TOPOLOGY PROCEEDINGS
Volume 41, 2013
Pages 153–165

http://topology.auburn.edu/tp/

Countability Properties of the σ-Compact-Open Topology on $C^*(X)$

by

S. Kundu and Vipra Pandey

Electronically published on August 18, 2012

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS
Volume 41 (2013)
Pages 153-165

E-Published on August 18, 2012

COUNTABILITY PROPERTIES OF THE σ-COMPACT-OPEN TOPOLOGY ON $C^*(X)$

S. KUNDU AND VIPRA PANDEY†

ABSTRACT. The main goal of this paper is to study the countability properties, such as the separability, second countability and Lindelöf property of the σ-compact-open topology on $C^*(X)$, the set of all bounded real-valued continuous functions on a Tychonoff space X.

1. INTRODUCTION

The set $C(X)$ of all real-valued continuous functions as well as the set $C^*(X)$ of all bounded real-valued continuous functions on a Tychonoff space X has a number of natural topologies. Two commonly used among them are the compact-open topology k and the topology of uniform convergence u. While the topology of uniform convergence on $C(X)$ has been used for more than a century as the proper setting to study uniform convergence of sequences of functions, the compact-open topology on $C(X)$ made its appearance in 1945 in a paper by Ralph H. Fox [10] and soon after it was developed by Richard F. Arens in [2] and by Arens and James Dugundji in [3]. This topology was shown in [15] to be the proper setting for studying sequences of functions which converge uniformly on compact subsets. But soon, it also turned out to be a natural and interesting locally convex topology on $C(X)$ from the measure-theoretic viewpoint. In fact, continuous functions and Baire measures on Tychonoff spaces are linked by the process of integration. A number of natural locally convex topologies on spaces of continuous functions have been studied in order to clarify this relationship. For more information on these topologies see [28].

2010 *Mathematics Subject Classification.* 54C35, 54D20, 54D99, 54E20.

Key words and phrases. Function space, separability, $\aleph_0$-bounded, the countable chain condition, Lindelöf, $\aleph_0$-space, cosmic, second countable.

†This works was done when the second author worked for his Ph.D under the supervision of the first author at the Indian Institute of Technology Delhi, New Delhi 110016, India.

The compact-open topology and the topology of uniform convergence on $C(X)$ (or on C^*(X)) are equal if and only if X is compact. Since compactness is such a strong condition, there is a considerable gap between these two topologies. This gap has been especially felt in topological measure theory; consequently in the last five decades, there have been quite a few topologies introduced that lie between k and u, such as the strict topology, the σ-compact-open topology, the topology of uniform convergence on σ-compact subsets and the topology of uniform convergence on bounded subsets. (See for example [5], [8], [11], [12] ,[16], [17], [18], [24], [25]).

The σ-compact-open topology σ is one such natural and interesting locally convex topology on $C^*(X)$, from the viewpoint of both topology and measure theory. The space $C^*(X)$ with the topology σ is denoted by $C^*_\sigma(X)$. Actually this topology was first introduced in [12] from the viewpoint of measure theory and functional analysis. According to Gulick himself, his "paper arose from an attempt to define a natural topology which would serve as the Mackey Topology for the strict topology" on $C^*(X)$. Later on, this topology which can also be considered as the topology of uniform convergence on σ-compact subsets of X, has been studied in [17], but on $C(X)$, instead of $C^*(X)$. In [19], the metrizability and uniform completeness of $C^*_\sigma(X)$ have been studied in detail. But another important family of properties, the countability properties of $C^*_\sigma(X)$, is yet to be studied. In this paper, we plan to do exactly that. More precisely, we would like to study the separability, $\aleph_0$-boundedness, countable chain condition, second countability and Lindelöf property of $C^*_\sigma(X)$. In section 2 of this paper, in addition to studying separability, we study the possibilities of $C^*_\sigma(X)$ having the properties of $\aleph_0$-boundedness and countable chain condition. In Sections 3 and 4, we study the second countability and Lindelöf property of $C^*_\sigma(X)$ respectively.

Throughout this paper, all spaces are Tychonoff and $\mathbb{R}$ denotes the space of real numbers with the usual topology. The constant zero function defined on X is denoted by 0, more precisely by 0_X. We call it the constant zero function in $C^*(X)$. The symbols ω_0 and ω_1 denote the first infinite and the first uncountable ordinal respectively. If X and Y are two spaces with the same underlying set, then we use $X = Y$, $X \leq Y$ and $X < Y$ to indicate, respectively, that X and Y have the same topology, that the topology on Y is finer than or equal to the topology on X and that the topology on Y is strictly finer than the topology on X.

