are 'duals' (as with the generalization of KBD). There is the *Banach-Kuratowski dichotomy* (for which see [3, Satz 1], [66, Ch. VI. 13. XII]; cf. [60, Ch. 6 Pblm P] ; cf. [6, Cor. 1.1.4] and also [5] and [4] for the measure variant), where the context is a group G and the subset is a subgroup H (so invariant under the translation action of H), and there is the *Kuratowski-McShane dichotomy* ([64], [66] Ch 1 §13.XI, [71, Cor. 1]), where the context is a topological space and the premise requires a transitive action of a subgroup of autohomeomorphisms such that each action either leaves the subset invariant or shifts it into disjointness (for bibliography, see [21]). The latter result is highly thematic here.

The dichotomies below are in keeping with this, though they interpret large as 'total'. We give an application below – see [22] for others related to additivity, sub-additivity, and convexity (or for more detailed analyses: [7], [12], [14], [62]). As may be expected from the Banach-Kuratowski dichotomy, for totality one relies either on density or connectedness. The following direct proofs, based on the Squared Pettis Theorem, are inspired by a close reading of work by Hoffman-Jørgensen ([54] p. 355), where the subgroup theorem is implicitly used in a topological group context. Here less than completeness is assumed.

Theorem 5.3 (Subgroup Theorem - density version)**.** *In an almost-complete normed group G, if H is a dense non-meagre subgroup with the Baire property, then $H = G$.*

Proof. We interpret the statement in the right norm topology. By the Squared Pettis Theorem, $H = (H^{-1}H)(H^{-1}H)$ is an open nhd of e_G in G, as H is Baire non-meagre. For any $g \in G\backslash H$ one has $H \cap Hg = \emptyset$ (as otherwise $h_1 = h_2 g$ for $h_1, h_2 \in H$ implies $g = h_2^{-1}h_1 \in H$) and so Hg is a nhd of g avoiding H. So H is closed in G; being dense in G, it is G. $\square$

In [21] we relied on a weak Archimedean property in lieu of density to derive a similar result, whereas in [22] we used Kronecker's Theorem to show that in the additive group $\mathbb{R}$ a non-meagre subgroup is dense. In the absence of density, the argument above still goes through when the group is connected, as then the Archimedean property holds in regard to H:

Theorem 5.4 (Subgroup Theorem – connected group version)**.** *In a connected almost-complete normed group G, if H is a non-meagre subgroup with the Baire property, then $G = \bigcup_{n\in\mathbb{N}} H^n$ and so $H = G$.*

Proof. Since, as before, H is an open nhd of e_G in G, pick $B := B_\varepsilon(e) \subseteq H$. As $BS = \bigcup_{s\in S} Bs$ is open for any set S and B is symmetric, the set $C := \bigcup_{n\in\mathbb{N}} B^n$ is an open subgroup with $\emptyset \neq C \subseteq H \subseteq G$. As before, $C \cap Cg = \emptyset$ for any $g \in G\backslash C$, i.e. C is also closed in G, so the whole of G, by connectedness, hence $H = G$. $\square$

Theorem 5.5 (Loy [69], Hoffman-Jørgensen [54]). *A non-meagre analytic topological group is Polish.*

Proof. An analytic topological group H, being separable, may be densely embedded (by completion) in a complete separable topological group G, but now H is a non-meagre subgroup with the Baire property (being analytic, by §1), so is all of G by Th. 5.3. $\square$

The argument above does not embrace general normed groups, since a normed group that can be extended to a complete normed group is necessarily a topological group (cf. [21] Th. 3.38). However, see §8 for an analogue.

6. Applications of shift-compactness

This section illustrates how easy it is to derive results using shift-compactness. A deeper result is the Semi-Polish Theorem of §8.

6.1. **Uniform Boundedness Theorem.**

Theorem 6.1. *For X a non-meagre topological vector space and F a family of continous linear functionals, if for each x the set $\{||f(x)|| : f \in F\}$ is bounded, then $\{||f(x)|| : f \in F\}$ is bounded on a nhd of* 0.

Proof. Suppose otherwise. Then, for $n \in \mathbb{N}$, there exist $x_n \in X$ converging to 0 and $f_n \in F$ such that $||f_n(x_n)|| \geq n$. As f is continuous, $\{x : ||f(x)|| \leq n\}$ is closed, and so is $A_n := \bigcap_{f\in F}\{x : ||f(x)|| \leq n\}$, so has the Baire property. By assumption $X = \bigcup_n A_n$; since X is non-meagre, there is N such that A_N is non-meagre. By Prop. 4.4, as x_n is convergent, there are $t \in A_N$ and infinite $\mathbb{M}_t$ such that $x_m + t \in A_N$ for $m \in \mathbb{M}_t$. For $m \in \mathbb{M}_t$ one has

$$||f_m(x_m)|| = ||f_m(x_m + t) - f_m(t)|| \leq ||f_m(x_m + t)|| + ||f_m(t)|| \leq 2N,$$

so $\{||f_m(x_m)|| : m \in \mathbb{M}_t\}$ is bounded, a contradiction. $\square$

Of course the above really proves a theorem about continuous homomorphisms on a non-meagre normed topological group.

6.2. **Steinhaus-Piccard Theorem.** The following result refers to condition (wcc) studied in §9, which is satisfied for $\mathcal{E}$ the Euclidean topology and $\mathcal{D}$ the density topology on the line. The proof resembles that used by Solecki [94]: indeed, there is a connection between our shift-compactness and his ‘amenability at 1’.

