TOPOLOGY PROCEEDINGS
Volume 41, 2013
Pages 337–360

http://topology.auburn.edu/tp/

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Electronically published on November 28, 2012

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/

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Volume 41 (2013)
Pages 337-360

E-Published on November 28, 2012

ON CUT POINTS PROPERTIES OF SUBCLASSES OF CONNECTED SPACES

DEVENDER KUMAR KAMBOJ AND VINOD KUMAR

Abstract. We investigate, from the view point of cut points, properties of some subclasses of connected spaces. We prove, without assuming any separation axiom, if a connected space X has (*) i.e., X has a closed R(i) subset H such that there is no proper regular closed, connected subset of X containing H, then there is no proper connected subset of X containing all non-cut points. This is used to show that a connected space having at most two non-cut points and (*) is a COTS with endpoints; also the converse holds. If, in addition, the space is locally connected, then it is compact. In an R(i) connected space, each component of the complement of a cut point is found to contain a non-cut point of the space. For an R(i) connected space X, it is also shown that if the removal of every two-point disconnected set leaves the space disconnected, then, for a, b ∈ X and a separating set H of X − {a,b}, H∪{a,b} is a $T_{\frac{1}{2}}$ R(i) COTS with endpoints. Also we obtain some other characterizations of COTS with endpoints, and some characterizations of the closed unit interval.

1. Introduction

The idea of the concept of a cut point in a topological space dates back to 1920's ([see 14, 15]). One of the reasons that the theory of cut points in topological spaces has been gaining importance is because it has found applications in computer science (see e.g. [8]). For the study of cut points, a topological space is assumed to be connected. Herein, by a space we mean a topological space. A point x of a connected space X is a cut point if X − {x} is disconnected. So far, for the study of cut points, the space is assumed to be nondegenerate i.e., has at least two points. If a space contains only two points, then both points are non-cut points. We suppose that a space has at least three points.

2010 *Mathematics Subject Classification.* 54F05, 54F15.

Key words and phrases. Cut point, Non-cut point existence theorem, R(i) connected space, closed point, COTS.

The set of the integers Z, equipped with the topology whose base is $\{\{2n-1, 2n, 2n+1\} : n \in Z\} \cup \{\{2n+1\} : n \in Z\}$, is the Khalimsky line ([1]). While studying cut points in spaces it is important to know about non-cut points of the space. There are many connected spaces like the real line, the Khalimsky line etc., where the removal of any one point of the space leaves the subspace disconnected, i.e., such spaces have no non-cut points. There is a concept of COTS introduced by E. D. Khalimsky, R. Kopperman, P. R. Meyer in [8]. In a COTS, by Proposition 2.5 of [8], there are at most two non-cut points, which turn out to be end points, and every other point is a cut point.

If there exist at least two non-cut points in a connected space, it is said that the non-cut point existence theorem holds for the space. One of the questions in the study of cut point properties of connected spaces is finding subclasses of connected spaces where the non-cut point existence theorem holds for every member of the subclass. As seen above, assuming the space to be just connected is not sufficient for the non-cut point existence theorem to hold for the space. To find such a subclass there is a need for its members to have another property. One way is "enriching connectedness" as is done e.g., in [12], [2] and [5]. Swingle in [12] defines a connected space to be a widely connected space if every nondegenerate connected subset intersects every nonempty open subset. It can be proved using Problem 1Q of [7] and Remark 2.3 (below) that a widely connected space has only open cut points and it has at most one cut point. The concept of extremally connected (E.C.), (if the closure of every two nonempty open subsets intersect) appears in [2]. The class of regularly closed super (r.c.s.) connected spaces, i.e., spaces which are connected and every two nonempty regular closed connected subsets intersect ([5]), is larger than that of widely connected spaces. The non-cut point existence theorem is obtained in [5] for E.C. spaces and r.c.s. conected spaces. Another way is that members of the class have some "compact-like" property defined using open covers and/or open filters in some form. For the subclass of connected spaces consisting of compact Hausdorff spaces, the non-cut point existence theorem was proved by R. L. Moore in 1920's (See [13]). G. T. Whyburn [15] proved the non-cut point existence theorem for the class of T_1 compact connected spaces. In view of the fact that many connected spaces that play an important role in the study and applications of cut points are not T_1, the assumption of separation axioms is avoided as much as possible. The concept of H(i) introduced by C.T. Scarborough and A.H. Stone in [9] (see also [11]) is weaker than compactness and no separation axiom is assumed. The non-cut point existence theorem, strengthening Theorem 3.8 of B. Honari and Y. Bahrampour [1], is proved by D. K. Kamboj and V. Kumar in [3] for H(i) connected spaces. The non-cut point existence theorem is proved by D. K. Kamboj and V. Kumar in [4] for connected spaces having only finitely many closed points. Another concept that of R(i), weaker than H(i), is introduced by C.T. Scarborough and A.H. Stone in [9]. R. M. Stephenson, Jr. in [10] proved the non-cut point existence theorem for two subclasses of T_1 connected spaces, one of which consists of R(i) spaces.

In this paper, we investigate some properties of some subclasses of connected spaces from the view point of cut points. The class of R(i) connected spaces is one of them and no separation axiom is assumed. Notation and definitions are given in Section 2. The main results of the paper appear in Sections 3, 4 and 5. In Section 3, we prove without assuming any separation axiom, that if a connected space X has a closed R(i) subset H such that there is no proper regular closed, connected subset of X containing H, then there is no proper connected subset of X containing all non-cut points; therefore such a space has at least two non-cut points i.e., the non-cut point existence theorem holds for the class of these spaces. This strengthens Theorems 2.3 and 2.5 of R. M. Stephenson, Jr. [10], and Theorem 3.4 of D. K. Kamboj and V. Kumar [3]. Further we prove that if a connected space X having at most two non-cut points has a closed R(i) subset H such that there is no proper regular closed, connected subset of X containing H, then X is a COTS with endpoints; also the converse holds. If in addition, the space is locally connected, then it is compact. In an R(i) connected space, each component of the complement of a cut point is found to contain a non-cut point of the space. It is shown that if X is an R(i) connected space such that the removal of any two-point disconnected set leaves the space disconnected, then, for a, b $\in$ X and a separating set H of X $-$ {a,b}, H$\cup${a,b} is a $T_{\frac{1}{2}}$ R(i) COTS with endpoints. Also we obtain some other characterizations of COTS with endpoints and some characterizations of the closed unit interval.

In Section 4, we prove that if X is a non-indiscrete connected space with at most two non-cut points such that the set of all closed points of X is finite, then X is homeomorphic to a finite subspace of the Khalimsky line. Using this result, it is shown that if X is a non-indiscrete connected space with at most two non-cut points such that every regular open filter base on X is finite, then X is homeomorphic to a finite subspace of the Khalimsky line. In Section 5, we obtain results about members of a subclass of connected spaces which contains the class of R(i) connected spaces, and the class β considered in [10].