2. Separability, $\aleph_0$-boundedness and countable chain condition

The σ-compact open topology on C^*(X) can be viewed in three different ways. First we can view the σ-compact open topology as a "set-open" topology in the following manner.
For any subset A of X and any open subset V of $\mathbb{R}$, define

$$[A,V]=\{f \in C^*(X) : \overline{f(A)} \subseteq V\}.$$

Now let $\sigma(X)$ be the family of all σ-compact subsets of X, and let $\mathbb{B}$ be the set of all bounded open intervals in $\mathbb{R}$. For the σ-compact-open topology on C^*(X), we take as a subbase, the family $\{[A, B] : A \in \sigma(X), B \in \mathbb{B}\}$ and we denote the corresponding space by C^*_σ(X).

The second way we can view the σ-compact open topology is as a "uniform topology". For each A $\in \sigma$(X) and $\epsilon > 0$, let

$$A_\epsilon = \{(f, g) \in C^*(X) \times C^*(X) : |f(x) - g(x)| < \epsilon \; \forall \; x \in A\}.$$

Then it can be verified that the collection $\{A_\epsilon : A \in \sigma(X), \epsilon > 0\}$ is a base for some uniformity on C^*(X). This uniformity induces the topology of uniform convergence on σ-compact subsets of X and is the same as the σ-compact open topology defined earlier.
For each $f \in C^*$(X), $A \in \sigma$(X) and $\epsilon > 0$, let $< f, A, \epsilon > = \{g \in C^*(X) : |f(x) - g(x)| < \epsilon \; \forall \; x \in A\}$. Then for each $f \in C^*(X)$, the collection $\{< f, A, \epsilon >: A \in \sigma(X), \epsilon > 0\}$ forms a neighborhood base at f in $C^*_\sigma(X)$. Since the topology comes from a uniformity, $C^*_\sigma(X)$ is completely regular and since the topology σ is finer than the topology of pointwise convergence, σ is Hausdorff. Consequently $C^*_\sigma(X)$ is a Tychonoff space.

The third way we can view the σ-compact open topology is as a locally convex topology generated by the collection of seminorms $\{p_A :$ A $\in \sigma$(X)$\}$ where for each A $\in \sigma$(X), the seminorm p_A on $C^*(X)$ is defined by $p_A(f) = \{sup|f(x)| : x \in A\}$. Also for each $A \in \sigma(X)$ and $\epsilon > 0$, let

$$V_{A,\epsilon} = \{f \in C^*(X) : p_A(f) < \epsilon\}.$$

Let $\mathcal{V} = \{V_{A,\epsilon} : A \in \sigma(X), \epsilon > 0\}$. It can be easily shown that for each $f \in C^*(X)$, $f + \mathcal{V} = \{f + V : V \in \mathcal{V}\}$ forms a neighborhood base at f. Since this topology is generated by a collection of seminorms, it is locally convex.

The uniform topology u on $C^*(X)$ is generated by the complete supremum metric ρ, where for $f, g \in C^*(X)$, $\rho(f, g) = sup\{|f(x) - g(x)| : x \in X\}$ and the corresponding topological space is denoted by $C^*_u(X)$. We also denote $\rho(f, g)$ by $||f - g||_\infty$. Usually we use the notation $C^*_\infty(X)$ in place of $C^*_u(X)$.

We now study a topological property of $C^*_\sigma(X)$, which is weaker than separability. The property is known as the countable chain condition. The precise definition follows.

Definition 2.1. A space X is said to have the *countable chain condition* (called ccc in brief) if any family of pairwise disjoint nonempty open subsets of X is countable. The ccc is also known as the Souslin property.

In the first result of this section, we show that pseudocompactness of X is a necessary condition for $C^*_\sigma(X)$ to have ccc.

Theorem 2.1. *If $C^*_\sigma(X)$ has the countable chain condition, then X is pseudocompact.*

Proof. Suppose that X is not pseudocompact. Then there is a closed C-embedding $\phi : \mathbb{N} \longrightarrow X$. This function ϕ induces a continuous function $\phi^* : C^*_\sigma(X) \longrightarrow C^*_\sigma(\mathbb{N})$ where ϕ^* is defined by $\phi^*(f) = f \circ \phi \ \forall f \in C^*(X)$. Since a C-embedding is a C^*-embedding, ϕ is a C^*-embedding and consequently ϕ^* is a surjection.

We show that $C^*_\sigma(\mathbb{N})$ does not have the countable chain condition. Then since ϕ^* is a continuous surjection, $C^*_\sigma(X)$ will also not have the countable chain condition. Let $U = (0, 1)$ and $V = (1, 2)$, and for each A in the power set $\mathcal{P}(\mathbb{N})$ of $\mathbb{N}$, define

$S_A = [A, U] \bigcap [\mathbb{N} \setminus A, V]$

where $[A, V] = \{f \in C^*(X) : \overline{f(A)} \subseteq V\}$.

