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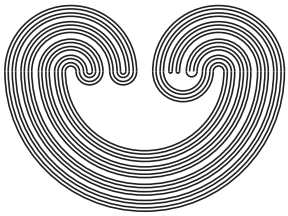

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by

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LIFTING THE COLLINS–ROSCOE PROPERTY BY CONDENSATIONS

VLADIMIR V. TKACHUK

Abstract. We show that every strongly monotonically monolithic Lindelöf Σ-space has a countable base. We also establish that the Collins–Roscoe property and monotonic κ-monolithity have a nice behavior with respect to condensations of Lindelöf Σ-spaces, i.e., if a Lindelöf Σ-space X condenses onto a monotonically κ-monolithic space (onto a space with the Collins–Roscoe property), then X itself is monotonically κ-monolithic (has the Collins–Roscoe property). We prove that a monotonically monolithic perfectly normal compact space is metrizable; this provides another method for constructing a Corson compact space which is not monotonically monolithic. Answering a question of Gary Gruenhage, we give an example of a compact space which fails to be Gul'ko but has the Collins–Roscoe property.

Introduction

It is a textbook result that every condensation (i.e., a continuous bijection) of a compact Hausdorff space is a homeomorphism, so if a compact space X condenses onto a space with a property $\mathcal{P}$, then X itself has $\mathcal{P}$. The class of Lindelöf Σ-spaces is an extension of the class of compact spaces; this extension has been widely studied in general topology as well as in functional analysis and descriptive set theory. In particular, it was proved that every Lindelöf Σ-space X is stable (see the beginning of

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Section 2 for the definition); this implies that X is cosmic if X condenses onto a cosmic space. In this paper we will show that there exist other important properties $\mathcal{P}$ such that a Lindelöf Σ-space has $\mathcal{P}$ whenever it condenses onto a space with the property $\mathcal{P}$. We will show that some consequences of these results are well-known difficult theorems of Gary Gruenhage.

Recall that, for an infinite cardinal κ, a space X is called *κ-monolithic* if $nw(\overline{A}) \leq \kappa$ for every set $A \subset X$ with $|A| \leq \kappa$. The space X is *monolithic* if it is κ-monolithic for any infinite κ. These concepts, introduced by A. V. Arhangel'skii in [5], proved to be very useful both for the theory of cardinal invariants and C_p-theory.

V. V. Tkachuk introduced in [18] the concept of a (strongly) monotonically monolithic space and proved that every subspace of a monotonically monolithic space has the D-property. It was also shown in [18] that $C_p(X)$ is monotonically monolithic for any Lindelöf Σ-space X and every space with a point-countable base is strongly monotonically monolithic. In particular, any metrizable space is strongly monotonically monolithic. Therefore, the class of monotonically monolithic spaces is reasonably large; it was also established in [18] that monotonic monolithity is preserved by countable products, subspaces, and closed maps. In [1] monotonic κ-monolithity was introduced for any infinite cardinal κ; it was proved, in particular, that monotonic κ-monolithity is preserved by countable products and σ-products.

It was asked in [1] whether every monotonically ω-monolithic compact space is Corson compact. Gruenhage, in [12], gave a positive answer to this question; he also considered a property introduced by P. J. Collins and A. W. Roscoe in [9]. It turned out that every Gul'ko compact has the Collins–Roscoe property which, in turn, implies its monotonic monolithity, so it was asked in [12] whether these two properties coincide.

Tkachuk studied the Collins–Roscoe property systematically in [19]. Answering a question of Gruenhage, he gave an example of a monotonically monolithic space without the Collins–Roscoe property. He also proved that the Collins–Roscoe property is preserved by closed maps, countable products, and σ-products and established that if a Lindelöf Σ-space X has a weakly σ-point-finite T_0-separating family of cozero subsets, then X has the Collins–Roscoe property.

In this paper we prove that both monotonic monolithity and the Collins–Roscoe property are " lifted" by condensations in Lindelöf Σ-spaces and generalize Gruenhage's result on Collins–Roscoe property in Gul'ko compact spaces in the context of condensations into Σ_s-products (see the definition of a Σ_s-product in Section 3 in the introductory text to Theorem 3.2). It turns out that monotonically ω-monolithic space of countable

tightness is monotonically monolithic while any strongly monotonically ω-monolithic space is strongly monotonically monolithic. Answering Question 3.4 of [12], we give an example of a compact space which fails to be Gul'ko compact but has the Collins–Roscoe property.