2. Notation, Definitions and Preliminaries

For completeness, we have included some of the standard notation and definitions. Let X be a space. For H ⊂ Y ⊂ X, $\mathbf{cl}_Y\mathbf{(H)}$ and $\mathbf{int}_Y\mathbf{(H)}$ denote the closure of H and interior of H in Y respectively. If $A|B$ form a separation of X, then we say that each one of A and B is a **separating set** of X. X is called $T_{\frac{1}{2}}$ ([8]) if every singleton set is either open or closed. A point x ∈ X is a **closed point** (**open point**) of X ([1]) if {x} is closed (open). A point x ∈ X called a **cut point** if there exists a separation of X − {x}. **ctX** is used to denote the set of all cut points of X. Let x ∈ ctX. A separation $A|B$ of X − {x} is denoted by $A_x|B_x$ if the dependence of the separation on x is to be specified. A_x^* is used for the set A_x∪{x}. For x ∈ ctX and a separation $A_x|B_x$ of X − {x}, if Y is a connected subset of X − {x}, then A_x(Y) or B_x(Y) and respectively A_x(−Y) or B_x(−Y) are used to denote the separating subset of X − {x} containing Y and not containing Y. A_x^*(Y) is used for the set A_x(Y)∪{x}. If Y = {a}, we write A_x(a) for A_x(Y), A_x(−a) for A_x(−Y), A_x^*(a) for A_x^*(Y) and A_x^*(−a) for A_x^*(−Y). A connected space with X = ctX is called a **cut point space** ([1]). Let a, b ∈ X. A point x ∈ ctX − {a,b}, is said to be a **separating point** between a and b or x **separates** a and b if there exists a separation $A_x|B_x$ of X − {x}, with a ∈ A_x and b ∈ B_x. **S(a,b)** is used to denote the set of all separating points between a and b. For x ∈ S(a,b), we shall write X − {x} = A_x(a)∪B_x(b) for a separation $A_x|B_x$ of X − {x}. If we adjoin the points a and b to S(a,b), then the new set is denoted by **S[a,b]**. A space X is called a **space with endpoints** if there exist a and b in X such that X = S[a,b].

A space X is called **H(i)** ([9]) if every open cover of X has a finite subcollection such that the closures of the members of that subcollection cover X. A filter base γ on a space X is said to be (i) trivial if γ contains only one member, (ii) fixed if $\bigcap\{A : A \in \gamma\} \neq \phi$, (iii) **o-filter base** if its members are open subsets of X. An o-filter base γ is **regular o-filter base** if each member of γ contains the closure of some member of γ. A space X is called **R(i)** ([9]) if every regular o-filter base on X is fixed.

A topological property P is said to be **cut point hereditary** ([6]) if whenever a space X has property P, then, for every x ∈ ctX there exists some separation $A_x|B_x$ of X − {x} such that each of A_x^* and B_x^* has property P. By the definition of a cut point x of a space X, there exists a separation $A_x|B_x$ of X − {x}. But the separation need not be unique. To obtain a unique separation, a condition on separating sets of X − {x} needs to be imposed. That gives rise to a concept stronger than that of a cut point, referred to as strong cut point in [8].

A cut point x of a space X is called a **strong cut point** if there exist separating sets of X − {x} that are connected, or are components of X − {x}. A space X is called **connected ordered topological space** (**COTS**) ([8]) if it is connected and has the property: if Y is a three-point subset of X, then there is a point x in Y such that Y meets two connected components of X − {x}.

In the following lemma, since we start with any space and any point of it, its results can be used elsewhere also. Theorem 3.2 of [1] follows as its corollary. It improves a part of Lemma 3.1 of [3].

Lemma 2.1. *Let X be a space, $x \in X$ and $H \subset X - \{x\}$. Then*
(a) $cl_X(H) \in \{cl_{X-\{x\}}(H),\ cl_{X-\{x\}}(H)\cup\{x\}\}$. (a) $int_X(H\cup\{x\}) \in \{int_{X-\{x\}}(H)\cup\{x\},\ int_{X-\{x\}}(H)\}$ (b) $cl_X(H) = cl_{X-\{x\}}(H)\cup\{x\}$ iff $x \in cl_X(H)$. (c) $cl_X(H) = cl_{X-\{x\}}(H)$ iff $x \in cl_X(H)$. (d) If $\{x\}$ is open in X, then $cl_X(H) = cl_{X-\{x\}}(H)$. (e) If $K \subset X - \{x\}$, then $cl_X(H)\cap cl_X(K) \subset (cl_{X-\{x\}}(H)\cap cl_{X-\{x\}}(K))\cup\{x\}$. (f) If H is closed in X − {x}, then (i) either H or $H\cup\{x\}$ is closed in X; (ii) either $x \in cl_X(H)$ or H is closed in X. (g) Let H be closed in X − {x} and {x} closed in X. Then $H\cup\{x\}$ is closed in X.*

Proof. Since $cl_{X-\{x\}}(H) = cl_X(H)\cap(X - \{x\})$, $cl_{X-\{x\}}(H) \subset cl_X(H) \subset cl_{X-\{x\}}(H)\cup\{x\}$, so (a) follows. To prove (a*), apply (a) to (X − {x}) - H. (b), (c) and (d) follow from (a). (e) holds as $cl_X(H) \subset cl_{X-\{x\}}(H)\cup\{x\}$. To prove (i) and (ii) of (f), use (a). (g) is a consequence of (i) of (f). □

Lemma 2.2. *Let X be a space, $x \in X$, $H \subset X - \{x\}$, and $K = (X - \{x\})$ - H. Let H be clopen in X − {x}. Then (a) Either H or $H\cup\{x\}$ is open in X. (b) {x} is open in X iff H and K are closed in X. (c) If X is connected, then exactly one of H and $K\cup\{x\}$ is open in X. (d) If X is connected and H is closed in X, then $H\cup\{x\}$ is open in X. (e) If X is connected, then either H and K are closed in X, or $H\cup\{x\}$ and $K\cup\{x\}$ are closed in X. (f) If $x \in cl_X(H)\cap cl_X(K)$, then {x}, $H\cup\{x\}$ and $K\cup\{x\}$ are closed in X. (g) If X is connected and {x} is closed in X, then H and K are open in X and $x \in cl_X(H)\cap cl_X(K)$.*

Proof. (a) Either H or H∪{x} is open in X, as, by Lemma 2.1(f)(i), either K or K∪{x} is closed in X. (b) follows using Lemma 2.1(d). (c) holds using (a). For (d), use (a). To prove (e), we use (c). If H is open, by Lemma 2.1(f)(i), H∪{x} is closed, so K is open, now K∪{x} is closed, by Lemma 2.1(f)(i); otherwise, since, K∪{x} is open, H is closed, and by (d), K is closed. (f) By Lemma 2.1(e), $cl_X(H)\cap cl_X(K) \subset \{x\}$, so {x} is closed in X, now, by Lemma 2.1(g), H∪{x} and K∪{x} are closed in X. (g) Since by Lemma 2.1(g), H∪{x} and K∪{x} are closed in X, H and K are open in X. $x \in cl_X(H)\cap cl_X(K)$, by Lemma 2.1(a). □

Remark 2.3. If x is a cut point of a connected space X, then there exists a clopen set H in X – {x}. Using Lemma 2.2(e), {x} is either open or closed in X. If {x} is open in X, then, by (d) of Lemma 2.1, H and (X – {x}) – H are closed in X. If {x} is closed in X, then, by (g) of Lemma 2.1, H and (X – {x}) – H are open in X. Thus Theorem 3.2 of [1] follows. By (a*) of Lemma 2.1, H $\subset$ int_X(H$\cup${x}). There follows Lemma 3.1 of [3].

The following result is stated using the notation used here. The first part which finds mention (with proof) at many places (e.g. Lemma 3.4, [1]) is a particular case of Problem 1Q of [7]; second part is a slight modification of Lemma 2.3(a) of [8], (i) and (ii) of second part easily follow from the first part, (iii) of second part follows from (i) and (ii).

Remark 2.4. (a) If X is connected, x $\in$ X and H is clopen in X – {x}, then H$\cup${x} is connected. (b) For a connected space X and x, y and z different points of ctX, let A_x, A_y and A_z be separating sets of X – {x}, X – {y} and X – {z} respectively. Then (i) A_y^*(–x) $\subset$ A_x iff y $\in$ A_x; (ii) A_x^* $\subset$ A_y(x) iff y $\notin$ A_x. (iii) x $\in$ A_z and y $\notin$ A_z iff A_x^*(–z) $\subset$ A_z $\subset$ A_y(z).

