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A NOTE ON THE MAIN INCLUSION THEOREM OF LUXEMBURG AND ZAANEN

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Abstract. We give a more elementary proof of the main statement of the proof of the main result in [8]. Following the idea of [8], we re-prove a result of Luxemburg and Zaanen, that is, an Archimedean Riesz space is Dedekind complete if and only if it is uniformly complete and has the band projection property.

1. Introduction

For the standard definition and terminology of Riesz space theory, we refer to [3], [4], or [9]. The Riesz space of real-valued continuous functions on a topological space is denoted by $C(X)$. A topological space X is called *extremally disconnected* if the closure of every open subset of X is also open; X is said to be *basically disconnected* if the closure of $\text{cozero}(f)$ is open for each $f \in C(X)$, where

$$\text{zero}(f) := f^{-1}(\{0\}) \qquad \text{and} \qquad \text{cozero}(f) := X \setminus \text{zero}(f).$$

One of the important theorems in Riesz space theory is known as the Main Inclusion Theorem, which is given in [4, pp. 137 & 174]. The present paper deals with a part of the Main Inclusion Theorem, following the idea of [8].

The following theorem, whose proof can be found in [1, Theorem 1.50], is due to Nakano (cf. [5]).

Theorem 1.1 (Nakano). *Let X be a topological space.*

(i) *If X is extremally disconnected, then the Riesz space $C(X)$ is Dedekind complete.*
(ii) *If $C(X)$ is Dedekind complete and X is completely regular, then X is extremally disconnected.*

A similar result to the above theorem can be given as follows (for a proof, see [7, Theorem 2.1.5]).

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Theorem 1.2. *Let X be a topological space.*

(i) *If X is basically disconnected, then the Riesz space $C(X)$ is Dedekind σ-complete.*

(ii) *If $C(X)$ is Dedekind σ-complete and X is completely regular, then X is basically disconnected.*

In Riesz space theory, as a part of the main inclusion theorem (see [1] or [4]), one of the most important theorems is the following: an Archimedean Riesz space E is Dedekind σ-complete if and only if it is uniformly complete and has the band projection property. This result is due to Luxemburg and Zaanen and their proof depends on Freudenthal's spectral theorem. An elementary proof of this fact without using Freudenthal spectral theory is given in [6] (see also [2]). Yet another proof of it is given in [8] as the main result.

2. Main Results

For a subset D of a Riesz space E, the *disjoint complement*, D^d, of D is defined by

$$D^d = \{x \in E \mid x \perp y \text{ for all } y \in D\}.$$

The interior of a subset A of a subset of a topological space X is denoted by A°. A proof of the following fact for completely regular spaces can be found in [9, p. 41].

Theorem 2.1. *Let X be a topological space and A be a subset of $C(X)$. Then*

$$A^d = \{f \in C(X) \mid f(X \setminus N^\circ) = 0\},$$

where

$$N = \bigcap_{f \in A} zero(f).$$

Proof. If $g \in C(X)$ is such that $g(X \setminus N^\circ) = 0$, then, clearly, one has $g \in A^d$. Now suppose that $0 \leq f \in A^d$. Let $x \in X \setminus N^\circ$. Suppose further that $0 < f(x)$. Since $x \notin N^\circ$, we have

$$f^{-1}\left(f(x) - \frac{1}{n}, f(x) + \frac{1}{n}\right) \nsubseteq N$$

for each n, where $f(x) - 1/n > 0$ (as $f(x) > 0$, such an n exists). Then there exists $g \in A$ such that $g(x) \neq 0$, so that $|f(x)| \wedge |g(x)| > 0$. This contradicts f being in A^d, and since A^d is an ideal, the proof is complete. □

The following fact is one of the most important theorems in Riesz space theory.

Theorem 2.2. *A Riesz space E is Dedekind σ-complete if and only if it is uniformly complete and has the band projection property.*

Theorem 2.2 is due to Luxemburg and Zaanen, and a proof of it can be found in [4] (cf. [2] and [6] for different proofs). Another proof of this result, without using Freudenthal's theorem, is given by M.A. Toumi in [8] as the main result. The main argument of [8] is the following: if X is a compact Hausdorff space and $C(X)$ has the principal band projection property, then X is basically disconnected. An easy proof of this statement for completely regular spaces will now be given. Before proceeding, note first that if $U \subset X$ is open, then

$$\{f \in C(X) \mid f(U) = 0\}^d = \{f \in C(X) \mid f(X \setminus \overline{U}) = 0\}.$$

Theorem 2.3. *Let X be a completely regular Hausdorff space. Then, the following satements are equivalent:*