Theorem 6.2 (Fine Topology Interior Point Theorem, [15]). *Let $\mathbb{R}$ be given a shift-invariant topology $\mathcal{T}$ under which it is a Baire space and suppose the homeomorphisms $h_n(x) = x + z_n$ satisfy (wcc), whenever $\{z_n\} \to 0$ is a null sequence (in the Euclidean topology). For S a non-meagre Baire subset under $\mathcal{T}$, the difference set $S-S$ contains an interval around the origin.*

Proof. Suppose otherwise. Then for each positive integer n we may select $z_n \in (-1/n, +1/n)\backslash(S-S)$. Since $\{z_n\} \to 0$ (in the Euclidean topology), the Category Embedding Theorem (Th. 9.2) applies, and gives an $s \in S$ and an infinite $\mathbb{M}_s$ such that $\{h_m(s) : m \in \mathbb{M}_s\} \subseteq S$. Then $s + z_m \in S$ for any $m \in \mathbb{M}_s$, i.e. $z_m \in S - S$, a contradiction. □

6.3. **The Effros Open Mapping Theorem.**

Definition 6.3. A group $G \subset \mathcal{H}(X)$ acts *weakly* on a space X if $(g, x) \to g(x)$ is continuous *separately* in g and in x.
A group $G \subset \mathcal{H}(X)$ acts *transitively* on a space X if for each x, y in X there is g in X such that $g(x) = y$.
The group acts *micro-transitively* on X if for U open in G and $x \in X$ the set $\{h(x) : h \in U\}$ is a nhd of x.

Theorem 6.4 (The Effros Open Mapping Principle, [37], [100]). *Let G be a Polish topological group acting transitively on a separable metrizable space X. The following are equivalent.*
(i) G acts micro-transitively on X,
(ii) X is Polish,
(iii) X is non-meagre.
*More generally, for G an analytic **normed group** acting transitively on a separable metrizable space X: (iii) $\Longrightarrow$ (i), i.e., if X is non-meagre, then G acts micro-transitively on X.*

Remark 6.5. Jan van Mill [100] gave the more general result here for G an analytic *topological* group, but actually his proof only assumes in effect a **normed** group structure. Of interest is the following result: [83] discusses a non-separable variant.

Theorem 6.6 (Effros Theorem – Baire variant, [83]). *Let the normed group G have separately continuous and transitive action on X. If under either norm topology G is analytic and a Baire space, and X is non-meagre, then the action of G is micro-transitive. That is, for U an open nhd of e_G and for arbitrary $x \in X$ the set $Ux := \{u(x) : u \in U\}$ is a nhd of x, so that the point-evaluation maps $g \to g(x)$ are open for all x.*

Remark 6.7. Analyticity here is needed principally to ensure that images of certain sets have the Baire property: since an analytic set is the continuous image of a Polish space, a continuous image of an analytic set is an analytic set, so has the Baire property.

Lemma 6.8 ([83]). *If G is a separable normed group, acting transitively on a non-meagre space X, then for each non-empty open U in G and each $x \in X$ the set Ux is non-meagre in X.*

Proof. G is separable under either norm topology. We work in the right norm topology first. Suppose that $u \in U$ and so without loss of generality assume that $U = B_\varepsilon(u) = B_\varepsilon(e_G)u$ (for some $\varepsilon > 0$); put $y := ux$ and $W := B_\varepsilon(e_G)$. Then $Ux = Wy$. Next work, exceptionally, in the left norm topology (for which $W = B_\varepsilon(e_G)$ is a nhd of e_G); as each set hW for $h \in G$ is open (since now the left shift $\lambda_h : g \to hg$ is a homeomorphism), the open family $\{gW : g \in G\}$ covers G, and so has a countable sub-cover, $\{g_nW : n \in \omega\}$ say. As G is transitive, $X = Gy$ and so X is covered by $\{g_nWy : n \in \omega\}$. So for some n, the set g_nWy is non-meagre. As g_n^{-1} is a homeomorphism of X, the set $Wy = Ux$ is also non-meagre in X. $\square$

Proof of the Effros Theorem, ([83]). Assume G acts transitively on a non-meagre space X. For some $x \in X$ and some $B := B_\varepsilon(e_G)$ suppose Bx is not a nhd of x. Then there is $x_n \to x$ with $x_n \notin Bx$ for each n. Take $A := B_{\varepsilon/2}(e_G)$ and note that A is a symmetric open set ($A^{-1} = A$, by the inversion axiom). By Lemma 6.8 Ax is non-meagre, as G, being analytic, is separable. Since the point-evaluation map $g \to g(x)$ is continuous Ax is analytic and so a Baire set; by Prop. 4.4 there are $a \in A$ (being open, has the Baire property) and a co-finite $\mathbb{M}_a$ such that $ax_m \in Ax$ for $m \in \mathbb{M}_a$. For any such m choose $b_m \in A$ with $ax_m = b_mx$. Now $a^{-1} \in A$, by symmetry, so $x_m = a^{-1}b_mx \in A^2x \subseteq Bx$, a contradiction. $\square$

7. Analyticity

The theme here is to argue persuasively for analytic spaces to be recognized as a mainstream topological tool, in view of the completeness arguments they adduce. Though p-spaces, riding harmoniously with α-favourability (refining a key feature of Baire spaces), have apparently overtaken analyticity, nonetheless even there Fremlin's Čech-analyticity (see below) comes through as a necessary tool: see the papers by Bouziad ([26], [27], cf. the concluding discussion in [78]). After their classical phase in the hands of the founding fathers of topology, analytic sets have been a central topic for logicians, whose 'neoclassical' contributions have been spectacular (especially Silver's Theorem – see [89] p. 463).