Remark 2.5. By Proposition 2.5 of [8], in a COTS, there are at most two end points and every other point is a cut point. The definition of COTS does not explicitly give the ordering, as also mentioned in [8]. It is proved in [8, Theorem 2.7] that there is a total order on every COTS. We need the explicit ordering of a COTS. Starting with the method of obtaining a partial order on a COTS as given in the proof of Theorem 2.7 of [8], we give below, in the symbols used here, a method proving the partial order is total that also directly relates the two definitions of a COTS.

Let X be a COTS. By definition of COTS, every cut point of X is a strong cut point. Fix a cut point z of X. Let X– {z} = $A_z \cup B_z$. Let x be any cut point of X different from z. We can write X– {x} = A_x(z)$\cup$$A_x$(–z).

Let $L_x = A_x$(z) if x $\notin$ A_z, $L_x = A_x$(–z) if x $\in$ A_z.

We take $L_z = A_z$.

The relation <, where for two different cut points x and y of X, x < y if $L_x \subset L_y$, gives a partial order on X, with endpoint(s), if any, at the extreme(s) (as seen below).

($\sharp$) Let x be any cut point of X different from z. If $L_x = A_x$(z), i.e., x $\notin$ A_z, then $A_z = A_z$(–x). By Remark 2.4(b)(ii), A_z(–x) $\subset$ A_x(z), thus $L_z \subset L_x$. If $L_x = A_x$(–z), i.e., x $\in$ A_z, then $A_z = A_z$(x). Thus, using Remark 2.4(b)(i), $L_x \subset L_z$. If $L_x \subset L_z$, then $L_x = A_x$(–z), so x $\in$ A_z. Therefore L_z = {x : $L_x \subset L_z$}.

Let x and y be any two different cut points of X. We prove that either x < y or y < x. If z equals x or y, then, by ($\sharp$), we are done. Otherwise either z separates y and x, or (we can suppose that) x separates y and z.

In view of uniqueness of separating sets involved, in the first case, either x $\in A_z$ and y $\notin A_z$, or x $\notin A_z$ and y $\in A_z$. It follows using Remark 2.4(b)(iii) and (♯), either $L_x \subset L_z \subset L_y$ or $L_y \subset L_z \subset L_x$. If x separates y and z, we can suppose that z $\in A_x$ and y $\notin A_x$. By Remark 2.4 (b)(iii), $A^*_z(-x) \subset A_x \subset A_y(x)$. Thus $A_x = A_x(z)$. So $L_z \subset L_x$. Since z $\in A_y$(x), $L_y = A_y(x)$ if y $\notin A_z$; then $L_z \subset L_x \subset L_y$. In case y $\in A_z$, $A^*_y(\text{-z}) \subset A_z(y) \subset A_x(z)$, by Remark 2.4(b)(iii), as x $\notin A_z$; therefore $L_y \subset L_z \subset L_x$. Hence either x < y or y < x.

In view of Remark 2.4(b), z separates y and x iff either x < z < y or y < z < x, also x separates y and z iff either z < x < y or y < x < z.

If X has a non-cut point a, then we take $A_z = A_z(a)$. Let x be any cut point of X different from z. In this situation, we note that a $\in L_x$. If x $\in A_z(a)$, then x separates z and a, as X is COTS and a is non-cut point. Therefore a $\in A_x(\text{-z}) = L_x$. If x $\notin A_z(a)$, then by (♯), $L_z \subset L_x$. This implies that a $\in L_x$. Using this we have:

(♯♯) x $\in A_z(a)$, iff x separates z and a; x $\notin A_z(a)$ iff z separates x and a.

We define a < x for every cut point x of X, and x < b if b is the other non-cut point of X.

Using the description of total order of a COTS given in the Remark 2.4 and (♯♯) we have the following, the converse of which can also be proved following the proof of the converse of Theorem 2.7 of [8].

Lemma 2.6. *Let X be a COTS. There are exactly two orders on X with the property that for any three distinct points x, y, z of X, x and z are in separate components of X – {y} iff x < y < z or z < y < x. Each of these orders is the reverse of the other.*

Throughout the paper < means an order with this property.

Lemma 2.7. *Let X be a COTS and x, y $\in$ X with x < y. Then the following are equivalent.*

(a) y is an immediate successor of x.

(b) One of {x} and {y} is closed, the other is open.

(c) One of x and y is in the closure of the other.

Proof. (a) ⇒ (b) ⇒ (c) follows by Proposition 2.9 and Lemma 2.8(c) of [8]. We prove (c) ⇒ (a). If y is not an immediate successor of x, then there exists z $\in$ X such that x < z < y. In view of Lemma 2.6, the two components of X − z are t : z < t and t : t < z. Using Lemma 2.2(a), y $\in \text{int}_X(\{t : z < t\} \cup \{z\})$ and x $\in \text{int}_X(\{t : t < z\} \cup \{z\})$. This implies that y $\notin \text{cl}_X(\{x\})$ and x $\notin \text{cl}_X(\{y\})$ which is not possible. Thus y is an immediate successor of x. □

3. R(i) Connected Spaces and Cut Points

Lemma 3.1. *Let x be cut point of a space X. Then (i)* $cl_X(A_x) \subset A_x^*$*;* $cl_X(A_x) = A_x^*$ *iff x is closed point, iff* A_x^* *is regular closed. (ii)* A_x *is regular open iff x is closed point. (iii) If* $y \neq x$ *is a closed cut point of X, then* $cl_X(A_y(-x)) \subset A_x$ *iff* $y \in A_x$*. (iv) If* $(A_y(-x)) \cap A_x \neq \phi$*, then* $y \in A_x$*.*

Proof. (i). $\text{cl}_X(A_x) \subset A_x^*$ follows from Lemma 2.1(a). For connected X, if x is closed point, using Lemma 2.1(b) and Lemma 2.2(g), $\text{cl}_X(A_x) = A_x^*$. If x is not closed point, then, by Lemma 2.2(e), x is an open point, which is not possible in view of Lemma 2.1(d). (ii). Suppose x is closed point. By (i) $\text{cl}_X(A_x) = A_x^*$. Now by Lemma 2.1(a*), $\text{int}_X(\text{cl}_X(A_x))$ equals $\text{cl}_X(A_x)$ or A_x. Since X is connected, $\text{int}_X(\text{cl}_X(A_x)) \neq \text{cl}_X(A_x)$. Therefore A_x is regular open. The converse follows using (i), as $x \in \text{cl}_X(A_x)$, by Lemma 2.1 (a). (iii). Use (i) and Remark 2.4(b)(i). (iv). If $y \notin A_x$, then $y \in B_x$. Therefore by (iii), $\text{cl}_X(A_y(-x)) \subset B_x$, which is not possible. □

Lemma 3.2. *Let X be a connected space. For* $T \subset ctX$*, let* $\varsigma = \{H : H = A_x$ *for some* $x \in T$*, x a closed point of X*$\}$*. If* ς *is a non-trivial filter base, then* ς *is a regular o-filter base.*

Proof. By Lemma 2.2(g), each member of ς is open. Therefore ς is an o-filter base. For A_x, $A_z \in \varsigma$, $z \neq x$, there exists $A_y \in \varsigma$ such that $A_y \subset A_x \cap A_z$. Since $x \notin A_y$, $A_y = A_y(-x)$. Therefore, by (iv) of Lemma 3.1, $y \in A_x$. Now by (iii) of Lemma 3.1, $\text{cl}_X(A_y(-x)) \subset A_x$. □

Corollary 3.3. *Let X be an R(i) connected space. For* $T \subset ctX$*, if* $\varsigma = \{H : H = A_x$ *for some* $x \in T$*, x a closed point of X*$\}$ *is a filter base, then* ς *is fixed.*

Remark 3.4. If ς in Lemma 3.2 is a chain containing at least two members, then being a filter base, is a regular o-filter base. Lemma 2.2 of [10] is proved for a T_1 connected space and for a collection which is a chain. Thus Lemma 3.2 is proved under much weaker assumption than those in Lemma 2.2 of [10].

