TOPOLOGY
PROCEEDINGS
Volume 41, 2013
Pages 261–266
http://topology.auburn.edu/tp/

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Electronically published on September 11, 2012

Topology Proceedings

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E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/
TOPOLOGY PROCEEDINGS
Volume 41 (2013)
Pages 261-266
E-Published on September 11, 2012

THE GROUPS S^3 AND $\mathsf{SO}(3)$ HAVE NO INVARIANT BINARY k-NETWORK

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ABSTRACT. A family $\mathcal{N}$ of closed subsets of a topological space X is called a *closed k-network* if for each open set $U \subset X$ and a compact subset $K \subset U$ there is a finite subfamily $\mathcal{F} \subset \mathcal{N}$ with $K \subset \bigcup\mathcal{F} \subset U$. A compact space X is called *supercompact* if it admits a closed k-network $\mathcal{N}$ which is *binary* in the sense that each linked subfamily $\mathcal{L} \subset \mathcal{N}$ is centered. A closed k-network $\mathcal{N}$ in a topological group G is *invariant* if $xAy \in \mathcal{N}$ for each $A \in \mathcal{N}$ and $x, y \in G$. According to a result of Kubiś and Turek [3], each compact (abelian) topological group admits an (invariant) binary closed k-network. In this paper we prove that the compact topological groups S^3 and $\mathsf{SO}(3)$ admit no invariant binary closed k-network.

1. Introduction

In this note we shall discuss the problem of the existence of invariant binary k-networks for compact G-spaces and compact topological groups.

A family $\mathcal{A}$ of subsets of a set X is called

- *linked* if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{A}$;
- *centered* if $\cap\mathcal{F} \neq \emptyset$ for any finite subfamily $\mathcal{F} \subset \mathcal{A}$;
- *binary* if each linked subfamily of $\mathcal{F}$ is centered.

A family $\mathcal{A}$ of subsets of a topological space X is called a *k-network* if for any open set $U \subset X$ and a compact subset $K \subset U$ there is a finite subfamily $\mathcal{F} \subset \mathcal{A}$ with $K \subset \bigcup\mathcal{F} \subset U$, see [2, §11]. If each set $A \in \mathcal{A}$ of a k-network is closed in X, then $\mathcal{A}$ will be called a *closed k-network*.

2010 *Mathematics Subject Classification.* 54D30, 22C05.

Key words and phrases. Compact topological group, supercompact space, invariant binary k-network.

The first author has been partially financed by NCN means granted by decision DEC-2011/01/B/ST1/01439.

A compact space X is called *supercompact* if X admits a subbase of the topology such that each cover of X by elements of the subbase contains a two-element subcover, see [5]. The following useful characterization of the supercompactness can be derived from Lemma 3.1 of [3]:

Theorem 1. *A compact Hausdorff space X is supercompact if and only if X admits a binary closed k-network.*

In [4] C. Mills proved that each compact topological group G is supercompact, that is G admits a binary closed k-network $\mathcal{N}$. This result was reproved by Kubiś and Turek [3] who observed that for an abelian compact topological group G one can construct $\mathcal{N}$ so that it is *left-invariant* in the sense that $xA \in \mathcal{N}$ for each $A \in \mathcal{N}$ and $x \in G$. They also asked if such a left-invariant binary k-network can be constructed in each compact topological group.

It is natural to consider this problem in the more general context of G-spaces. By a G-space we understand a topological space X endowed with a continuous action $\alpha : G \times X \to X$ of a topological group G. A family $\mathcal{F}$ of subsets of a G-space X will be called *G-invariant* if $gF \in \mathcal{F}$ for each $F \in \mathcal{F}$ and each $g \in G$.

A compact G-space X will be called *G-supercompact* if X admits a G-invariant binary closed k-network.

Problem 1. *Which compact G-spaces are G-supercompact?*

We shall resolve this problem for the unit sphere $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ in the Euclidean space $\mathbb{R}^{n+1}$, endowed with the natural action of the group $\mathsf{SO}(n+1)$ (of orientation preserving linear isometries of $\mathbb{R}^{n+1}$).

Example 1.

(1) The 0-sphere $S^0 = \{-1, 1\}$ in $\mathbb{R}$ is $\mathsf{SO}(1)$-supercompact because the family $\mathcal{F}_0 = \{\{-1\}, \{1\}\}$ of singletons is an $\mathsf{SO}(1)$-invariant binary closed k-network for S^0.

(2) The 1-sphere S^1 is $\mathsf{SO}(2)$-supercompact because the family $\mathcal{F}_1$ of all closed connected subsets of diameter less than $\sqrt{3}$ in S^1 is an $\mathsf{SO}(2)$-invariant binary closed k-network for the circle S^1.