Early interest was shown by F. B. Jones, who showed that a Hamel basis for the reals as a vector space over the rationals cannot be analytic (as did also W. Sierpiński), and applied this to an analysis of the Hamel pathology of additive functions in [57].

Recall that Cantor's Theorem on the intersection of a nested sequence of closed (or compact, as appropriate) sets has two formulations: (i) referring to vanishing diameters (in a complete-space setting), and (ii) to (countable) compactness. Our first aim in this section is to give a topological version that is in this same spirit but appropriate to an analytic, rather than complete or compact, context.

For X a Hausdorff space write $\mathcal{K} = \mathcal{K}(X)$ for the family of compact subsets of a space X, and $\wp(X)$ for the power set. Following the the notation of [55], write I for $\mathbb{N}^{\mathbb{N}}$ endowed with the product topology (treating $\mathbb{N}$ as discrete), and with $i|n := (i_1, ..., i_n)$, for $i \in I$ and $n \in \mathbb{N}$, put $I(i|n) = \{j \in I : j|n = i|n\}$, a basic open nhd in I. For X Hausdorff, a map $K : I \to \wp(X)$ is called *compact-valued* if $K(i)$ is compact for each $i \in I$, and *singleton-valued* if each $K(i)$ is a singleton. K is *upper semi-continuous* if, for each $i \in I$ and each open U in X with $K(i) \subseteq U$, there is $n \in \mathbb{N}$ with $K(i_1, ..., i_n) := K(I(i_1, ..., i_n)) = \bigcup_{i \in I(i_1,...,i_n)} K(i) \subseteq U$. A subset of X is $\mathcal{K}$-*analytic* if it is the image $K(I)$ under an upper semicontinuous compact-valued map. By a theorem of Jayne (cf. [55, §2.8]) this is equivalent to two other definitions studied by: Choquet (1951), Sion (1960). Fremlin (1980) defines a more general notion of *Čech-analyticity* (for which see Hansell [48, Th. 5.3], [56, §8], [49] or Fremlin's website.)

The following result is implicit in a number of situations, and goes back to Frolík's characterization of completely regular Čech-complete spaces as $\mathcal{G}_\delta$ in some compactification ([44]; see [39] §3.9). See [78] for this and other versions. It is the 'completeness-compactness' property below that motivates the Choquet-style α-favourability perspective on analytic sets, and exposes their inherently topological nature.

Theorem 7.1 (Analytic Cantor Theorem, [78]). *Let X be a Hausdorff space and $A = K(I)$ be $\mathcal{K}$-analytic in X, with K compact-valued and upper-semicontinuous.*
If F_n is a decreasing sequence of (non-empty) closed sets in X such that $F_n \cap K(I(i_1, ..., i_n)) \neq \emptyset$, for some $i = (i_1, ...) \in I$ and each n, then $K(i) \cap \bigcap_n F_n \neq \emptyset$.
Equivalently, if there are open sets V_n in I with $\mathrm{cl}V_{n+1} \subseteq V_n$ and $\mathrm{diam}_I V_n \downarrow 0$ such that $F_n \cap K(V_n) \neq \emptyset$, for each n, then
(i) $\bigcap_n \mathrm{cl}V_n$ is a singleton, $\{i\}$ say;
(ii) $K(i) \cap \bigcap_n F_n \neq \emptyset$.

Proof. If not, then $\bigcap_n K(i) \cap F_n = \emptyset$ and so, by compactness, $K(i) \cap F_p = \emptyset$ for some p, i.e. $K(i) \subseteq X \backslash F_p$. So by semicontinuity $F_p \cap K(I(i_1, ..., i_n)) = \emptyset$ for some $n \geq p$, yielding the contradiction $F_n \cap K(I(i_1, ..., i_n)) = \emptyset$. □

The next theorem (Th. 7.3) is crucial, and may be deduced from the Convergence Criterion, an immediate corollary of Th. 7.1. (For an alternative approach – see also [78].)

Proposition 7.2 (Convergence Criterion, [79])**.** *In a normed group, for* $r_n \searrow 0$ *and* $\alpha_n = a_n \cdot ... \cdot a_1$ *with* $\mathrm{cl}B_{r_{n+1}}(a_{n+1}) \subseteq B_{r_n}(e)a_n$, *if* $X = K(I)$ *is an analytic subset and* $K(i_1, ..., i_n) \cap B_{r_n}(\alpha_n) \neq \emptyset$ *for some* $i \in I$ *and all* n, *then the sequence* $\{\alpha_n\}$ *is convergent.*

Proof. Indeed, $\alpha_n \to \alpha$, if $\{\alpha\} := K(i) \cap \bigcap_n F_n$ for $F_n = \mathrm{cl}(B_{r_n}(\alpha_n))$. □

Theorem 7.3 (Analytic Baire Theorem, [81])**.** *In a normed group* X *under* d_R^X, *if* X *contains a non-meagre analytic set, then* X *is Baire. In fact, up to a meagre set,* X *is analytic (and separable).*

As a corollary we obtain the following.

Theorem 7.4 (Characterization Theorem for Almost completeness, [81])**.** *In a separable normed group* X *under* d_R^X, *the following are equivalent:*
(i) X *is a non-meagre absolute* $\mathcal{G}_\delta$ *modulo a meagre set (i.e. is almost complete);*
(ii) X *contains a non-meagre analytic subset;*
(iii) X *is non-meagre and almost analytic (i.e. analytic modulo a meagre set).*

Armed with these facts we may state a result that relies on analyticity.