The following theorem proved without assuming any separation axiom strengthens Theorem 2.5 of [10], and also Theorem 3.10 of [3].

Theorem 3.5. *If a connected space X has a closed R(i) subset H such that there is no proper regular closed, connected subset of X containing H, then there is no proper connected subset of X containing all non-cut points.*

Proof. Assume the contrary, and let Y be a proper connected subset containing all the non-cut points of X. As X − Y ⊂ ctX, by Lemma 3.11 of [4], there exists an infinite chain α of proper connected sets of the form A_x^*(Y) where x ∈ X − Y, x a closed point of X, covering X. Since α is an infinite chain, $\gamma = \{B_x : B_x$ is the other separating set of X − {x} corresponding to A_x(Y) with $A_x^*(Y) \in \alpha\}$ is a non-trivial filter base on X . By Lemma 3.2, γ is a regular o-filter base on X. Let $\gamma^* = \{H \cap B_x : B_x \in \gamma\}$. A_x^* is proper connected regular closed subset (by Lemma 3.1(i)) of X. For each $B_x \in \gamma$, H∩B_x is non-empty as there is no proper regular closed connected subset of X and containing H. γ^* is an o-filter base on H as γ is an o-filter base. Since H is closed and γ is a regular o-filter base on X, γ^* is a regular o-filter base on H. Therefore γ^* is fixed as H is R(i). But X = $\bigcup\{A_x^*(Y) : A_x^*(Y) \in \alpha\}$, therefore $\bigcap \{B_x : B_x \in \gamma\} = \phi$. This leads to a contradiction. The proof is complete. □

Stephenson in [10] proved that a T_1 R(i) connected space has at least two non-cut points. It is proved in [3] that an H(i) connected space has at least two non-cut points. In the following theorem no separation axiom is assumed, thus it strengthens Theorem 2.3 of [10] and also Theorem 3.4 of [3].

Theorem 3.6. *(a) If X is a connected space having a closed R(i) subset H such that there is no proper regular closed, connected subset of X containing H, then X has at least two non-cut points. (b) A connected R(i) space X has at least two non-cut points.*

Proof. Use Theorem 3.5. □

Corollary 3.7. *(i) A cut point space is always non-R(i); (ii) An R(i) COTS is a space with endpoints.*

Proof. For (ii), we need Proposition 2.5 of [8]. □

Lemma 3.8. *Let X be a connected space with endpoints a and b. Then (i) there is no proper connected subset of X containing {a,b}. (ii) cl_X({a,b}) contains at most four points.*

Proof. (i) Assume the contrary, and let Y be a proper connected subset of X containing {a,b}. Then by Theorem 3.1 of [4], X − Y ⊂ ctX. Since Y is proper subset of X, there exists x ∈ X − Y. Then x is a cut point of X. Therefore X − {x} = A_x(a)∪B_x(b) as X is a connected space with endpoints. Now Y ⊂ X − {x} implies that either Y ⊂ A_x(a) or Y ⊂ B_x(b) as Y is connected. But this is not possible as {a,b} ⊂ Y. Thus there is no proper connected subset of X containing {a,b}.

(ii) By Theorem 3.2 of [4], X is a COTS with endpoints. So by Theorem 2.7 of [8], there is a total order $<$ on X with a $<$ b. Let x $\in$ X − {a,b}. a $<$ x $<$ b. If x is not an immediate successor of a, then x $\notin$ $\mathrm{cl}_X(\{a\})$ by Lemma 2.7. Similarly, if x is not immediate predecessor of b, then x $\notin$ $\mathrm{cl}_X(\{b\})$ using Lemma 2.7. Thus each of $\mathrm{cl}_X(\{a\})$ and $\mathrm{cl}_X(\{b\})$ contains at most two points. Hence $\mathrm{cl}_X(\{a,b\}) = \mathrm{cl}_X(\{a\}) \cup \mathrm{cl}_X(\{b\})$ contains at most four points. □

It is proved in [3, Theorem 4.4] that an H(i) connected space with at most two non-cut points is a COTS with endpoints. As a reply to a question in [3], it is proved in [6, Theorem 4.4] that a connected space with at most two non-cut points is a COTS with endpoints iff it has an H(i) subset such that there is no proper connected subset of the space containing the H(i) subset. In the following theorem, we obtain another characterization of the class of spaces where every member is a COTS with end points.

Theorem 3.9. *(a) A connected space X has at most two non-cut points and a closed R(i) subset H such that there is no proper regular closed, connected subset of X containing H iff X is a COTS with endpoints. (b) A connected R(i) space with at most two non-cut points is a COTS with endpoints.*

Proof. (a) If X has a closed R(i) subset H such that there is no proper regular closed, connected subset of X containing H, then, by Theorem 3.6, X has at least two non-cut points. Therefore, by the given condition, X has exactly two non-cut points, say, a and b. Let x $\in$ X − {a,b}. Then x $\in$ ctX. By Remark 2.4(a), each of A_x^* and B_x^* is connected. By Theorem 3.5, there is no proper connected subset of X containing X − ctX. But X − ctX = {a,b}. So a $\in A_x$ and b $\in B_x$ or conversely. This implies that x $\in$ S(a,b). Hence X = S[a,b]. Now by Theorem 3.2 of [4], X is a COTS with endpoints a and b. Conversely suppose that X is a COTS with endpoints. Then X is a connected space with endpoints, say, a and b. Therefore by Lemma 3.8(i), there is no proper connected subset of X containing {a,b}. This implies that there is no proper regular closed, connected subset of X containing $\mathrm{cl}_X(\{a,b\})$. By Lemma 3.8(ii), $\mathrm{cl}_X(\{a,b\})$ contains at most four points. Let H = $\mathrm{cl}_X(\{a,b\})$, then H is compact; so is R(i). Thus X has a closed R(i) subset H such that there is no proper regular closed, connected subset of X containing H. The proof is complete. (b) follows from (a). □

The following remark is an improvement on the Remark 4.6 of [3] and follows using Proposition 2.9 of [8] and Theorem 3.9.

Remark 3.10. If X is a connected space having at most two non-cut points and a closed R(i) subset H such that there is no proper regular closed, connected subset of X containing H, then X is $T_{\frac{1}{2}}$.

Theorem 4.5 of [4] is proved for the class of H(i) spaces. The following theorem is proved for a larger class consisting of R(i) spaces.

Theorem 3.11. *If a connected space X having at most two non-cut points and a closed R(i) subset H such that there is no proper regular closed, connected subset of X containing H is locally connected, then X is a compact COTS with endpoints.*

Proof. By Theorem 3.9, X is a COTS with endpoints. Now the theorem follows by Theorem 4.4 of [4]. □

The question of a space being homeomorphic with the closed unit interval is of considerable importance. We give below one such result.

Theorem 3.12. *If a T_1 separable connected and locally connected space X has at most two non-cut points and a closed R(i) subset H such that there is no proper regular closed, connected subset of X containing H, then X is homeomorphic with the closed unit interval.*

Proof. By Theorem 3.11, X is a compact COTS with endpoints. X being a T_1 COTS is Hausdorff, using Proposition 2.9 of [8]. Since a separable compact connected Hausdorff space with exactly two non-cut points is homeomorphic with the closed unit interval (see [13, Theorem 122]), X is homeomorphic with the closed unit interval. □

Lemma 3.13. *If a subset A of a space X is R(i), then, for $x \in X$, $A \cup \{x\}$ is R(i).*

Proof. Let ς be a regular o-filter base on A∪{x}. We prove that ς is fixed. If $\{x\} \in \varsigma$, then ς is fixed. So we suppose that $\{x\} \notin \varsigma$. Then since each member of ς is a non-empty subset of A∪{x}, $B \cap A \neq \phi$ for every $B \in \varsigma$. Therefore $\varsigma' = \{G : G = B \cap A \text{ for some } B \in \varsigma\}$ is a regular o-filter base on A. Since A is R(i), $\bigcap\{G : G \in \varsigma'\} \neq \phi$, therefore $\bigcap\{B : B \in \varsigma\} \neq \phi$. This completes the proof. □

The following is a vastly improved version of Lemma 2.4 of [10].