Then S_A is a nonempty basic open set in $C^*_\sigma(\mathbb{N})$. Suppose that $A, B \in \mathcal{P}(\mathbb{N})$ with $A \neq B$. Without loss of generality, say there is some $x \in A \setminus B$. If $f \in S_A$, then $f(x) \in U$; but if $f \in S_B$, then $f(x) \in V$. Since $U \bigcap V = \emptyset$, $S_A \bigcap S_B = \emptyset$. Therefore $\{S_A : A \in \mathcal{P}(\mathbb{N})\}$ is an uncountable pairwise disjoint family of non-empty open subsets of $C^*_\sigma(\mathbb{N})$. □

Remark 2.1. If a space X has a dense subspace having ccc, then X itself has ccc. In [17], it has been shown that if $C_\sigma(X)$ (the space $C(X)$ equipped with the σ-compact-open topology) has ccc, then X is pseudocompact. From this result, Theorem 2.1 immediately follows. Actually the proof of Theorem 2.1 is the same as the proof of this result.

Theorem 2.2. *For a space X, the following are equivalent.*

(*a*) *$C^*_\infty(X)$ is separable.*
(*b*) *$C^*_\sigma(X)$ is separable.*
(*c*) *X is compact and metrizable.*

Proof. $(a) \Rightarrow (b)$ This is immediate since $C^*_\sigma(X) \leq C^*_\infty(X)$.

$(b) \Rightarrow (c)$ If $C^*_\sigma(X)$ is separable, then $C^*_k(X)$ is also separable since $C^*_k(X) \leq C^*_\sigma(X)$. But $C^*(X)$ is dense in $C_k(X)$ (the space $C(X)$ equipped with the compact open topology). Hence $C_k(X)$ is separable. But then by Theorem 5 in [27], X is submetrizable. Again, $C^*_\sigma(X)$ being separable, has ccc and hence by Theorem 2.1, X is pseudocompact. But a pseudocompact submetrizable space is compact and metrizable, see Lemma 4.3 in [17].

$(c) \Rightarrow (a)$ Suppose X is compact and metrizable. Let d be a compatible metric inducing the topology of X. Since (X, d) is compact, it is separable. Let $\{x_1, x_2, ..., x_n, ...\}$ be a countable dense subset of X. For each n, let $f_n : X \longrightarrow \mathbb{R}$ be the function defined by $f_n(x) = d(x, x_n)$ for each $x \in X$. Each f_n is continuous. Since X is compact, each f_n is bounded and hence $f_n \in C^*(X)$. It can be easily shown that if $x, y \in X$ with $x \neq y$, then there exists $n \in \mathbb{N}$ such that $f_n(x) \neq f_n(y)$. But this implies that the algebra generated by $\{1_X, f_1, f_2, ...\}$ separates the points of X. Here 1_X denotes the constant function 1 defined on X that is $1_X(x) = 1\ \forall x \in X$. Now by the Stone-Weierstrass theorem (see Theorem 11.5, page 89 in [1]), this algebra must be dense in $C^*_\infty(X)$. Now, consider the countable collection $\mathcal{C}$ of all finite products of the countable collection $\{1_X, f_1, f_2, ...\}$. Let $\mathcal{C} = \{g_1, g_2, ...\}$. It can be verified that the finite linear combinations of $\{1, g_1, g_2, ...\}$ with rational coefficients form a countable dense subset of $C^*_\infty(X)$. □

Example 2.1. Let $X = \mathbb{R}$. Then since $\mathbb{R}$ is not compact, $C^*_\sigma(\mathbb{R})$ is not separable.

Example 2.2. Let $X = \mathbb{R}_l$, the real line equipped with the lower limit topology. We also call this space the Sorgenfrey line. In $\mathbb{R}_l$, since every compact subset is countable, $\mathbb{R}_l$ is not even σ-compact and consequently $C^*_\sigma(\mathbb{R}_l)$ is not separable.

Example 2.3. Since $[0, \omega_1)$ is countably compact, but not compact, $[0, \omega_1)$ is not metrizable. Hence neither $C^*_\sigma([0, \omega_1))$ nor $C^*_\sigma([0, \omega_1])$ is separable.

Now we would like to study another topological property of $C^*_\sigma(X)$, which is weaker than separability. The property is known as being $\aleph_0$-bounded. The precise definition follows.

Definition 2.2. Let G be a topological group (under addition). Then G is said to be $\aleph_0$-*bounded* provided that for each neighborhood U of the identity element in G, there exists a countable subset S of G such that $G = S + U = \{s + u : s \in S, u \in U\}$.

Arhangel'skii studied $\aleph_0$-bounded topological groups in the Section 9 of [4] in a more general setting of τ-bounded topological groups. According to Arhangel'skii, the τ-bounded topological groups were first studied by Guran in [14]. We would like to state the following interesting and significant results on $\aleph_0$-bounded topological groups mentioned in [4].