Theorem 7.5 (Analytic Conjugate Shift Theorem, [21])**.** *In a normed group* X *under the topology of* d_R^X, *for* $z_n \to e_X$ *null,* A *analytic and non-meagre:*
for a non-meagre set of $a \in A$ *(in fact with co-meagre Baire envelope), there is an infinite set* $\mathbb{M}_a$ *and points* $a_n \in A$ *converging to* a *such that*

$$\{aa_m^{-1}z_m a_m : m \in \mathbb{M}_a\} \subseteq A$$

(a transconjugate null sequence, i.e. translated conjugate null sequence). Moreover, if the normed group is topological, for quasi-all $a \in A$, *there is an infinite set* $\mathbb{M}_a$ *such that*

$$\{az_m : m \in \mathbb{M}_a\} \subseteq A.$$

7.1. **Fine Analytic Baire Theorem.** The Gandy-Harrington Theorem asserts that Baire's Theorem holds for $\mathcal{GH}$, the fine topology of the reals generated by declaring the effective analytic sets to be open ([89] p. 466). We generalize this theorem to embrace a wide family of fine topologies. The key idea is of course appropriate analytic generation of the topology. Refinement topologies are on the whole regarded by topologists as 'strange beasts' (unless one studies function spaces where the interplay of weak and strong topologies is common). But refinement is an important tool; let us briefly survey its use.

One says that the density topology $\mathcal{D}$ (cf. §4 above) is a *fine* topology on $\mathbb{R}$. Fine topologies on $\mathbb{R}$ capture different notions of 'typicality' or 'randomness'. The alternative intuitive view is that they allow in more 'exceptions' – as e.g. in *Denjoy continuity.* Other natural examples come from Analysis: whilst $\mathcal{D}$ is most widely known, there is also *the* 'fine topology', a standard tool in potential theory for dimension ≥ 2. Lesser known topologies used to advantage include O'Malley's [76] *r-topology* $\mathcal{R}$ on $\mathbb{R}$ (or *resolvable*-topology – for this term see [66] §12. III, V), which he used to study approximate differentiability of real-valued functions; $\mathcal{R} \subseteq \mathcal{D}$ and is generated by taking as base $\mathcal{B}:= \mathcal{D} \cap \mathcal{G}_\delta \cap \mathcal{F}_\sigma$, the sets of $\mathcal{D}$ that are ambiguously both $\mathcal{G}_\delta$ and $\mathcal{F}_\sigma$ in the real line. Another is *Scheinberg's maximal topology* $\mathcal{U} \supseteq \mathcal{D}$ (see [92]), which has the following important lifting property: any bounded measurable real-valued function f is equal a.e. to a unique $\mathcal{U}$-continuous function $\tilde{f}$. (His modification refers to an ultrafilter of measurable sets extending the filter $\mathcal{D}_0 := \{D \in \mathcal{D} : 0 \in \mathcal{D}\}$.) Whilst these are submetrizable, we recall the important role of metrizable refinement topologies in the study of Borel sets (remetrized to be complete, see e.g. [59, Th. 13.6]), in the semi-Polish Theorem Th. 8.1, and *Staiger's* complete metric refinement of the Cantor space used in [29] to show that the Law of Large Numbers holds co-meagerly (and to characterize random sequences in the Martin-Löf sense). We also note that Frolík and Holický [45] use fine uniformities to study non-separable analyticity.

Mathematical Logic has been particularly successful in exhibiting various natural forms of 'typicality' in the form of 'genericity' constructed through 'forcing' methods. Refinements of the usual (Euclidean) topology of the reals play a special role in 'neo-classical' descriptive set theory. Set-theoretic topologists are already familiar with the Ellentuck topology $\mathcal{E}l$ in relation to the Ramsey property: $\mathcal{E}l$ corresponds to Mathias forcing (and Mathias reals). Less well-known, alas, are the Gandy "reals" (Gandy-Sacks degrees), which pre-dated and motivated the Gandy-Harrington topology $\mathcal{GH}$ – introduced as a proof vehicle for Silver's Theorem, mentioned already. Perhaps topologists might profit from reviewing the best-loved Logic examples: the Cohen reals (generic in the Euclidean topology) and the Solovay reals (generic in the density topology); their relation to the respective σ-ideals $\mathcal{M}$ (the meagre sets) and $\mathcal{N}$ (the null sets) is well known, so perhaps the specialized infinitary combinatorics associated with these reals would bear scrutiny by reference to the established *ideal-neglecting topologies.* In the group context, if a σ-ideal $\mathcal{I}$ is translation-invariant and satisfies the localization property (see below), then a topology that neglects members of $\mathcal{I}$ may be generated by having G open in the *ideal-neglecting topology* iff $G = U\backslash Z$ for some U open and $Z \in \mathcal{I}$; see [70] p. 25. The case $\mathcal{I} = \mathcal{N}$, studied in [92], gives a topology $\mathcal{T}$ with $\mathcal{E} \subseteq \mathcal{T} \subseteq \mathcal{D}$.

Our viewpoint here is that Harrington's $\mathcal{GH}$ and Ellentuck and Louveau's $\mathcal{E}l$ (see [38], [68], or [59, §29.B]) should both be viewed as spectacular examples of a canonical *analytic* construction of refinement encompassing the classical examples quoted above. We need a definition and an old result of Kuratowski. (The term 'heavy' is established, going back to [28]; van Mill [100] calls a dense, heavy set 'fat'. See also [96] for a general 'kernel' approach.)