Lemma 3.14. *Let X be an R(i) space. If $Y \subset X$ is such that there is an open subset E and a closed subset F of X such $E \subset Y$, $F \subset Y$ and $Y - (E \cap F)$ is non-empty and finite, then Y is R(i).*

Proof. Let ς be a regular o-filter base on Y. Let H = E∩F. We suppose that $(\bigcap\{B : B \in \varsigma\}) \cap (Y - H) = \phi$. Since Y – H is finite and ς is a regular o-filter base on Y, there exists some V $\in \varsigma$ such that $(Y - H) \cap \mathrm{cl}_Y(V) = \phi$. Thus $\varsigma' = \{G \in \varsigma : G \subset V\}$ is a family of open subsets of X as H $\subset$ E $\subset$ Y and E is open in X. Since ς is a regular o-filter base on Y, for G $\in \varsigma'$, there is some B $\in \varsigma$ such that $\mathrm{cl}_Y(B) \subset G$. Since B $\subset$ F $\subset$ Y and F is closed in X, $\mathrm{cl}_X(B) = \mathrm{cl}_Y(B)$. So $\mathrm{cl}_X(B) \subset G$. This proves that ς' is a regular o-filter base on X. Also for each B $\in \varsigma$, there is some G $\in \varsigma$ such that G $\subset$ B∩V; since G $\in \varsigma'$, so $\bigcap\{G : G \in \varsigma'\} \subset \bigcap\{B : B \in \varsigma\}$. But ς' is fixed as X is R(i). Thus Y is R(i). □

Lemma 3.15. *Let X be an R(i) connected space and $x \in ctX$. Then A_x^* is R(i).*

Proof. By Lemma 2.2(c) either A_x is open in X, and then A_x^* is closed in X or A_x is closed in X and, then A_x^* is open in X. Now the result follows by Lemma 3.14. □

The following two theorems strengthen Theorems 3.9 and 3.12 of [3] respectively.

Theorem 3.16. *Let X be an R(i) connected space and $x \in ctX$. Then A_x contains a non-cut point of X.*

Proof. By Lemma 3.15, A_x^* is R(i). By Remark 2.4(a), A_x^* is connected. Therefore by Theorem 3.6, A_x^* has at least two non-cut points in A_x^*. Now the theorem follows by Lemma 3.8 of [3]. □

Theorem 3.17. *Let X be an R(i) connected space such that $ctX \neq \phi$. Then there is a two-point disconnected subspace {a,b} of X such that X – {a,b} is connected.*

Proof. Let x $\in$ ctX. Let A_x and B_x be separating sets of X – {x}. By Lemma 3.15, A_x^* is R(i). By Remark 2.4(a), A_x^* is connected. Therefore using Theorem 3.6, A_x^* has at least two non-cut points in A_x^*. Let a be a non-cut point of A_x^* other than x. Then A_x^* – {a} is connected and a $\in A_x$. Similarly, there exists b $\in B_x$ such that B_x^* – {b} is connected. Now $(A_x^* - \{a\}) \cup (B_x^* - \{b\})$ is connected. Therefore X – {a,b} is connected. Since $A_x \cap \{a, b\} = A_x^* \cap \{a, b\} = \{a\}$, {a} is closed in {a, b}, by Lemma 2.1(f)(i). Similarly, {b} is closed in {a, b}. Therefore {a,b} is disconnected. □

The following corollary strengthens Corollary 3.13 of [3].

Corollary 3.18. *If X an R(i) connected space such that the removal of every two-point disconnected space leaves the space disconnected, then $ctX = \phi$.*

Lemma 3.19. *Let X be a connected space. Let a, b $\in$ X – ctX. Let H be a separating set of X – {a,b}. If X is R(i), then H$\cup${a,b} is R(i).*

Proof. By Lemma 3.11(I) of [5], there is an open subset E and a closed subset F of X such that H $\subset$ E $\subset$ H$\cup${a,b} and H $\subset$ F $\subset$ H$\cup${a,b}. Since H$\cup${a,b} − E$\cap$F is finite, the result follows by Lemma 3.14. □

Theorem 3.20. *Let X be an R(i) connected space such that the removal of every two-point disconnected set leaves the space disconnected. For a, b $\in$ X, let H be a separating set of X – {a,b}. Then H$\cup${a,b} is a $T_{\frac{1}{2}}$ R(i) COTS with endpoints.*

Proof. In view of Theorem 3.9 and Remark 3.10, it suffices to prove that H$\cup${a,b} is an R(i) connected space with at most two non-cut points. Since ctX = ϕ by Corollary 3.18, X − {a} and X − {b} are connected. H is a separating subset of (X − {b}) − {a} as well as (X − {a}) − {b}, so by Remark 2.4(a), H$\cup${a} and H$\cup${b} are connected; therefore their union H$\cup${a,b} is connected. Now by Lemma 3.19, H$\cup${a,b} is R(i). Since H$\cup${a} and H$\cup${b} are connected, a and b are non-cut points of H$\cup${a,b}. Let K be the other separating set of X − {a,b}. Similarly K$\cup${a,b} is R(i), and a and b are non-cut points of K$\cup${a,b}. Let p $\in$ H be a non-cut point of H$\cup${a,b}. If q $\in$ K is a non-cut point of K$\cup${a,b}, then (H$\cup${a,b}) − {p} and (K$\cup${a,b}) − {q} are connected; therefore their union X − {p,q} is connected which is not possible. Therefore K$\cup${a,b} is a space with endpoints a and b by Theorem 3.9. Let z $\in$ K. Since z is cut point of K$\cup${a,b}, in view of Theorem 3.1 of [4], there exists a separation A|B of K$\cup${a,b} − {z} in K$\cup${a,b} with A and B connected. We can suppose that a $\in$ A and b $\in$ B. Therefore A$\cup$((H$\cup${a,b}) − {p})$\cup$B is connected. Since X − {p,z} = A$\cup$((H$\cup${a,b}) − {p})$\cup$B, we arrive at a contradiction. Thus H$\cup${a,b} has exactly two non-cut points. This completes the proof. □

Corollary 3.21. *Let X be an R(i) connected space such that the removal of every two-point disconnected set leaves the space disconnected. Then for every two point disconnected subset x,y of X, there exist subsets M and N of X such that M$\cup$N = X, M$\cap$N = {x,y}, and both M and N are $T_{\frac{1}{2}}$ R(i) COTS with endpoints.*

Theorem 3.22. *Let X be a connected space that has an H(i) subset H such that there is no proper regular closed connected subset of X containing H. Then there is no proper connected subset of X containing all non-cut points.*

Proof. Assume the contrary, and let Y be a proper connected subset containing all the non-cut points of X. As X − Y $\subset$ ctX, by Lemma 3.11 of [4], there exists an infinite chain α of proper connected sets of the form A_x^*(Y), x $\in$ X − Y, x a closed point of X, covering X.

By Lemma 3.3 of [3], interiors of members of α cover X. So H = $\bigcup\{\text{int}_X(A_x^*)\cap\text{H} : A_x^* \in \alpha\}$. Therefore, since H is H(i), there exists a finite subfamily α' of α such that H = $\bigcup\{\text{cl}_H(\text{int}_X(A_x^*)\cap\text{H}) : A_x^* \in \alpha'\}$. This implies that H $\subset \text{cl}_X(A_p^*)$ for some $A_p^* \in \alpha'$. Since p is closed, by Lemma 2.1(g), A_p^* is closed. Therefore H $\subset A_p^*$ which is a contradiction to the given condition as A_p^* is regular closed by Lemma 3.1(i). The proof is complete. □

The following result is the non-cut point existence theorem for a subclass of connected spaces.