(a) The product of any family of $\aleph_0$-bounded topological groups is $\aleph_0$-bounded.
(b) Any subgroup of an $\aleph_0$-bounded topological group is $\aleph_0$-bounded.
(c) A topological group having a dense $\aleph_0$-bounded topological subgroup is itself $\aleph_0$-bounded.
(d) The image of an $\aleph_0$-bounded topological group under a continuous homomorphism is $\aleph_0$-bounded.
(e) The class of $\aleph_0$-bounded groups contains all subgroups of compact Hausdorff groups.
(f) A topological group is $\aleph_0$-bounded if and only if it is topologically isomorphic to a subgroup of a topological group having ccc. Then obviously a topological group having ccc is itself $\aleph_0$-bounded.
(g) A Lindelöf topological group is $\aleph_0$-bounded.
(h) A metrizable $\aleph_0$-bounded space is separable.

In [21], an $\aleph_0$-bounded topological group has been called totally $\aleph_0$-bounded. Here we would like to put a note of caution. In [7], Arhangel'skii has used the term '$\aleph_0$-bounded' for an entirely different concept. In [7], a topological space has been called $\aleph_0$-bounded if the closure of every countable subset of X is compact.

The next result gives a necessary condition for $C^*_\sigma(X)$ to be $\aleph_0$-bounded.

Theorem 2.3. *For a space X, assume that $C^*_\sigma(X)$ is $\aleph_0$-bounded. Then every C^*-embedded σ-compact subset of X is metrizable and compact.*

Proof. Let A be a C^*-embedded σ-compact subset of X. Now first we will show that $C^*_\sigma(A)$ is $\aleph_0$-bounded. Since A is σ-compact, $C^*_\sigma(A)$ is metrizable. To avoid confusion, let us denote for each $f \in C^*(A)$ and $\epsilon > 0$

$$\langle f, A, \epsilon\rangle_A = \{g \in C^*(A) : |f(x) - g(x)| < \epsilon \ \forall\ x \in A\}.$$

Then for each f in $C^*(A)$, the collection $\{\langle f, A, \epsilon\rangle_A : \epsilon > 0\}$ forms a neighborhood base of f in $C^*_\sigma(A)$. Let $\langle 0_A, A, \epsilon\rangle_A$ be a basic neighborhood of the zero function 0_A in $C^*_\sigma(A)$. Now $\langle 0_X, A, \epsilon\rangle$ is a basic neighborhood of the zero function 0_X in $C^*_\sigma(X)$. Since $C^*_\sigma(X)$ is $\aleph_0$-bounded, there exists a countable set B in $C^*_\sigma(X)$ such that $C^*_\sigma(X) = B\ + \langle 0_X, A, \epsilon\rangle$. Now let $B_A = \{f|_A : f \in B\}$. Let $f \in C^*(A)$. Since A is C^*-embedded in X, there exists a continuous extension f^* of f to X. Then $f^* = h + g$ where $h \in B$ and $g \in \langle 0_X, A, \epsilon\rangle$. Then $h|_A \in B_A$, $g|_A \in \langle 0_A, A, \epsilon\rangle$ and $f = g|_A + h|_A$. Thus $C^*_\sigma(A)$ is $\aleph_0$-bounded. Now, $\aleph_0$-bounded metrizable groups are separable. Therefore, $C^*_\sigma(A)$ is separable. By Theorem 2.2, A is metrizable and compact. □

Corollary 2.4. *If $C^*_\sigma(X)$ is either Lindelöf or has ccc, then every C^*-embedded σ-compact subset of X is metrizable and compact.*

Corollary 2.5. *If X is σ-compact and $C^*_\sigma(X)$ is Lindelöf or has ccc, then X is compact and metrizable.*

By using Theorem 2.2, an alternate proof of Corollary 2.5 can be given as follows. If X is σ-compact, then $C^*_\sigma(X) = C^*_\infty(X)$ and consequently $C^*_\sigma(X)$ is metrizable. So in addition, if $C^*_\sigma(X)$ is Lindelöf, then $C^*_\sigma(X)$ would be separable and consequently by Theorem 2.2, X would be metrizable and compact.

3. Second Countability

In this section, we study the second countability of $C^*_\sigma(X)$. We begin this study by first observing that $C^*_\sigma(X)$ is second countable if and only if $C^*_\infty(X)$ is so.

Theorem 3.1. *For a space X, $C^*_\sigma(X)$ is second countable if and only if $C^*_\infty(X)$ is second countable. Moreover, in either case $C^*_\sigma(X) = C^*_\infty(X)$.*

Proof. If either $C^*_\sigma(X)$ or $C^*_\infty(X)$ is second countable, then it is separable and consequently by Theorem 2.2, X is compact and metrizable. Hence in this case, $C^*_\sigma(X) = C^*_\infty(X)$. □

Theorem 3.1 can be strengthened further as follows. In particular, the next result shows that $C^*_\sigma(X)$ is separable if and only if it is second countable. Recall that a space X is called cosmic if it has a countable network. A space X is called almost cosmic if it has a dense cosmic subspace. Also by Proposition 10.2 of [22], a space X is cosmic if and only if it is a continuous image of a separable metric space. In particular, a separable metric space is cosmic and a cosmic space is separable.