Definition 7.6. For $\mathcal{I}$ a σ-ideal, say that S is **$\mathcal{I}$-heavy**, resp. *$\mathcal{I}$-heavy on G*, in X if $S \cap U \notin \mathcal{I}$ for every open set $U \subseteq X$ meeting S, resp. for every open U meeting $S \cap G$.
The **$\mathcal{I}$-light part** of A is defined to be the set $L_{\mathcal{I}}(A) := \bigcup\{V \cap A : V$ open and $A \cap V \in \mathcal{I}\}$. The **heavy part** of A is the complementary set $H_{\mathcal{I}}(A) = A\backslash L_{\mathcal{I}}(A)$. So A is **heavy** (*heavy* on G) iff $L_{\mathcal{I}}(A)$ is empty (iff $L_{\mathcal{I}}(A) \cap G$ is empty).
Say that $\mathcal{I}$ has the **localization property** if $L_{\mathcal{I}}(A) \in \mathcal{I}$ for any A.

Remark 7.7. 1. In a second-countable space any σ-ideal has the localization property. In a metric space the σ-ideal $\mathcal{M}$ of meagre sets has the localization property: this is Banach's Category Theorem, quoted in Prop. 3.5 and used there to identify an $\mathcal{M}$-heavy set. In a locally compact metric group equipped with a Haar measure the σ-ideal $\mathcal{N}$ of null sets has the localization property (this follows from the appropriate Lebesgue Density Theorem, or its generalization – see [21]).
2. Under such circumstances, $H = H_{\mathcal{I}}(A)$ is $\mathcal{I}$-heavy, since $H \subseteq A$. (Otherwise there is open U with $\emptyset \neq U \cap H \in \mathcal{I}$, so $U \cap A \subseteq (U \cap H) \cup L_{\mathcal{I}}(A)) \in \mathcal{I}$; then $U \cap A \subseteq L_{\mathcal{I}}(A)$ so $(U \cap A) \cap H = \emptyset$, and so $\emptyset \neq U \cap H = U \cap H \cap A = \emptyset$, a contradiction.)

When non-empty, the heavy part is dense-in-itself, and so some work of Kuratowski [65], concerning closed sets that are dense-in themselves relative to some open set, can be easily amended to give the following results. (Compare also [55, §2.4].)

Lemma 7.8. *In a separable metric space, let $\mathcal{I}$ be a σ-ideal of subsets with the localization property. For G open and F closed with $F \cap G$ non-empty and $\mathcal{I}$-heavy, there exist for each $\delta > 0$ non-empty, closed $\mathcal{I}$-heavy subsets H_i of G with $d(H_i) < \delta$ such that*

$$F \cap G = \bigcup\nolimits_i H_i \text{ with each } H_i \backslash \bigcup\nolimits_{j<i} H_j \text{ non-empty and } \mathcal{I}\text{-heavy.}$$

Theorem 7.9 (Generalized Kuratowski Representation)**.** *For a separable metric space X and any σ-ideal of subsets $\mathcal{I}$ with the localization property, for any non-empty absolute-$\mathcal{G}_\delta$ subset A that is $\mathcal{I}$-heavy, there exists an upper semicontinuous representation $K : I \to X$ with each $K(I(i|n))$ an intersection of an open set $G(i|n)$ and a closed $\mathcal{I}$-heavy subset $F(i|n)$ of A of diameter at most 2^{-n}.*

Since all points of a topologically complete metric space dense-in-itself are condensation points, Kuratowski's original theorem refers de facto to properties of $\mathcal{I}_\omega$-heavy sets, for $\mathcal{I}_\omega$ the σ-ideal of countable subsets.

Take $\mathcal{I} = \mathcal{N}$ and note that for the density topology $\mathcal{D}$ the family $\mathcal{H} = \mathcal{D} \cap \mathcal{F} \subseteq \mathcal{G}_\delta$ is a weak base (see below) comprising $\mathcal{N}$-heavy sets. So these have a Kuratowski representation with each $K(I(i|n)) = G(i|n) \cap F(i|n)$, a $\mathcal{D}$-open set (since each $G(i|n)$ is also $\mathcal{D}$-open). This motivates a definition.

Definition 7.10. ($\mathcal{K}$-analytically heavy topologies). 1. For $(X, \mathcal{T})$ a topological space denote by $\mathcal{A}(\mathcal{T})$ the family of $\mathcal{K}$-analytic subsets of $(X, \mathcal{T})$.
2. $\mathcal{B}$ is a *weak base* (or π-base) for a topology $\mathcal{T}$ if for each non-empty $V \in \mathcal{T}$ there is $B \in \mathcal{B}$ with $\emptyset \neq B \subseteq V$.
3. Let $(X, \mathcal{T})$ be a regular Hausdorff space and $\mathcal{T}' \supseteq \mathcal{T}$ a refinement topology. We say $\mathcal{T}'$ is *analytically heavy*, or *weakly $\mathcal{K}$-analytically generated* in $\mathcal{T}$, if $\mathcal{T}'$ possesses a weak base $\mathcal{H} \subseteq \mathcal{A}(\mathcal{T})$ comprising sets that are $\mathcal{K}$-analytic sets in $\mathcal{T}$ with a *$\mathcal{T}'$-open representation*, i.e. an upper semicontinuous representation $K : I \to \mathcal{K}(X)$ with $K(U) \in \mathcal{T}'$ for U open in I.

Remark 7.11. The term *analytically heavy* is only suggestive, since for any open $G \neq \emptyset$, there is an analytic A with $G \supseteq A \neq \emptyset$. We are now able to give a very general Baire Theorem.