Theorem 3.23. *Let X be a connected space that has an H(i) subset H such that there is no proper regular closed connected subset of X containing H. Then X has at least two non-cut points of X.*

Proof. Suppose to the contrary. Then X – ctX has at most one element. There exists some x ∈ X such that X – ctX ⊂ {x}. This is a contradiction in view of Theorem 3.22. □

Theorem 3.24. *Let X be a connected space. Then X is a COTS with endpoints if and only if X has at most two non-cut points and has an H(i) subset H such that there is no proper regular closed connected subset of X containing H.*

Proof. If X has an H(i) subset H such that there is no proper regular closed connected subset of X containing H, then, by Theorem 3.23, X has at least two non-cut points. Therefore, by given condition, X has exactly two non-cut points, say, a and b. Let x ∈ X – {a,b}. Then x ∈ ctX. By Remark 2.4(a), each of A_x^* and B_x^* is connected. There is no proper connected subset of X containing X – ctX by Theorem 3.22. But X – ctX = {a,b}. So a ∈ A_x and b ∈ B_x or conversely. This implies that x ∈ S(a,b). Hence X = S[a,b]. Now by Theorem 3.2 of [4], X is a COTS with endpoints a and b. Conversely suppose that X is a COTS with endpoints. Then X is a connected space with endpoints, say, a and b. Therefore by Lemma 3.8(i), there is no proper connected subset of X containing {a,b}. This implies that there is no proper regular closed, connected subset of X containing {a,b}. Let H = {a,b}; then H is compact, so is H(i). The contradiction arrived at completes the proof. □

Call a chain α of a partially ordered set ζ **infinite upwards** if given A ∈ α, there exists B ∈ α such that B ≠ A and B ⊃ A.

Lemma 3.25. *Let X be a connected space having a cut point hereditary property P. If X has a non-empty proper connected subset Y such that X – Y ⊂ ctX, then there exists an infinite upwards chain of proper regular closed, connected subsets of the form $A_x^*(Y)$, x ∈ X – Y, x a closed point of X and $A_x^*(Y)$ has property P, covering X.*

Proof. Let x $\in$ X − Y. Since x $\in$ ctX and P is a cut point hereditary property, there exists a separation of X − {x} such that each of A_x^* and B_x^* has property P. Since Y $\subset$ X − {x} = $A_x \cup B_x$ and Y is connected, either Y $\subset A_x$ or Y $\subset B_x$. We suppose that Y $\subset A_x$ i.e., $A_x^* = A_x^*$(Y). This implies that $B_x \subset$ ctX. If x is not a closed point of X, then, in view of Remark 2.3, B_x is closed in X, and therefore B_x contains a closed cut point of X using Lemma 2.2(e). Therefore ς = {A : A =A_x^*(Y) for some separating set A_x(Y) of X − {x}, x $\in$ X − Y, x a closed point of X and A_x^*(Y) has property P} is a non-empty partially ordered set under set inclusion. Members of ς are connected by Remark 2.4(a) and regular closed by Lemma 3.1. Using the Hausdorff Maximal Principle, there exists a maximal chain α in ς. W the union of members of α is a connected subset of X as α is a chain. Let z $\in$ X − W. X − W $\subset$ ctX as Y $\subset$ W. Proceeding as above we get a closed cut point y of X such that y $\in$ X − W. Then y $\in$ X − Y. A_y^*(W) is connected and has property P. As Y $\subset$ W, A_y^*(W) = A_y^*(Y). Thus A_y^*(Y) $\in \varsigma$. Since W $\subset A_y^*$(Y), $\alpha \cup \{A_y^*(\mathrm{Y})\}$ is a chain in containing α properly as y $\notin$ W. This proves that X = W. It now follows that α is infinite upwards, as members of α are proper subsets of X. □

Theorem 3.26. *Let X be a connected space having a cut point hereditary property P. If X has an H(i) subset H such that there is no proper regular closed, connected subset of X, having property P and containing H, then there is no proper connected subset of X containing all non-cut points.*

Proof. Assume the contrary, and let Y be a proper connected subset containing all the non-cut points of X. As X − Y $\subset$ ctX, by Lemma 3.25, there exists an infinite upwards chain α of proper connected regular closed subsets of the form A_x^*(Y), x $\in$ X − Y, x a closed point of X and A_x^*(Y) has property P, covering X. By Lemma 3.3 of [3], interiors of members of α cover X. Therefore H = $\bigcup\{\mathrm{int}_X(A_x^*) \cap \mathrm{H} : A_x^* \in \alpha\}$. Since H is H(i), there exists a finite subfamily α' of α such that H = $\bigcup\{\mathrm{cl}_H(\mathrm{int}_X(A_x^*) \cap \mathrm{H}) : A_x^* \in \alpha'\}$. This implies that H $\subset \mathrm{cl}_X(A_p^*)$ for some $A_p^* \in \alpha'$. But A_p^* is closed, so H $\subset A_p^*$. This is a contradiction to the given condition as A_p^* is regular closed, connected set, having property P. The proof is complete. □

The following result is the non-cut point existence theorem for a subclass of connected spaces having a cut point hereditary property P.

Theorem 3.27. *Let X be a connected space having a cut point hereditary property P. If X has an H(i) subset H such that there is no proper regular closed connected subset of X, having property P and containing H, then X has at least two non-cut points of X.*

Proof. Suppose to the contrary. Then X − ctX has at most one element. There exists some x ∈ X such that X − ctX ⊂ {x}. This contradicts Theorem 3.26. Then there are at least two non-cut points of X. □

Theorem 3.28. *Let X be a connected space having a cut point hereditary property P. Then X is a COTS with endpoints if and only if X has at most two non-cut points and has an H(i) subset H such that there is no proper regular closed connected subset of X, having property P and containing H.*

Proof. If X has an H(i) subset H such that there is no proper regular closed connected subset of X, having property P, containing H, then, by Theorem 3.27, X has at least two non-cut points. Therefore, by given condition, X has exactly two non-cut points, say, a and b. Let x ∈ X − {a,b}. Then x ∈ ctX. By Remark 2.4(a), each of A_x^* and B_x^* is connected. There is no proper connected subset of X containing X − ctX by Theorem 3.26. But X − ctX = {a,b}. So a ∈ A_x and b ∈ B_x or conversely. This implies that x ∈ S(a,b). Hence X = S[a,b]. Now by Theorem 3.2 of [4], X is a COTS with endpoints a and b. Conversely suppose that X is a COTS with endpoints. Then X is a connected space with endpoints, say, a and b. Therefore by Lemma 2.8(i), there is no proper connected subset of X containing {a,b}. This implies that there is no proper regular closed, connected subset of X containing {a,b}. Let H = {a,b}, then H is compact; so is H(i). This completes the proof. □

Theorem 3.29. *Let X be a connected space having a cut point hereditary property P. If X has a closed R(i) subset H such that there is no proper regular closed connected subset of X, having property P and containing H, then there is no proper connected subset of X containing all non-cut points.*

Proof. Let Y be a proper connected subset containing all the non-cut points of X. As X − Y ⊂ ctX, by Lemma 3.25, there exists an infinite upwards chain α of proper connected regular closed subsets of the form A_x^*(Y), x ∈ X − Y, x a closed point of X and A_x^*(Y) has property P, covering X. Since α is an infinite chain, $\gamma^* = \{B_x : B_x$ is the other separating set of X − {x} corresponding to A_x(Y) with $A_x^*(Y) \in \alpha\}$ is a non-trivial filter base on X . By Lemma 3.2, γ^* is a regular o-filter base on X. Let $\gamma = \{H \cap B_x : B_x \in \gamma^*\}$. A_x^* is a proper regular closed connected subset of X, having property P. For each $B_x \in \gamma^*$, H∩B_x is non-empty as there is no proper regular closed connected subset of X, having property P and containing H. γ is a regular o-filter base on H as γ^* is a regular o-filter base on X and H is closed. Therefore γ is fixed as H is R(i). But $X = \bigcup\{A_x^*(Y) : A_x^*(Y) \in \gamma\}$, therefore $\bigcap\{B_x : B_x \in \gamma^*\} = \phi$. This leads to a contradiction. The proof is complete. □

The following result is the non-cut point existence theorem for another subclass of connected spaces having a cut point hereditary property.