For the next result on the second countability of $C^*_\sigma(X)$, we need the definition of π-base.

Definition 3.1. A family of nonempty open sets in a space X is called a π-base for X if every nonempty open set in X contains a member of this family.

The routine proof of the following lemma is omitted.

Lemma 3.2. *Let D be a dense subset of a space X. Then D has a countable π-base if and only if X has a countable π-base.*

Theorem 3.3. *For a space X, the following assertions are equivalent.*

(*a*) *$C^*_\sigma(X)$ contains a dense subspace which has a countable π-base.*
(*b*) *$C^*_\sigma(X)$ has a countable π-base.*

(c) $C^*_\sigma(X)$ *is second countable.*
(d) $C^*_\infty(X)$ *is second countable.*
(e) $C^*_\infty(X)$ *is separable.*
(f) $C^*_\sigma(X)$ *is separable.*
(g) $C^*_\sigma(X)$ *is cosmic.*
(h) $C^*_\sigma(X)$ *is almost cosmic.*
(i) X *is compact and metrizable.*

Proof. By Lemma 3.2, $(a) \Leftrightarrow (b)$. By Theorem 2.2, $(e) \Leftrightarrow (f) \Leftrightarrow (i)$ and by Theorem 3.1, $(c) \Leftrightarrow (d)$.
$(d) \Rightarrow (e)$ and $(g) \Rightarrow (h)$. These are immediate.
$(b) \Rightarrow (c)$. If $C^*_\sigma(X)$ has a countable π-base, then by Theorem 2.4 in [19], $C^*_\sigma(X)$ is metrizable. But a space having a countable π-base is separable and a separable metrizable space is second countable.
$(h) \Rightarrow (f)$. If $C^*_\sigma(X)$ is almost cosmic, that is, if it has a dense cosmic subspace, then $C^*_\sigma(X)$ has a dense separable subspace. Consequently $C^*_\sigma(X)$ itself would be separable.
$(i) \Rightarrow (g)$ and $(i) \Rightarrow (b)$. If X is compact and metrizable, then $C^*_\sigma(X) = C^*_\infty(X)$ is metrizable and separable. But a separable metrizable space is cosmic as well as second countable. □

Corollary 3.4. *If X is pseudocompact, then $C^*_\sigma(X)$ is second countable if and only if X is second countable.*

Proof. If $C^*_\sigma(X)$ is second countable, then by Theorem 3.3, X is compact and metrizable. But a compact metrizable space is second countable. Conversely, If X is second countable, then X is metrizable. But a pseudocompact metrizable space is compact. Hence by Theorem 3.3, $C^*_\sigma(X)$ is second countable.

□

4. Lindelöf Property

In the last section of this paper, we study the situations when possibly $C^*_\sigma(X)$ can be Lindelöf. Since $C^*_p(X) \leq C^*_k(X) \leq C^*_\sigma(X)$, any necessary condition for either $C^*_p(X)$ to be Lindelöf or $C^*_k(X)$ to be Lindelöf also becomes necessary for $C^*_\sigma(X)$ to be Lindelöf. Therefore it becomes expedient to search for criteria in terms of topological properties of X so that $C^*_p(X)$ becomes Lindelöf. But here we should mention that though many well-known mathematicians have researched and studied several such criteria for $C_p(X)$ to be Lindelöf, no satisfactory intrinsic characterisation of the space X, for which $C_p(X)$ is Lindelöf is yet to emerge. Likewise it appears that the situations when possibly $C^*_p(X)$ can be Lindelöf are difficult to find. In the literature there has hardly been any direct reference to the situation when $C_p^*(X)$ is Lindelöf. Consequently, we have not been able much to make significant contribution to the study of the situations for $C_\sigma^*(X)$ to be Lindelöf. But in addition to giving one necessary condition for $C_\sigma^*(X)$ to be Lindelöf in terms of tightness of X, we show that $C_j^*(X)$ (j=p, k, σ) is Lindelöf if and only if $C_j(X,[0,1])$ is Lindelöf. Here $C_j(X,[0,1])$ is the space $C(X,[0,1])$ equipped with the topology j (j=p, k, σ) and $C(X,[0,1]) = \{f \in C(X) : f(X) \subseteq [0,1]\}$. In general, for a closed interval [a,b] in $\mathbb{R}$, $C_j(X,[a,b])$ is the space $C(X,[a,b])$ equipped with the topology j=p, k, σ and $C(X,[a,b]) = \{f \in C(X) : f(X) \subseteq [a,b]\}$. Note $C(X,[a,b]) = C^*(X,[a,b])$. To be precise, here the topologies p, k and σ on $C^*(X,[a,b])$ denote the topology of pointwise convergence, the topology of uniform convergence on compact subsets of X and the topology of uniform convergence on σ-compact subsets of X respectively.