Theorem 7.12 (Fine Analytic Baire Theorem, Generalized Gandy-Harrington Theorem, cf. [55, p. 466], [59, 25.18, 25.19], [78, Th.3])**.** *In a regular Hausdorff space $\mathcal{T}$, if $\mathcal{T}'$ is a refinement topology of $\mathcal{T}$, possessing a weak base $\mathcal{H} \subseteq \mathcal{A}(\mathcal{T}) \cap \mathcal{T}'$ whose members have an analytic representation that is $\mathcal{T}'$-open, then $\mathcal{T}'$ is Baire.*
In particular, this applies to a Polish space, the Gandy-Harrington $\mathcal{GH}$, the density $\mathcal{D}$, the Ellentuck $\mathcal{E}l$ and O'Malley $\mathcal{R}$ topologies.

8. Semi-Polish Theorem

We sketch below the underlying theme of the proof of the

Theorem 8.1 (Semi-Polish Theorem, [82])**.** *For a normed group X under d_R^X, if the space X is non-meagre and semi-analytic (i.e. is analytic under the symmetrized metric d_S, e.g. Polish under d_S), then it is a Polish topological group.*

Remark 8.2. This asserts that the topology determined by d_R^X is Polish and is *admissible*, i.e. under this topology X is a topological group.

The proof applies the Baire Homomorphism Theorem of §5, the following Open Mapping Theorem and shift-compactness.

Lemma 8.3 (Levi's Open Mapping Theorem, [67])**.** *Let X be a regular analytic space. Then X is a Baire space iff $X = f(P)$ for some f, continuous and defined on some Polish space P, with the property that there exists a set X' which is a dense metrizable $\mathcal{G}_\delta$ in X such that for $P' = f^{-1}(X')$ the **restriction map** $f|P' : P' \to X'$ **is open**.*

Lemma 8.4. *For X a normed group, if (X, d_S) is Polish and (X, d_R) non-meagre, then there is a subset Y of X which is a dense absolute-$\mathcal{G}_\delta$ in (X, d_R), and on which the d_S and d_R topologies agree.*

Proof. As d_S is Polish, the continuous embedding $j : (X, d_S) \to (X, d_R)$ with $j(x) = x$ makes (X, d_R) analytic, and being non-meagre it is a Baire space.
Apply Levi's Theorem to $f = j$ to obtain a set Y that is a *dense* absolute $\mathcal{G}_\delta$ in (X, d_R), such that every open set in (Y, d_S) is open in (Y, d_R). Every open set in (Y, d_R) is already open in (Y, d_S), since d_S is a refinement of d_R. Thus the two topologies agree on the $\mathcal{G}_\delta$ subset Y.
As Y is a $\mathcal{G}_\delta$ subset of (X, d_R), it is also a $\mathcal{G}_\delta$ subset in the complete space (X, d_S), and so (Y, d_S) is topologically complete. So (Y, d_R) is an absolute $\mathcal{G}_\delta$, being homeomorphic to (Y, d_S). Working in Y, we thus have $y_n \to_R y$ iff $y_n \to_S y$ iff $y_n \to_L y$. □

Lemma 8.5. *If in the setting of Lemma 8.4 the three topologies d_R, d_L, d_S agree on a dense absolutely-$\mathcal{G}_\delta$ set Y of (X, d_R), then for any $\tau \in Y$ the conjugacy $\gamma_\tau(x) := \tau x \tau^{-1}$ is continuous.*

The proof uses Th. 7.5 (the analytic shift theorem) to show first that $\gamma(x) := \tau^{-1} x \tau$ is continuous for $\tau \in Y$ (as $\gamma|Y$ is continuous on Y). Hence $\gamma_\tau = \gamma^{-1}$, being a Baire homomorphism, is continuous for $\tau \in Y$ (by Th. 5.2). We can now prove the main result by reference to the topological centre (§2.1).

Proof of the Semi-Polish Theorem. Under d_R, the topological centre $Z_\Gamma := \{x : \gamma_x \text{ is continuous}\}$ is a *close*d (subsemigroup) of X ([21, Prop. 3.43]). So $X = \mathrm{cl}_R Y \subseteq Z_\Gamma$, i.e. γ_x is continuous for all x, and so X is a topological group under the topology of d_R^X, by the Equivalence Theorem (Th. 2.5). So $x_n \to_R x$ iff $x_n^{-1} \to_R x^{-1}$ iff $x_n \to_L x$ iff $x_n \to_S x$. So the topology of d_R^X is Polish as the topology of d_S^X is a Polish. $\square$

9. Life without translations

The density topology $\mathcal{D}$ on $\mathbb{R}$ was described in §4. It is a refinement of the Euclidean topology $\mathcal{E}$ (§6.2), i.e. $\mathcal{E} \subseteq \mathcal{D}$. So $\mathcal{D}$ is submetrizable (for which see [47]), and translations are homeomorphisms, making $\mathcal{D}$ semitopological but not paratopological ([92] Prop. 1.9). The key idea of this section now follows.

Definition 9.1. (**weak category convergence**). A sequence of homeomorphisms h_n of a topological space $(X, \mathcal{T}_X)$ satisfies the *weak category convergence* condition (wcc) if for any non-meagre open set U there is an non-meagre open set $V \subseteq U$ such that, for each $k \in \omega$,

$$\bigcap\nolimits_{n \geq k} V \backslash h_n^{-1}(V) \text{ is meagre.} \tag{wcc}$$

Equivalently, there is a meagre set M such that, for each $k \in \omega$ and $t \notin M$,

$$t \in V \Longrightarrow (\exists n \geq k)\ h_n(t) \in V.$$

The condition (wcc) may variously be interpreted as a topological convergence condition: see below and the next subsection.