Theorem 3.30. *Let X be a connected space having a cut point hereditary property P. If X has a closed R(i) subset H such that there is no proper regular closed connected subset of X, having property P and containing H, then X has at least two non-cut points of X.*

Proof. If the result is not true, then exists some x ∈ X such that X − ctX ⊂ {x} which is contrary to Theorem 3.29. Then there are at least two non-cut points of X. □

Theorem 3.31. *Let X be a connected space having a cut point hereditary property P. Then X is a COTS with endpoints if and only if X has at most two non-cut points and a closed R(i) subset H such that there is no proper regular closed connected subset of X, having property P and containing H.*

Proof. If X has a closed R(i) subset H such that there is no proper regular closed connected subset of X, having property P, containing H, then, by Theorem 3.30, X has at least two non-cut points. Therefore, by given condition, X has exactly two non-cut points, say, a and b. Let x ∈ X − {a,b}. Then x ∈ ctX. By Remark 2.4(a) each of A_x^* and B_x^* is connected. By Theorem 3.29, there is no proper connected subset of X containing X − ctX = {a,b}. So a ∈ A_x and b ∈ B_x or conversely. This implies that x ∈ S(a,b). Hence X = S[a,b]. Now by Theorem 3.2 of [4], X is a COTS with endpoints a and b. Conversely suppose that X is a COTS with endpoints. Then X is a connected space with endpoints, say, a and b. Therefore by Lemma 3.8(i), there is no proper connected subset of X containing {a,b}. This implies that there is no proper regular closed, connected subset of X having property P containing $\mathrm{cl}_X(\{a,b\})$. By Lemma 3.8(ii), $\mathrm{cl}_X(\{a,b\})$ contains at most four points. Let H = $\mathrm{cl}_X(\{a,b\})$, then H is compact; so is R(i). This completes the proof. □

Theorem 3.32. *Let X be a connected and locally connected space with at most two non-cut points. If X has an H(i) subset H such that there is no proper regular closed connected subset of X containing H, then X is a compact COTS with endpoints.*

Proof. By Theorem 3.24, X is a COTS with endpoints. Now the theorem follows by Theorem 6.1(ii) of [6]. □

Now we get the following corollary, using Corollary 6.2(i) of [6] and Theorem 3.32.

Corollary 3.33. *Let X be a T_1 separable, connected and locally connected space with at most two non-cut points. If X has an H(i) subset H with the property that there is no proper regular closed connected subset of X containing H, then X is homeomorphic with the closed unit interval.*

Theorem 3.34. *Let X be a connected and locally connected space with at most two non-cut points and having a cut point hereditary property P. If X has an H(i) subset H such that there is no proper regular closed connected subset of X, having property P, containing H, then X is a compact COTS with endpoints.*

Proof. By Theorem 3.28, X is a COTS with endpoints. Now the theorem follows by Theorem 6.1(ii) of [6]. □

Theorem 3.34 coupled with Corollary 6.2(i) of [6] gives the following corollary.

Corollary 3.35. *Let X be a T_1 separable, connected and locally connected space with at most two non-cut points and having a cut point hereditary property P. If X has an H(i) subset H such that there is no proper regular closed connected subset of X, having property P, containing H, then X is homeomorphic with the closed unit interval.*

Theorem 3.36. *Let X be a connected and locally connected space with at most two non-cut points and having a cut point hereditary property P. If X has a closed R(i) subset H such that there is no proper regular closed connected subset of X, having property P, containing H, then X is a compact COTS with endpoints.*

Proof. By Theorem 3.31, X is a COTS with endpoints. Now the theorem follows by Theorem 6.1(ii) of [6]. □

Theorem 3.36 coupled with Corollary 6.2(i) of [6] gives the following corollary.

Corollary 3.37. *Let X be a T_1 separable, connected and locally connected space with at most two non-cut points and having a cut point hereditary property P. If X has a closed R(i) subset H such that there is no proper regular closed connected subset of X, having property P, containing H, then X is homeomorphic with the closed unit interval.*

We have the following two results as consequences of different characterizations of COTS.

Theorem 3.38. *The following are equivalent for a connected space X and having at most two non-cut points.*

(i) There is an H(i) subset H of X such that there is no proper connected subset of X containing H.

(ii) There is an H(i) subset H of X such that there is no proper regular closed connected subset of X containing H.

(iii) There is a closed R(i) subset H of X such that there is no proper connected subset of X containing H.

(iv) There is a closed R(i) subset H of X such that there is no proper regular closed connected subset of X containing H.

Theorem 3.39. *Let P be a cut point hereditary property. Let X be a connected space having the property P and has at most two non-cut points. Then the following are equivalent.*

(i) There is an H(i) subset H of X such that there is no proper connected subset of X, having property P and containing H.

(ii) There is an H(i) subset H of X such that there is no proper regular closed connected subset of X, having property P and containing H.

(iii) There is a closed R(i) subset H of X such that there is no proper connected subset of X, having property P and containing H.

(iv) There is a closed R(i) subset H of X such that there is no proper regular closed connected subset of X, having property P and containing H.

4. Regular O-filter Bases and the Khalimsky Line

By Proposition 2.5 of [8], a COTS has at most two end points and every other point is a cut point. The Khalimsky line is a COTS without end points; subspaces of the Khalimsky line are COTS with or without end points. In this Section, first we observe when a COTS or a connected space with at most two non-cut points is homeomorphic to a subspace of the Khalimsky line.

Lemma 4.1. *Let X be a COTS. Let K be the set of all closed points of X. If X – K is non-empty, then there is a one-one function from X – K to K∪E, where E is the set of end points of X.*

Proof. By Theorem 2.7 of [8], there are two total orders on X with the property that for each x $\in$ X, the two components of X $-$ {x} are {y : y $>$ x} and {y : y $<$ x}. Let a and b be the end points of X (if they exist). If X has only one end point, we can suppose that a $<$ x for every x $\in$ X $-$ {a}, otherwise we consider the other order of the COTS; b is the other end if X has two end points, and then a $<$ x $<$ b, for every x $\in$ X $-$ {a, b}. Let S be the set of points of X $-$ K not having a successor. Then S $= \phi$ or {b} if b exists. Let y $\in$ (X $-$ K) $-$ S. First suppose that y $\neq$ a. By Proposition 2.9 of [8], y is an open point. Therefore y has an immediate successor $y^+ \in$ K, by Lemma 2.8(b) and (c) of [8]. Put f(y) $= y^+$; if y $=$ a, then in view of Lemma 2.8(b) of [8], a has an immediate successor a^+. Defining f(y) $= y^+$, and f(b) $=$ a if the need arises, gives a function f : X – K $\longrightarrow$ K∪E, which is one-one. □

Remark 4.2. We note that, in Lemma 4.1, f : X − K ⟶ K, if E ⊂ {a}; otherwise f : X − K ⟶ K∪{a}. In view of Lemma 2.8(b) and (c) of [8], an element of K has pre-image under f iff it has an immediate predecessor.

Remark 4.3. Let X be an r.c.s. connected or E.C. space. By Lemma 3.1 of [5], X has at most one closed point. X has at least two non-cut points by Theorem 3.2 of [5]. If X is a COTS, then X has exactly two non-cut points. Now using Lemma 4.1, X contains at most four elements.