At the end of this section we give several examples in relation to the Lindelöf property of $C_\sigma^*(X)$. The first result of this section is the key step towards establishing the equivalence of Lindelöfness of $C_\sigma^*(X)$ and $C_\sigma^*(X,[0,1])$.

Theorem 4.1. *For any space X, $C_\sigma^*(X,[a,b])$ is homeomorphic $C_\sigma^*(X,[0,1])$.*

Proof. Let $\phi : [a,b] \to [0,1]$ be the homeomorphism given by $\phi(\alpha) = \frac{\alpha-a}{b-a}$ $\forall\ \alpha \in [a,b]$. Note that both ϕ and ϕ^{-1} are uniformly continuous. Define $\widehat{\phi} : C_\sigma^*(X,[a,b]) \to C_\sigma^*(X,[0,1])$ by $\widehat{\phi}(f) = \phi \circ f\ \forall\ f \in C^*(X,[a,b])$. It is easy to see that $\widehat{\phi}$ is a bijection and $(\widehat{\phi})^{-1} = \widehat{\phi^{-1}}$. Hence in order to check that $\widehat{\phi}$ is a homeomorphism, it is enough to show that $\widehat{\phi}$ is continuous. Let $(f_\lambda)_{\lambda\in\Lambda}$ be a net converging to f in $C_\sigma^*(X,[a,b])$. Choose any $\epsilon > 0$ and any $A \in \sigma(X)$. Since ϕ is uniformly continuous, $\exists\ \delta > 0$ (δ depends on ϵ only) such that if for any $\alpha, \beta \in [a,b]$, $|\alpha - \beta| < \delta$, then $|\phi(\alpha) - \phi(\beta)| < \epsilon$. Since $f_\lambda \to f$ in $C_\sigma^*(X,[a,b])$, $\exists\ \lambda_0 \in \Lambda$ such that $|f_\lambda(x) - f(x)| < \delta$ whenever $x \in A$ and $\lambda \geq \lambda_0$. Hence $|\phi(f_\lambda(x)) - \phi(f(x))| < \epsilon$ whenever $x \in A$ and $\lambda \geq \lambda_0$, that is, $|\widehat{\phi}(f_\lambda)(x) - \widehat{\phi}(f)(x)| < \epsilon$ whenever $x \in A$ and $\lambda \geq \lambda_0$. But this precisely means $\widehat{\phi}(f_\lambda) \to \phi(f_\lambda)$ uniformly on A. Since $A \in \sigma(X)$ was chosen arbitrarily, $\widehat{\phi}(f_\lambda) \to \widehat{\phi}(f)$ in $C_\sigma^*(X,[0,1])$. Hence $\widehat{\phi}$ is continuous. □

In a manner similar to Theorem 4.1, the following result can be proved.

Theorem 4.2. *For any space X, $C_j^*(X,[a,b])$ (j=p, k) is homeomorphic to $C_j^*(X,[0,1])$.*

Theorem 4.3. *For any space X, $C_j^*(X)$ (j=p, k, σ) is Lindelöf if and only if $C_j^*(X,[0,1])$ is Lindelöf.*

Proof. Note that $C^*(X) = \bigcup_{n=1}^{\infty} C^*(X, [-n, n])$ and each $C^*(X, [-n, n])$ is closed in $C_j^*(X)$ (j=p, k, σ). If $C_j^*(X)$ is Lindelöf, then each $C_j^*(X, [-n, n])$ is Lindelöf. Consequently, by Theorem 4.1 or by Theorem 4.2 (depending on whether j= σ or j=p, k) $C_j^*(X, [0, 1])$ is Lindelöf. Conversely, if $C_j^*(X, [0, 1])$ is Lindelöf, then by Theorem 4.1 or Theorem 4.2, each $C_j^*(X, [-n, n])$ is Lindelöf. But a countable union of Lindelöf subsets is again Lindelöf. Hence $C_j^*(X)$ is Lindelöf. □

Corollary 4.4. *If X is a metric space with a separable derived set, then $C_k^*(X)$ is Lindelöf.*

Proof. If X is a metric space with a separable derived set, then by Theorem 1 of [23], $C_k^*(X, [0, 1])$ is Lindelöf. □

Recall that a space X is said to have *countable tightness* if for each $x \in X$ and $A \subseteq X$ such that $x \in \overline{A}$, there exists a countable subset C of A such that $x \in \overline{C}$.