We also provide a method of construction of Corson compact spaces which are not monotonically ω-monolithic and prove that any strongly monotonically monolithic Lindelöf Σ-space is second countable, generalizing the respective result of J. Chaber (see [7]) on Lindelöf Σ-spaces with a point-countable base.

1. Notation and Terminology

All spaces are assumed to be Tychonoff. Given a space X, the family $\tau(X)$ is its topology; if $x \in X$, then $\tau(x, X) = \{U \in \tau(X) : x \in U\}$; besides, for any set $A \subset X$, we will need the family $\tau(A, X) = \{U \in \tau(X) : A \subset U\}$. We denote by $\mathbb{R}$ the real line with its natural topology and $\mathbb{I} = [0, 1] \subset \mathbb{R}$; furthermore, $\mathbb{D} = \{0, 1\}$ is the doubleton with the discrete topology.

Say that a family $\mathcal{F}$ of subsets of a space X is a network with respect to a cover $\mathcal{C}$ if, for any $C \in \mathcal{C}$ and $U \in \tau(C, X)$, there exists $F \in \mathcal{F}$ such that $C \subset F \subset U$. If $\mathcal{C} = \{\{x\} : x \in X\}$, then a network with respect to $\mathcal{C}$ is called a *network* of X. The network weight $nw(X)$ of a space X is the minimal cardinality of a network in X. A space that has a countable network is called *cosmic*.

A space X is Lindelöf Σ if there exists a countable family $\mathcal{F}$ of subsets of X such that $\mathcal{F}$ is a network with respect to a compact cover $\mathcal{C}$ of the space X. If X is a space, then a family $\mathcal{G}$ of subsets of X is called a *network (base) at a point* $x \in X$ if ($\mathcal{G} \subset \tau(X)$ and), for any $U \in \tau(x, X)$ there exists $G \in \mathcal{G}$ such that $x \in G \subset U$. Given a set A in a space X say that a family $\mathcal{N}$ of subsets of X is an external network (base) of A in X if (all elements of $\mathcal{N}$ are open in X and) $\mathcal{N}$ is a network at every $x \in A$. A family $\mathcal{B}$ is called an *outer base* of a set F in a space X if $F \subset \bigcap \mathcal{B}$ and, for every $U \in \tau(F, X)$, there exists $B \in \mathcal{B}$ such that $B \subset U$.

For an infinite cardinal κ, say that a space X is *(strongly) monotonically κ-monolithic* if, for any set $A \subset X$ with $|A| \leq \kappa$, we can assign an external network (base) $\mathcal{O}(A)$ to the set $\overline{A}$ in such a way that the following conditions are satisfied:

(a) $|\mathcal{O}(A)| \leq \kappa$;
(b) if $A \subset B$, then $\mathcal{O}(A) \subset \mathcal{O}(B)$;
(c) if $\lambda \leq \kappa$ is an ordinal and we have a family $\{A_\alpha : \alpha < \lambda\}$ of subsets of X such that $\alpha < \beta < \lambda$ implies $A_\alpha \subset A_\beta$, then $\mathcal{O}(\bigcup_{\alpha<\lambda} A_\alpha) = \bigcup_{\alpha<\lambda} \mathcal{O}(A_\alpha)$.

A space X is (strongly) monotonically monolithic if it is (strongly) monotonically κ-monolithic for any infinite cardinal κ.

Given a space X, assume that, for every point $x \in X$, a countable family $\mathcal{G}(x)$ of subsets of X is chosen. Say that $\{\mathcal{G}(x) : x \in X\}$ is a (strong) Collins–Roscoe collection if, for any $x \in X$ (we have $\mathcal{G}(x) \subset \tau(X)$) and for each $U \in \tau(x, X)$, we can find an open set V such that $x \in V \subset U$ and, for any $y \in V$, there exists a set $P \in \mathcal{G}(y)$ with $x \in P \subset U$. If a space X has a Collins–Roscoe collection, then we will say that X has the Collins–Roscoe property. Gruenhage proved in [12] that a collection $\{\mathcal{G}(x) : x \in X\}$ of countable families has the (strong) Collins–Roscoe property if and only if, for any set $A \subset X$, the family $\bigcup\{\mathcal{G}(x) : x \in A\}$ is a network (or base, respectively) at every point of $\overline{A}$.