Theorem 9.2 (Bitopological Shift-Theorem, aka Category Embedding Theorem, [16], [74])**.** *Let $\mathcal{T}_X$ be a submetrizable topology on X, i.e. a refinement topology of some metric topology $(X, \mathcal{T}_d)$.*
For a subgroup $G \subseteq \mathcal{H}(X, \mathcal{T}_d) \cap Auth(\mathcal{T}_X)$ under the right norm topology, put $\varphi(g, x) = g(x)$ for $g \in G$ and $x \in X$.
Then the mapping $\varphi^g : x \to g(x)$ is continuous.
Suppose further that for any $h_n \to e_G$ in norm, h_n satisfies the (wcc).
Let $A \subseteq X$ be a non-meagre Baire set under $\mathcal{T}_X$.
Then there exists $a \in A$ such that $h_n(a) \in A$ infinitely often.

Remark 9.3. 1. Generalizations exist with consecutive embeddings of fixed length 'in van der Waerden style' (e.g. both h_{2m} and h_{2m+1} in A) and with multiple embeddings (into a sequence of sets A_n – this is Kingman's theorem). For details [20].
2. If $\mathcal{T}_X$ is the density topology, the set A above may without loss of generality be a density-open set W. One may show under certain circumstances, which include the case of the real line under the density topology,

that (wcc) is a *continuity* condition (see [74] and Remark 9.9 below).
3. On the real line one proves that, for $z_n \to 0$, the shifts $h_n(t) = t + z_n$ satisfy wcc both (i) in the usual (Euclidean norm) topology and (ii) in the density topology.
Hence both versions of KBD follow from the last theorem.
4. Recall the **Birkhoff-Kakutani Theorem** (§2) that a metrizable *topological* group has a right-invariant metric. In this case:
i) if X is a Baire space under the norm topology, then the (wcc) holds under the norm topology,
ii) if, additionally, X is locally compact then X has a Haar measure and (wcc) can be verified for the *Haar-density topology.*

Definition 9.4. Say that $\{h_n\}$ *$\mathcal{I}$-converges to the identity* and write $\{h_n\} \rightrightarrows_{\mathcal{I}} e_G$ if for any open U on X there is a non-empty open $W \subseteq U$ such that for every increasing sequence $\{m(n)\}$ of natural numbers,

$$\bigcap\nolimits_n V \backslash h_{m(n)}^{-1}(V) \in \mathcal{I}.$$

(The double arrow is cautionary: a convergence structure need not generate a topology – see [35, p. 26], or [41].) Taking in particular $m(n) = n+k$, one retrieves (wcc) for $k = 1, 2, ...$ as part of a more demanding condition. For the group of translations on $\mathbb{R}^d$ the new condition is equivalent to (wcc), since $z_{m(n)}$ is a null sequence whenever z_n is a null sequence.

[74] studies when the convergence structure $\{h_n\} \rightrightarrows_{\mathcal{I}} e_G$ is topological, i.e. generates a topology $\mathcal{T}_{\mathcal{I}}$. One has the following:

Theorem 9.5 ([74])**.** *For the group G of translations of the real line, with $\mathcal{I} = \mathcal{N}$ or $\mathcal{I} = \mathcal{M}$, the topology $\mathcal{T}_{\mathcal{I}}$ on G is well-defined and is coarser than its right supremum-norm topology.*

9.1. **From weak category to coarse convergence.** This section is devoted to interpreting (wcc). Refining an argument developed in [16], one obtains the following improvement.

Theorem 9.6 (Convergence to the identity, cf. [16])**.** *Assume that the homeomorphisms $h_n : X \to X$ satisfy (wcc) and that X is a submetrizable Baire space. Then, for quasi-all t, there is an infinite $\mathbb{N}_t$ such that with $\mathcal{T}_X$ and $\mathcal{T}_d$ as in Th. 9.2*

$$\lim\nolimits_{m \in \mathbb{N}_t} h_m(t) = t \text{ under } \mathcal{T}_d.$$

Proof. Working first in $\mathcal{T}_d$, let $\mathcal{B} = \bigcup_{m \in \omega} \mathcal{B}_m$ be a basis with each $\mathcal{B}_n$ discrete. Now work in $\mathcal{T}_X$ until further notice. In this finer topology each $\mathcal{B}_n$ is still open and discrete, and so the members of each $\mathcal{B}_n$ may be assumed non-empty, so non-meagre (as the finer topology is Baire).

By (wcc), select for each $U \in \mathcal{B}_m$ and $k \in \omega$ a non-empty open $V_k(U) \subseteq U$ such that $M_k(U) := \bigcap_{n \geq k} V_k(U) \backslash h_n^{-1}(V_k(U))$ is a meagre subset of $V_k(U)$. For $k \in \omega$ and $W \in \mathcal{B}$ choose a maximal family $\mathcal{V}_k^W := \{V_k(U_i^W) : i \in I^W\}$ of disjoint non-empty open subsets of the form $V_k(U)$ for $U \subseteq W$. Let V_k^W denote its union and $F_k(W)$ the closure of V_k^W. Then $F_k(W) \subseteq \bar{W}$. Furthermore $W \subseteq F_k(W)$, otherwise $U := W \backslash F_k(W)$ is non-empty, open and disjoint from V_k^W, so that $V_k(U)$ is non-empty and disjoint from the members of $\mathcal{V}_k^W$, contradicting maximality.

We now construct two meagre sets N and M as follows.