It turns out that S^0 and S^1 are the unique examples of $\mathsf{SO}(n+1)$-supercompact spheres S^n.

Theorem 2. *The unit sphere S^n in the Euclidean space $\mathbb{R}^{n+1}$ is $\mathsf{SO}(n+1)$-supercompact if and only if $n \leq 1$.*

This theorem will be proved in Section 2. Now we shall apply this theorem for finding an example of a compact topological group that admits no invariant binary closed k-network.

A family $\mathcal{F}$ of subsets of a group G will be called

- *left-invariant* (resp. *right-invariant*) if for each $F \in \mathcal{F}$ and $g \in G$ we get $gF \in \mathcal{F}$ (resp. $Fg \in \mathcal{F}$);
- *invariant* if $\mathcal{F}$ is both left-invariant and right-invariant.

It is well-known that the 3-dimensional sphere S^3 has the structure of a compact topological group. Namely, S^3 is a group with respect to the operation of multiplication of quaternions (with the unit norm). It is known [1, §4.1] that for each isometry $f \in \mathsf{SO}(4)$ of S^3 there are quaternions $a, b \in S^3$ such that $f(x) = axb$ for all $x \in S^3$. This implies that a family $\mathcal{F}$ of subsets of the group S^3 is invariant if and only if it is $\mathsf{SO}(4)$-invariant. Now we see that Theorem 2 implies:

Corollary 1. *The compact topological group S^3 admits no invariant binary closed k-network.*

It is known that the quotient group $S^3/\{-1,1\}$ of S^3 by the two-element subgroup $\{-1,1\}$ is isomorphic to the special orthogonal group $\mathsf{SO}(3)$. Using this fact, we can deduce from Corollary 1 the following:

Corollary 2. *The compact topological group $\mathsf{SO}(3)$ admits no invariant binary closed k-network.*

Problem 2. *Does the group S^3 or $\mathsf{SO}(3)$ have a left-invariant binary k-network?*

Problem 3. *Let G be a compact abelian group and X be a compact metrizable G-space. Is X G-supercompact?*

Problem 4. *Let G be a metrizable (separable) abelian topological group. Does G have an invariant binary closed k-network?*

2. Proof of Theorem 2

First we fix some notation. By $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n x_i y_i$ we denote the standard inner product of the Euclidean space $\mathbb{R}^n$. This inner product generates the norm $\|\mathbf{x}\| = \langle \mathbf{x}, \mathbf{x} \rangle$. By $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ we shall denote the unit sphere in $\mathbb{R}^{n+1}$.

For an Euclidean space $E = \mathbb{R}^n$ let E^* be the dual space of E, i.e., the space of linear functionals on E endowed with the sup-norm. By Riesz's Representation Theorem, for each functional $y^* \in E^*$ there is a unique vector $y \in E$ such that $y^*(x) = \langle y, x \rangle$. So we can identify E^* with E.

A *convex body* in an Euclidean space E is a convex subset $C \subset E$ with non-empty interior in E. By ∂C we denote the boundary of C in E.

A functional $y^* \in E^*$ will be called a *support functional* to C at a point $c \in \partial C$ if

$$y^*(c) = \max x^*(C) > \inf y^*(C).$$

By the Hahn-Banach Theorem, each point $c \in \partial C$ of a convex body $C \subset E$ has a support functional y^* with unit norm. If such a support functional is unique, then c is called a *smooth point* of ∂C. It follows from the classical Mazur's Theorem [6, 1.20] on the differentiablity of continuous convex functions on E that the set of smooth points is dense in ∂C.

In an obvious way Theorem 2 follows from Example 1 and the following theorem:

Theorem 3. *For any $n \geq 2$ and any closed subset $A \subset S^n$ of diameter $0 < \operatorname{diam}(A) \leq 1$ there is an isometry $f \in \mathsf{SO}(n+1)$ such that the family $\{A, f(A), f^2(A)\}$ is linked but not centered.*

Proof. Let $E = \mathbb{R}^{n+1}$ and E^* be the dual space to E. By S^* we denote the unit sphere in E^*. □

Lemma 1. *There are distinct points $a_0, a_1 \in A$ and a vector $b \in S^*$ such that $\langle b, a_0\rangle = 0 = \max_{a\in A}\langle b, a\rangle$ and $\langle b, a_1\rangle > -\frac{1}{2}\|a_1 - a_0\|$.*

Proof. The lemma trivially holds if there are a vector $b \in S^*$ and two distinct points $a_0, a_1 \in A$ such that $\langle b, a_0\rangle = \langle b, a_1\rangle = \max_{a\in A}\langle b, a\rangle = 0$.