The rest of our terminology is standard and follows from [11].

2. Monotonic Monolithity and Condensations

In general, the fact that a space X has a weaker monotonically monolithic topology does not imply that X must be ω-monolithic. To see that, observe that the Sorgenfrey line is not ω-monolithic while it condenses onto a monotonically monolithic space $\mathbb{R}$. Therefore, even the hereditary Lindelöf property does not help to "lift" monolithity under a condensation. The situation is different if we look at Lindelöf Σ-spaces.

Recall that a space X is called *κ-stable* if, for any continuous image Y of the space X, if Y condenses onto a space of network weight $\leq \kappa$, then $nw(Y) \leq \kappa$. A space is stable if it is κ-stable for any infinite cardinal κ. The following result is an easy consequence of stability of Lindelöf Σ-spaces (see [6, Theorem II.6.21]); however, it seems to be new.

Proposition 2.1. *Given an infinite cardinal κ, suppose that a Lindelöf Σ-space X condenses onto a κ-monolithic space. Then X is κ-monolithic.*

Proof. Suppose that $f : X \to Y$ is a condensation and Y is κ-monolithic. If $A \subset X$ and $|A| \leq \kappa$, then $\overline{A}$ is a Lindelöf Σ-space which condenses into the space $Z = \overline{f(A)}$ with $nw(Z) \leq \kappa$, so we can apply stability of $\overline{A}$ to conclude that $nw(\overline{A}) \leq \kappa$, i.e., X is κ-monolithic. □

Corollary 2.2. *If a Lindelöf Σ-space X condenses onto a monolithic space, then X is monolithic.*

We will see later that the same results can be proved for monotonic monolithity and the Collins–Roscoe property. Our main tool is the following lemma.

Lemma 2.3. *Suppose that $f : X \to Y$ is a condensation of a Lindelöf Σ-space X onto a space Y. Fix a countable network $\mathcal{N}$ with respect to a*

compact cover $\mathcal{C}$ of the space X. Assume that $y \in Y$ and $\mathcal{F}$ is a network at y in Y. Then the family $\mathcal{E} = \{f^{-1}(F) \cap N : F \in \mathcal{F}$ and $N \in \mathcal{N}\}$ is a network at the point $x = f^{-1}(y)$ in X.

Proof. Take any $U \in \tau(x, X)$; there exists $C \in \mathcal{C}$ with $x \in C$. Since $\mathcal{F}$ is a network at y, we can choose a set $F \in \mathcal{F}$ such that $y \in F$ and $\overline{F} \cap f(C \setminus U) = \emptyset$. If $G = f^{-1}(F)$, then $\overline{G} \cap (C \setminus U) = \emptyset$; by normality of X there exists a set $V \in \tau(C \setminus U, X)$ such that $\overline{V} \cap G = \emptyset$. The set $V \cup U$ is an open neighborhood of C in X, so we can find $N \in \mathcal{N}$ for which $C \subset N \subset U \cup V$. It is straightforward that $E = G \cap N \in \mathcal{E}$ and $x \in E \subset U$, so $\mathcal{E}$ is network at the point x. □

Theorem 2.4. *Given an infinite cardinal κ, suppose that a Lindelöf Σ-space X can be condensed onto a monotonically κ-monolithic space. Then the space X is monotonically κ-monolithic.*

Proof. Fix a countable network $\mathcal{N}$ with respect to a compact cover of X and take a condensation $f : X \to Y$ of X onto a monotonically κ-monolithic space Y; let $\mathcal{O}$ be a κ-monolithity operator in Y. Take any set $A \subset X$ with $|A| \leq \kappa$ and let $\mathcal{G}(A) = \{f^{-1}(B) \cap N : N \in \mathcal{N}$ and $B \in \mathcal{O}(f(A))\}$. Then $|\mathcal{G}(A)| \leq \kappa$; it is immediate that the properties (b) and (c) of the definition of monotonic κ-monolithity also hold for the operator $\mathcal{G}$. Now, if $x \in \overline{A}$ and $y = f(x)$, then $\mathcal{O}(f(A))$ is a network at the point y, so we can apply Lemma 2.3 to conclude that the family $\mathcal{G}(A)$ is a network at the point x, i.e., $\mathcal{G}$ is a monotonic κ-monolithity operator on X. □

Corollary 2.5. *If a Lindelöf Σ-space X condenses onto a monotonically monolithic space, then X is monotonically monolithic.*

The following proposition generalizes Lemma 3.26 from [10] and gives another method for constructing a Corson compact space which is not monotonically ω-monolithic (see [12, Example 2.3]).