Proposition 4.5. *If $C_\sigma^*(X)$ is Lindelöf, then X has countable tightness.*

Proof. Suppose $C_\sigma^*(X)$ is Lindelöf, then $C_p^*(X)$ is also Lindelöf. Hence by Theorem 4 of [20], X has countable tightness. □

Theorem 4.6. *The space $C_\sigma^*(X)$ is not Lindelöf whenever either of the following is true:*

(*i*) *X is almost σ-compact, but not compact;*
(*ii*) *X is compact, but not metrizable.*

Proof. (i) If X is almost σ-compact, then $C_\sigma^*(X) = C_\infty^*(X)$ and $C_\sigma^*(X)$ is metrizable. If $C_\sigma^*(X)$ is Lindelöf, then it would be separable. But then by Theorem 2.2, X would be compact and metrizable. We arrive at a contradiction. Hence $C_\sigma^*(X)$ is not Lindelöf.
(ii) If X is compact and $C_\sigma^*(X)$ is Lindelöf, then $C_\sigma^*(X)$ would be separable and consequently again by Theorem 2.2, X would be metrizable. Hence $C_\sigma^*(X)$ is not Lindelöf. □

Corollary 4.7. *Let X be almost σ-compact. Then $C_\sigma^*(X)$ is Lindelöf if and only if X is compact and metrizable.*

The hypothesis that X be almost σ-compact cannot be omitted from Corollary 4.7. This can be seen by taking $X = [0, \omega_1)$, as in Example 4.4 below.

Example 4.1. (Example 3 in [20]). Let X be the interval [0,1) with the Sorgenfrey topology. It has been shown in [23] that $C_p^*(X, [0, 1])$ is not normal and consequently $C_p^*(X, [0, 1])$ is not Lindelöf. Hence by Theorem 4.3, $C_p^*(X)$ is not Lindelöf. Since $C_p^*(X) \leq C_\sigma^*(X)$, $C_\sigma^*(X)$ is not Lindelöf either. Yet X is a Lindelöf space. But since X is first countable, X^n has countable tightness for each $n \in \mathbb{N}$. In particular, this example shows that the converse of Proposition 4.5 need not to be true. Also note that for this space X, $C_p^*(X) < C_k^*(X) < C_\sigma^*(X) = C_\infty^*(X)$. Note that X, being separable, is almost σ-compact, but not compact. So by Theorem 4.6(i), we can also immediately conclude that $C_\sigma^*(X)$ is not Lindelöf.

Example 4.2. The "double arrow" space X is first countable and compact, but it is not metrizable. In literature, this space is also called "two arrows" space. For details on this space, see exercise 3.10C, page 212 in [9]. Also see page 30 in [6]. Note for this space X, $C(X) = C^*(X)$ and $C_p^*(X) < C_k^*(X) = C_\sigma^*(X) = C_\infty^*(X)$. Since X is not metrizable, by Theorem 4.6(ii), $C_\sigma^*(X)$ is not Lindelöf. Here again, since X is first countable, X^n has countable tightness for each $n \in \mathbb{N}$. So again this example shows that the converse of Proposition 4.5 need not be true.

Example 4.3. Let X be the ordinal space $[0,\omega_1]$. Since X does not have countable tightness, by Proposition 4.5 , $C_\sigma^*(X)$ is not Lindelöf. Also by Theorem 4.6(ii), we can immediately conclude that $C_\sigma^*(X)$ is not Lindelöf. Note for this space X, $C(X) = C^*(X)$ and $C_p^*(X) < C_k^*(X) = C_\sigma^*(X) = C_\infty^*(X)$

Example 4.4. Let X be the ordinal space $[0, \omega_1)$. Since X is countably compact, $C(X) = C^*(X)$. Each σ-compact subset of X has compact closure, but X does not contain a dense σ-compact subset. Hence $C_p^*(X) < C_k^*(X) = C_\sigma^*(X) < C_\infty^*(X)$. But since X is first countable, X^n has countable tightness for each $n \in \mathbb{N}$. So Proposition 4.5 cannot be used to check if $C_k^*(X) = C_\sigma^*(X)$ is Lindelöf. But in [20], it has been shown that $C_k^*(X)$ is indeed Lindelöf. Here we reproduce the arguments given in [20] to justify that $C_k^*(X)$ is Lindelöf. An implication of Theorem 2 of [13] says that if X is an invariant subspace of a Σ-product of separable metric spaces, then $C_k(X)$ is Lindelöf. Since $[0, \omega_1)$ can be embedded as a closed subspace of the Σ-product of ω_1 copies of $\mathbb{R}$ by means of the diagonal product map of $\{f_\alpha\}$ where $f_\alpha(\beta) = 0$ if $\beta \leq \alpha$ and $f_\alpha(\beta) = 1$ if $\beta > \alpha$, it follows that $C_k^*([0, \omega_1)) = C_\sigma^*([0, \omega_1))$ is Lindelöf.

Example 4.5. (Example 25 in [26]) The Fortissimo space F, does not have countable tightness and consequently by Proposition 4.5, $C_\sigma^*(F)$ is not Lindelöf.

Example 4.6. (Example 4 in [20]) The Tychonoff plank T is defined to be $[0, \omega_1] \times [0, \omega_0]$ where both ordinal spaces are given the order topology. The subspace $T_\infty = T - \{(\omega_1, \omega_0)\}$ is called the deleted Tychonoff Plank. The space T_∞ is not normal, but it is pseudocompact. Hence $C(T_\infty) = C^*(T_\infty)$. For more details on T_∞, see Example 87 in [26].