So, assume that no such vectors b, a_0, a_1 exist. Let L_A be the linear hull of the set A and $C \subset L_A$ be the closed convex hull of the set $A \cup \{0\}$ in L_A. Since the set $A \subset S^n$ contains more than one point, the linear space L_A has dimension $\dim L_A \geq 2$. It is clear that C is a convex body in L_A. By Mazur's Theorem [6, 1.20] the set of smooth points is dense in the boundary ∂C. Consequently, there is a smooth point $c \in \partial C$ such that $0 < \|c\| < 1$. Let $b^* \in L_A^*$ be the unique norm one support functional to C at the point c. Let $a_0 = \dfrac{c}{\|c\|}$ and observe that $a_0 \in \operatorname{conv}(A) \subset C$. Since b^* is a support functional at c, we get $b^*(c) = \max b^*(C) \geq b^*(0) = 0$. We claim that $b^*(c) = 0$. The strict inequality $b^*(c) > 0$ would imply $b^*(c) > 0 = \max b^*(C) \geq b^*(a_0) = \dfrac{b^*(c)}{\|c\|}$ and $\|c\| > 1$, which contradicts the choice of c.

Let us show that the point $a_0 = c/\|c\|$ belongs to the set A. Since $c \in \operatorname{conv}(A \cup \{0\}) \setminus \{0\}$ by the Caratheodory Theorem, there are pairwise distinct points $a_1, \dots, a_k \in A$ and positive real numbers $\lambda_1, \dots, \lambda_k$ such that $\sum_{i=1}^k \lambda_i \leq 1$ and $c = \sum_{i=1}^k \lambda_i a_i$. This equality and $b^*(c) = 0 = \max b^*(A)$ imply that $b^*(a_i) = 0$ for all $1 \leq i \leq k$. Now our assumption guarantees that $k = 1$ (otherwise, a_1 and a_2 are two distinct points with $b^*(a_1) = b^*(a_2) = \max b^*(A) = 0$, which is forbidden by our assumption). Therefore, $c = \lambda_1 a_1$ and hence $a_0 = c/\|c\| = c/\lambda_1 = a_1 \in A$.

Let $c^* \in L_A^*$ be any functional with the unit norm such that $c^*(a_0) = 0$ and $0 < \|b^* - c^*\| \le \frac{1}{2}$. Since the functional $c^* \ne b^*$ is not support at the point c, there is a point $a_1 \in A$ such that $c^*(a_1) > 0$.

Observe that

$$b^*(a_1) = b^*(a_1 - a_0) \ge c^*(a_1 - a_0) - \|c^* - b^*\| \cdot \|a_1 - a_0\| = \\ = c^*(a_1) - \frac{1}{2}\|a_1 - a_0\| > -\frac{1}{2}\|a_1 - a_0\| .$$

By Riesz's Representation Theorem, the functional b^* can be identified with a unique vector $b \in L_A \subset E$ such that $b^*(x) = \langle b, x\rangle$ for all $x \in L$. The vector b and the points a_0, a_1 have the properties required in Lemma 1. □

Let L be the 3-dimensional linear subspace of E generated by the vectors b, a_0, a_1 (from Lemma 1) and let $L^\perp \subset E$ be its orthogonal complement. Then the space E decomposes into the direct sum $L \oplus L^\perp$.

Find a (unique) point a_2 in the 2-sphere $L \cap S^n$ such that $\|a_2 - a_0\| = \|a_2 - a_1\| = \|a_1 - a_0\|$ and $\langle b, a_2\rangle > 0$. Let $c = \frac{1}{3}(a_0 + a_1 + a_2)$ be the center of the equilateral triangle $\triangle a_0, a_1, a_2$. It follows from $\langle b, a_0\rangle = 0$ and $0 \ge \langle b, a_1\rangle > -\frac{1}{2}\|a_0 - a_1\|$ that $\langle b, a_2\rangle > \frac{1}{2}\|a_0 - a_1\|$. Consequently,

$(*)$ $\langle b, c\rangle = \frac{1}{3}(\langle b, a_1\rangle + \langle b, a_2\rangle) > 0.$

Claim 1. *$\langle c, a\rangle > 0$ for each $a \in A$.*

Proof. Observe that $\langle a_2, a_0\rangle = \frac{1}{2}(\|a_2\|^2 + \|a_0\|^2 - \|a_2 - a_0\|^2) \ge \frac{1}{2}(1 + 1 - 1) = \frac{1}{2}$ and then $\|-a_2 - a_0\|^2 = \|a_2\|^2 + \|a_0\|^2 + 2\langle a_2, a_0\rangle \ge 3$, which implies that $-a_2 \notin A$ because $\operatorname{diam}(A) \le 1$. Then for each $a \in A$ we get $a_2 \ne -a$ and hence $\langle a_2, a\rangle > -\|a_2\| \cdot \|a\| = -1$.