(i) *The space $C(X)$ has the principal band projection property.*
(ii) *The space X is basically disconnected.*
(iii) *The space $C(X)$ is Dedekind σ-complete.*

Proof. We will first prove that (i) implies (ii). Let $f \in C(X)$ be given, and let $K := \text{zero}(f)$ and $U := \text{cozero}(f)$. Then, one has

$$\{f\}^d = \{g \in C(X) \mid g(\overline{U}) = 0\} \quad \text{and} \quad \{f\}^{dd} = \{g \in C(X) \mid g(\overline{K^\circ}) = 0\}.$$

Since $C(X)$ has the principal band projection property, one also has

$$1 = g\chi_{X\setminus\overline{U}} + h\chi_{X\setminus\overline{K^\circ}} = g\chi_{X\setminus\overline{U}} + h\chi_{\overline{U}^\circ}$$

for some f in $C(X)$. As

$$(X \setminus \overline{U}) \cap \overline{U}^\circ = \emptyset \quad \text{and} \quad (X \setminus \overline{U}) \cup \overline{U}^\circ = X,$$

it follows that $\overline{U}$ is closed-and-open, whence X is basically disconnected.

That (ii) implies (iii) follows from Theorem 1.2, and it is routine to see that (iii) implies (i). □

Now, following the idea of [8], we can give an elementary proof of the following fact, a part of the Main Inclusion Theorem, which is again due to Luxemburg and Zaanen (cf. [4]).

Theorem 2.4. *A Riesz space E is Dedekind complete if and only if it is uniformly complete and has the band projection property.*

Proof. If E is Dedekind complete, then, using simple arguments, it can be shown that E is uniformly complete and band projection propery. Conversely, suppose that E is uniformly complete and it has the band projection property. To see that E is Dedekind complete, it is enough to show that the ideal generated by $e \in E^+$ is Dedekind complete for every e.

Let $0 < e \in E$ be given. Then A_e has the band projection property (see [9, Theorem 12.4]). Since E is uniformly complete, the Kakutani Representation Theorem yields that, for some compact Hausdorff space X, the spaces A_e and $C(X)$ are Riesz isomorphic. By Theorem 1.1, then, it is enough to show that X is extremally disconnected. Let $U \subset X$ be an open set and let

$$B := \{f \in C(X) \mid f(\overline{U}) = 0\}.$$

Then B is a band of $C(X)$ and

$$B^d = \{f \in C(X) : f(\overline{X \setminus \overline{U}}) = 0\}.$$

Since $C(X)$ has the band projection property, one has $C(X) = B \oplus B^d$, which implies that

$$1 = f\chi_{X\setminus\overline{U}} + g\chi_{X\setminus\overline{X\setminus\overline{U}}} = f\chi_{X\setminus\overline{U}} + g\chi_{(\overline{U})^\circ}.$$

Clearly

$$(X \setminus \overline{U}) \cup (\overline{U})^\circ \quad \text{and} \quad (X \setminus \overline{U}) \cap (\overline{U})^\circ = X.$$

This yields $(\overline{U})^\circ = \overline{U}$, whence $\overline{U}$ is open. It then follows that X is extremally disconnected, which shows that $C(X)$ is Dedekind complete by Theorem 1.1. □

References

[1] C.D. Aliprantis & O. Burkinshaw, *Locally Solid Riesz Spaces with Applications to Economics*, 2nd ed., American Mathematical Society, Mathematical Surveys and Monographs, Vol. 105, Providence, RI, 2003.

[2] B.C. Anderson & H. Nakano, *Semi-continuous linear lattices*, Studia Math., **37** (1970/71), 191–195.

[3] E. de Jonge and A.C.M. van Rooij, *Introduction to Riesz spaces*, Mathematical Centre Tracts, No. 78. Mathematisch Centrum, Amsterdam, 1977.

[4] W.A.J. Luxemburg & A.C. Zaanen, *Riesz Spaces*, Vol. 1, North-Holland Publishing Co., Amsterdam, 1971.

[5] H. Nakano, *Uber das System aller stetigen Funktionen auf einem topologischen Raum*, Proc. Imp. Acad. Tokyo, **17** (1941), 308–310.

[6] H. Nakano, *On semicontinuous linear lattices*, Proc. Amer. Math. Soc., **34** (1972), 115–117.

[7] P. Meyer-Nieberg, *Banach Lattices*, Springer-Verlag, Berlin, Heidelberg, 1991.

[8] M.A. Toumi, *A simple proof for a theorem of Luxemburg and Zaanen*, J. Math. Anal. Appl., **322** (2006), no. 2, 1231–1234.

[9] A.C. Zaanen, *Introduction to Operator Theory in Riesz Spaces*, Springer-Verlag, Berlin, Heidelberg, 1997.

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