Observe that $N_k(W) := F_k(W) \backslash V_k^W$ is closed and nowhere dense as V_k^W is open; one has $N_k(W) \subseteq \bar{W}$. For $m \in \omega$, note that $B_m := \bigcup\{\bar{W} \backslash W : W \in \mathcal{B}_m\}$ is nowhere dense. (Any point x of X has a nhd G meeting at most one set $W \in \mathcal{B}_m$, say W_G, but otherwise for $W \in \mathcal{B}_m$ the set G misses W and so also $\bar{W}$. As $\bar{W}_G \backslash W_G$ is nowhere dense, there is a non-empty subset G' of G avoiding $\bar{W}_G \backslash W_G$.) By the Banach Category Theorem, since $\mathcal{B}_m$ is a discrete open family, $\bigcup\{N_k(W) \cap W : W \in \mathcal{B}_m\}$ is meagre. So $N_{k,m} := \bigcup\{N_k(W) : W \in \mathcal{B}_m\}$ is meagre, being a subset of $B_m \cup \bigcup\{N_k(W) \cap W : W \in \mathcal{B}_m\}$. So

$$N := \bigcup\nolimits_{k,m \in \omega} N_{km} = \bigcup\{N_k(W) : W \in \mathcal{B}_m \text{ and } k, m \in \omega\}$$

is meagre. Note that $W \backslash N_k(W) \subseteq F_k(W) \backslash N_k(W) \subseteq V_k^W$.

Next for $W \in \mathcal{B}$ and $k \in \omega$, put

$$M_k^W := \bigcup\{M_k(U_i^W) : i \in I^W\} \subseteq W,$$

which is meagre – again by the Banach Category Theorem, as $M_k(U_i^W) \subseteq V_k(U_i^W)$ and $\mathcal{V}_k^W$ is disjoint. Put

$$M := \bigcup\nolimits_{k,m \in \omega} \bigcup\{M_k^W : W \in \mathcal{B}_m\},$$

which is likewise meagre (by the discreteness of $\mathcal{B}_m$).

Consider $t \notin N \cup M$. For $W \in \mathcal{B}$ with $t \in W$ and $k \in \omega$, one has $t \in V_k(U_i^W)$ for some $U_i^W \subseteq W$ with $i \in I^W$, as $t \in W \backslash N_k(W) \subseteq V_k^W$. Also, since $t \notin M$, one has $t \in V_k(U_i^W) \backslash M_k(U_i^W)$, so $t \in h_m^{-1}(V_k(U_i^W))$ for some $m = m(t, k, W, i) \geq k$. So $h_m(t) \in V_k(U_i^W) \subseteq U_i^W \subseteq W$. Passing now to the coarser topology $\mathcal{T}_d$ in which $\mathcal{B}_t := \{W \in \mathcal{B} : t \in W\}$ is a basis for the nhds of t, it follows that there is an infinite set $\mathbb{N}_t$ of integers m for which $h_m(t) \to t$ in $\mathcal{T}_d$. □

9.2. **A generalized Pettis theorem.** Our final example of homeomorphisms replacing translations is motivated by a key property required in topological regular variation theory ([17]): for any convergent sequence of points x_n with limit x_0, existence of a corresponding sequence of bounded self-homeomorphisms, i.e. in $\mathcal{H}(X)$, converging to the identity (i.e. $\psi_n \to id_X$ under the supremum metric $\hat{d}^X$, as in (sup) above) with $\psi_n(x_0) = x_n$. We call the ψ_n a 'crimping sequence' and think of them as diminishing 'topological' shifts. This property is equivalent to the microtransitivity of the Effros Theorem ([37], [100], [83]).

The continuous analogue below of crimping yields a generalized Pettis theorem requiring only that $\mathcal{H}(X)$ act transitively on X (homogeneity).

Definition 9.7. Let $\{\psi_u : u \in U\}$ for U an open set in X be a family of homeomorphisms in $\mathcal{H}(X)$. Let $u_0 \in U$. Say that ψ_u *converges to the identity as* $u \to u_0$ if $\lim_{u\to u_0} \|\psi_u\|_{\mathcal{H}} = 0$.

Theorem 9.8 (Generalized Piccard-Pettis Theorem, [21])**.** *Let* (X, d^X) *be homogenous under* $\mathcal{H}(X)$. *Suppose that the homeomorphisms* ψ_u *converge to the identity as* $u \to u_0$, *and that* $A \subseteq X$ *is a non-meagre Baire subset. Then, for some* $\delta > 0$, *we have*

$$A \cap \psi_u(A) \neq \emptyset, \text{ for all } u \text{ with } d^X(u, u_0) < \delta,$$

or equivalently, for some $\delta > 0$

$$A \cap \psi_u^{-1}(A) \neq \emptyset, \text{ for all } u \text{ with } d^X(u, u_0) < \delta.$$

Example. Take $X = \mathbb{R}$, $u_0 = 0$ and $\psi_u(x) = x+u$, then $\lim_{u\to u_0} \|\psi_u\|_{\mathcal{H}} = 0$. For A Baire non-meagre, there is some $\delta > 0$ such that for each u with $|u| < \delta$ there is a_1, a_2 with $a_1 = a_2 + u \in A \cap (A + u)$. So $a_1, a_2 \in A$ and $u = a_1 - a_2$, i.e. $(-\delta, \delta) \subseteq A - A$.

Remark 9.9. Writing $\Psi(u)$ for the map $x \to \psi_u(x)$, the property here when A is an open set U may be rephrased as $B_\delta(u_0) \subseteq \Psi^{-1}\{h \in \mathcal{H}(X) : U \cap h(U) \neq \emptyset\}$ indicating that Ψ is continuous relative to the lower Fox-Mosco topology on $\mathcal{H}(X)$, as defined in [74].

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