On the other hand, for $i \in \{0, 1\}$ we get

$$\langle a_i, a\rangle = \frac{1}{2}(\|a_i\|^2 + \|a\|^2 - \|a_0 - a\|^2) \ge \frac{1}{2}(1 + 1 - 1) = \frac{1}{2}.$$

Then
$\langle c, a\rangle = \frac{1}{3}\langle a_0 + a_1 + a_2, a\rangle = \frac{1}{3}(\langle a_0, a\rangle + \langle a_1, a\rangle + \langle a_2, a\rangle) > \frac{1}{3}(\frac{1}{2} + \frac{1}{2} - 1) = 0.$ □

Let $R : L \to L$ be the rotation of the 3-dimensional Euclidean space L around the axis $\mathbb{R}c$ on the angle $2\pi/3$ such that $R(a_0) = a_1$, $R(a_1) = a_2$ and $R(a_2) = a_0$. Extend R to an isometry $f \in \mathsf{SO}(n+1)$ of $E = L \oplus L^\perp$ letting $f(x + y) = R(x) + y$ for $(x, y) \in L \times L^\perp$. It remains to prove:

Claim 2. *The system $\mathcal{L} = \{A, f(A), f^2(A)\}$ is linked but not centered.*

Proof. The linkedness of the system $\mathcal{L}$ follows from the inclusion $\{a_0, a_1\} \subset A$ and the linkedness of the system

$$\big\{\{a_0, a_1\}, \{a_1, a_2\}, \{a_2, a_0\}\big\} = \big\{\{a_0, a_1\}, f(\{a_0, a_1\}), f^2(\{a_0, a_1\})\big\}.$$

To see that $\mathcal{L}$ is not centered, consider the half-spaces $H_b = \{x \in E : \langle b, x\rangle \le 0\}$ and $H_c = \{x \in E : \langle c, x\rangle > 0\}$. The choice of the vectors b, a_0, a_1, a_2 guarantees that $a_0, a_1 \in A \subset H_b$ but $a_2, c \notin H_b$. By Claim 1, $A \subset H_c$.

Let $H_c^L = H_c \cap L$ and $H_b^L = H_b \cap L$. The inclusions $b, c \in L$ imply that $H_b = H_b^L \oplus L^\perp$ and $H_c = H_c^L \oplus L^\perp$.

It follows that $R(H_c^L) = H_c^L$ and hence $f(H_c) = H_c$. Observe that

$$A \cap f(A) \cap f^2(A) \subset H_c \cap H_b \cap f(H_b) \cap f^2(H_b) = (H_c^L \cap H_b^L \cap R(H_b^L) \cap R^2(H_b^L)) \oplus L^\perp.$$

Now to see that $A \cap f(A) \cap f^2(A) = \emptyset$ it suffices to prove that the intersection $H^L = H_c^L \cap H_b^L \cap R(H_b^L) \cap R^2(H_b^L)$ is empty. Assuming that this intersection contains some point h, we conclude that it contains its rotations $R(h)$ and $R^2(h)$ and also the center $c_h = \frac{1}{3}(h + R(h) + R^2(h))$ of the equilateral triangle $\{h, R(h), R^2(h)\}$ (by the convexity of H^L). The center c_h lies on the axis $\mathbb{R} \cdot c$ of the rotation R. Taking into account that $c_h \in H_c$, we conclude that $\langle c, c_h\rangle > 0$ and hence $c \in (0, +\infty) \cdot c_h \subset H_b$, which contradicts the inequality $(*)$. □

References

[1] J. Conway, D. Smith, *On quaternions and octonions: their geometry, arithmetic, and symmetry*, A K Peters, Ltd., Natick, MA, 2003.

[2] G. Gruenhage, *Generalized metric spaces*, in: Handbook of set-theoretic topology (K.Kunen, J.Vaughan, Eds.), 423–501, North-Holland, Amsterdam, 1984.

[3] W.Kubiś, S.Turek, *A decomposition theorem for compact groups with application to supercompactness*, Central European J. Math. **9**:3 (2011), 593–602.

[4] C. Mills, *Compact groups are supercompact*, seminar report from the Faculty of Mathematics, Free University of Amsterdam, September 1978.

[5] J. van Mill, *Supercompactness and Wallman spaces*, Math. Centre Tracts, No. 85. Math. Centrum, Amsterdam, 1977.

[6] R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, Springer-Verlag, Berlin, 1993.

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