Theorem 4.4. *Let X be a connected space with at most two non-cut points. If the set K of all closed points of X is finite, then X is homeomorphic to a finite subspace of the Khalimsky line.*

Proof. By Corollary 3.13 of [4], X has at least two non-cut points. So X has exactly two non-cut points. Now by Theorem 3.17 of [4], the result follows. □

Remark 4.5. Let X be a COTS. If the set of all closed points of X is countably infinite, then X is countably infinite.

Proof. Use Lemma 4.1. □

Theorem 4.6. *If the set of all closed points of a connected space X with endpoints is countable, then X is either homeomorphic to a finite subspace of the Khalimsky line or to a countably infinite, $T_{\frac{1}{2}}$ COTS with endpoints.*

Proof. By Theorem 3.2 of [4], X is a COTS with two non-cut points. Let K be the set of all closed points of X. If K is finite, then by Theorem 4.4, X is homeomorphic to a finite subspace of the Khalimsky line. Otherwise the proof is complete using Proposition 2.9 of [8] and Remark 4.5. □

Remark 4.7. Every member of a regular o-filter base on a space contains a regular open set, so there may be spaces, (out of the class of spaces having very few regular open sets) where every regular o-filter base on the space is finite. The next result is about spaces where every regular o-filter base is finite.

Theorem 4.8. *Let X be a connected space with at most two non-cut points such that every regular o-filter base on X is finite. Then X is homeomorphic to a finite subspace of the Khalimsky line.*

Proof. In view of Theorem 4.4, it suffices to show that the set K of all closed cut points of X is finite. Using given condition, Theorem 3.9 and Remark 2.5, $\varsigma = \{L_x : x \in K\}$ is a non-trivial filter base. By Lemma 3.2, ς is a regular o-filter base on X. By the given condition, ς is finite, therefore K is finite. □

5. About Non-cut Points of Members of a Subclass Ω of Connected Spaces

Let a and b be points of a space X. We say that a, b are **cut point separated** if S(a,b) $\neq \phi$; a and b are **completely separated** ([10]) if there is a real valued continuous function f on X with f(a) $\neq$ f(b). An o-filter base γ is called **completely regular o-filter base** ([10]) if for every G $\in \gamma$ there exists some S $\in \gamma$ with S $\subset$ G, and a continuous function f : X $\longrightarrow$ [0,1] such that f(S) = {0} and f(X − G) = {1}. The subclass β of connected T_1 spaces considered by R. M. Stephenson, Jr. in [10], can be defined, using the concept of cut point separated, as the class of connected T_1 spaces X such that every completely regular o-filter base on X is fixed and every pair of cut point separated points of X is completely separated.

We say that a space has **non-cut property** if the space has at least two non-cut points. For a connected space X, x $\in$ ctX and a separation $A_x|B_x$ of X − {x}, A_x^* and B_x^* are connected by Remark 2.4(a). So it is worth studying the non-cut property as cut point hereditary property. We consider the subclass Ω of connected spaces with the non-cut property, in which the non-cut property is also cut point hereditary. The members of Ω need not be T_1. By Theorem 3.6(b) and Lemma 3.15, the class of R(i) connected spaces is a subclass of Ω. We note below that Ω contains β.

Remark 5.1. If X $\in \Omega$ is a COTS, then, X is a space with endpoints by Proposition 2.5 of [8].

Lemma 5.2. *Let $X \in \Omega$. For $x \in ctX$, there exists a separation $A_x|B_x$ of X − {x} such that each one of A_x and B_x contains a non-cut point of X.*

Proof. Since the non-cut property is a cut point hereditary property, there exists a separation $A_x|B_x$ of X − {x} such that A_x^* (B_x^*) has at least two non-cut points in A_x^* (B_x^*). Now using Lemma 3.8 of [3], each one of A_x and B_x contains a non-cut point of X. □

Lemma 5.3. *Let $X \in \Omega$. If X has at most two non-cut points, then every cut point of X is a strong cut point and X is a space with endpoints.*

Proof. Using given condition, X has exactly two non-cut points, say a and b. Thus ctX = X − {a,b}. Let x $\in$ ctX. Using that the non-cut property is a cut point hereditary property and Lemma 3.8 of [3], A_x^* has exactly two non-cut points, and x is one of them. So A_x is connected. Similarly B_x is connected. Therefore x is a strong cut point. In view of Lemma 5.2, we can suppose that a $\in A_x$ and b $\in B_x$. This implies that x $\in$ S(a,b). Therefore X = S[a,b]. Thus X is a space with endpoints. □

Theorem 5.4. *Let $X \in \Omega$. If X is T_1 and has at most two non-cut points, then X is a Hausdorff COTS with endpoints.*

Proof. By Lemma 5.3, X is a connected space with endpoints. So by Theorem 3.2 of [4], X is a COTS with endpoints. Since X is T_1, X is Hausdorff by Proposition 2.9 of [8]. □

Theorem 5.5. *Let $X \in \Omega$. If $ctX \neq \phi$, then there is a two-point disconnected subspace {a,b} of X such that X – {a,b} is connected.*

Proof. Let x ∈ ctX. Since the non-cut property is a cut point hereditary property, A_x^* has at least two non-cut points in A_x^*. Let a be a non-cut point of A_x^* other than x. Then A_x^* − {a} is connected and a ∈ A_x. Similarly, there exists b ∈ B_x such that B_x^* − {b} is connected. Now (A_x^* − {a})∪(B_x^* − {b}) is connected. Therefore X − {a,b} is connected. Since A_x∩{a, b} = A_x^*∩{a, b} = {a}, {a} is closed in {a, b}, by Lemma 2.1(f)(i). Similarly, {b} is closed in {a,b}. Therefore {a,b} is disconnected. □

Corollary 5.6. *Let $X \in \Omega$. If the removal of any two-point disconnected space leaves the space disconnected, then $ctX = \phi$.*

Theorem 5.7. *Let $X \in \Omega$ be locally connected. If X has at most two non-cut points, then X is a compact COTS with endpoints.*

Proof. By Lemma 5.3, X is a space with endpoints. Now the theorem follows using Theorems 3.2 and 4.4 of [4]. □

The following is another characterization of the closed unit interval.

Theorem 5.8. *Let $X \in \Omega$ be T_1 separable and locally connected. If X has at most two non-cut points, then X is homeomorphic with the closed unit interval.*

Proof. By Theorems 5.4 and 5.7, X is a compact Hausdorff COTS with endpoints. Since by Theorem 122 of [13], a separable compact connected Hausdorff space with exactly two non-cut points is homeomorphic with the unit interval, X is homeomorphic with the closed unit interval. □

About non-cut points of members of the class β

For ready reference Theorem 3.5 of [10] is stated below.

Theorem 3.5 ([10]). Let X ∈ β. Then X has at least two non-cut points i.e., X has non-cut property.

Lemma 5.9. *The non-cut property is a cut point hereditary property for the members of β.*

Proof. Let X $\in \beta$. Let x $\in$ ctX. There exists a separation $A_x|B_x$ of X − {x}. By Remark 2.4(a), each one of A_x^* and B_x^* is connected. By Lemma 3.3 of [10], A_x^*, $B_x^* \in \beta$, therefore each one of A_x^* and B_x^* has at least two non-cut points in A_x^*. □

Remark 5.10. Using Theorem 3.5 of [10] and Lemma 5.9, $\beta \subset \Omega$. Therefore all the above results i.e., 5.1–5.8 hold for members of β.

Remark 5.11. The subject raised in Section 5 slightly generalizes to connected spaces without cut points, and this subject is worth further study.

Acknowledgement. The authors are thankful to the referee for his very valuable suggestions which resulted in the improvement of the paper. In particular, Remark 5.11 is due to him.

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Kamboj - Govt. College, Karnal, Haryana, India
E-mail address: kamboj.dev81@rediffmail.com

Kumar - JCDM College of Engineering, Sirsa, Haryana, India
E-mail address: vkvinod96@yahoo.co.in