It has been shown in [20] that $C_p(T_\infty) = C_p^*(T_\infty)$ is not Lindelöf. Hence $C_\sigma^*(T_\infty)$ is not Lindelöf either. We know that T_∞ is almost σ-compact. Since T_∞ is not normal, it is not compact. So by Theorem 4.6(i), we can also immediately conclude that $C_\sigma^*(T_\infty)$ is not Lindelöf.

References

1. Charalambos D. Aliprantis and Owen Burkinshaw, *Principles of real analysis*, third ed., Academic Press Inc., San Diego, CA, 1998.
2. R. Arens, *A topology for spaces of transformations*, Ann. of Math. **47** (1946), 480–495.
3. R. Arens and J. Dugundji, *Topologies for function spaces*, Pacific J. Math. **1** (1951), 5–31.
4. A. V. Arhangel′skiĭ, *Classes of topological groups*, Russian Math. Surveys **36** (1981), no. 3, 151–174.
5. ———, *On linear homeomorphisms of function spaces*, Soviet Math. Dokl. **25** (1982), 852–855.
6. ———, *C_p-theory*, Recent Progress in General Topology, North Holland (1992), 1–56.
7. ———, *Topological function spaces*, Mathematics and its Applications (Soviet Series), vol. 78, Kluwer Academic Publishers Group, Dordrecht, 1992, Translated from the Russian by R. A. M. Hoksbergen.
8. R. C. Buck, *Bounded continuous functions on a locally compact space*, Michigan Math. J. **5** (1958), 95–104.
9. R. Engelking, *General topology*, revised and completed ed., Heldermann Verlag, Berlin, 1989.
10. R. H. Fox, *On topologies for function spaces*, Bull. Amer. Math. Soc. **51** (1945), 429–432.
11. R. Giles, *A generalization of the strict topology*, Trans. Amer. Math. Soc. **161** (1971), 467–474.
12. D. Gulick, *The σ-compact-open topology and its relatives*, Math. Scand. **30** (1972), 159–176.
13. S. P. Gul'ko, *On properties of subsets of Σ-products*, Soviet Math. Dokl. **18** (1977), no. 6, 1438–1442.
14. I. L. Guran, *On embeddings of topological groups*, L'vorsk Univ., L'vov, 1981, (manuscript deposited at VINITI).
15. J. R. Jackson, *Comparison of topologies on function spaces*, Proc. Amer. Math. Soc. **3** (1952), 156–158.
16. S. Kundu, *$C_b(X)$ revisited: induced map and submetrizability*, Quaest. Math. **27** (2004), 47–56.
17. S. Kundu and R. A. McCoy, *Topologies between compact and uniform convergence on function spaces*, Internat. J. Math. Math. Sci. **16** (1993), 101–109.
18. S. Kundu, R. A. McCoy, and A. B. Raha, *Topologies between compact and uniform convergence on function spaces. II*, Real Anal. Exchange **18** (1992/93), 176–189.
19. S. Kundu and Vipra Pandey, *The metrizability and completeness of the σ-compact-open topology on $C^*(X)$*, Topology Appl. **159** (2012), no. 3, 593–602.
20. R. A. McCoy, *Necessary conditions for function spaces to be Lindelöf*, Glasnik Matematički **15(35)** (1980), 163–168.

21. R. A. McCoy and I. Ntantu, *Topological properties of spaces of continuous functions*, Lecture Notes in Mathematics, vol. **1315**, Springer-Verlag, Berlin, 1988.
22. E. Michael, $\aleph_0$-*spaces*, J. Math. Mech. **15** (1966), 983–1002.
23. R. Pol, *Normality in function spaces*, Fund. Math. **84** (1974), no. 2, 145–155.
24. J. Schemets, *Espaces de fonctions continues*, Lecture Notes in Math., vol. **519**, Springer-Verlag, New York, 1976.
25. F. D. Sentilles, *Bounded continuous functions on a completely regular space*, Trans. Amer. Math. Soc. **168** (1972), 311–336.
26. L. A. Steen and J. A. Seebach, Jr., *Counterexamples in topology*, second ed., Springer-Verlag, New York, 1978.
27. Seth Warner, *The topology of compact convergence on continuous function spaces*, Duke Math. J. **25** (1958), 265–282.
28. Robert F. Wheeler, *A survey of Baire measures and strict topologies*, Expo. Math. **2** (1983), 97–190.

S. Kundu: Department Of Mathematics, Indian Institute of Technology Delhi, New Delhi 110016, India.
E-mail address: skundu@maths.iitd.ernet.in

Vipra Pandey: Foreign Service Institute, Baba Gang Nath Marg, Old JNU Campus, New Delhi 110067, India.
E-mail address: viprapandey@gmail.com