Proposition 2.6. *If X is a monotonically ω-monolithic perfectly normal compact space, then X is metrizable.*

Proof. Observe that X must be Corson compact by [12, Corollary 2.2], but there exist, at least consistently, perfectly normal Corson non-metrizable compact spaces. Let $\mathcal{O}$ be an operator that witnesses the monotonic ω-monolithity of X. There is no loss of generality to assume that all elements of $\mathcal{O}(A)$ are closed in X for every $A \subset X$. Choose a countable outer base $\mathcal{B}_F$ for every closed set $F \subset X$. Fix a set $A \subset X$ and observe that the family $\mathcal{G}(A) = \bigcup\{\mathcal{B}_F : F \in \mathcal{O}(A)\} \subset \tau(X)$ is countable. It is standard to verify that the operator $\mathcal{G}$ witnesses the strong monotonic

ω-monolithity of the space X. Therefore, by [1, Theorem 2.6], X must be metrizable. □

Recall that X is called an *L-space* if X is hereditarily Lindelöf and non-separable. Compact L-spaces exist under CH and do not exist under $\mathsf{MA}+\neg\mathsf{CH}$. Gruenhage constructed in [12] a ZFC example of a Corson compact space which is not monotonically ω-monolithic. Gruenhage's construction is very difficult, but if CH is assumed, then it is easy to obtain such an example.

Corollary 2.7. *If X is a compact L-space, then X can be irreducibly mapped onto a Corson compact space which is not monotonically ω-monolithic.*

Proof. Since $t(X) = \omega$, we can apply [4, Theorem 3.2.4] to find a Corson compact space K such that X can be irreducibly mapped onto K. If K is monotonically ω-monolithic, then it is metrizable by Proposition 2.6; every closed irreducible preimage of a separable space is separable so X is separable, which is a contradiction. □

Gruenhage proved the following characterization of monotonic monolithity.

Theorem 2.8 ([12, Theorem 3.2])**.** *An arbitrary space X is (strongly) monotonically monolithic if and only if, for any finite set $H \subset X$, we can choose a countable family $\mathcal{A}_H$ of (open) subsets of X such that, for every $A \subset X$, the family $\bigcup\{\mathcal{A}_H : H \in [A]^{<\omega}\}$ is an external network for $\overline{A}$.*

We will apply Theorem 2.8 to show that in some cases monotonic κ-monolithity implies monotonic monolithity. It is a folklore fact that an ω-monolithic space of countable tightness is monolithic. Our purpose is to show that an analogous result holds for monotonic ω-monolithity.

Lemma 2.9. *Let $\mathcal{O}$ be an operator of (strong) monotonic κ-monolithity on a space X. For any finite set $H \subset X$, let $\mathcal{A}_H = \mathcal{O}(H)$ and consider the family $\mathcal{G}(A) = \bigcup\{\mathcal{A}_H : H \in [A]^{<\omega}\}$ for every set $A \subset X$. Then $\mathcal{G}(A)$ coincides with $\mathcal{O}(A)$ for every $A \subset X$ with $|A| \leq \kappa$.*

Proof. It is immediate that $\mathcal{G}(A) \subset \mathcal{O}(A)$ for any $A \subset X$ with $|A| \leq \kappa$, so we need only to show that $\mathcal{O}(A) \subset \mathcal{G}(A)$. Proceeding by transfinite induction, assume first that $|A| \leq \omega$; then it is possible to represent A as $\bigcup\{H_n : n \in \omega\}$ where the set H_n is finite and $H_n \subset H_{n+1}$ for every $n \in \omega$. It follows from the properties of the operator $\mathcal{O}$ that $\mathcal{O}(A) = \bigcup\{\mathcal{O}(H_n) : n \in \omega\} = \bigcup\{\mathcal{A}_{H_n} : n \in \omega\} \subset \mathcal{G}(A)$.

Now assume that $\lambda \leq \kappa$ is a cardinal and we proved, for any $\mu < \lambda$, that if $A \subset X$ and $|A| \leq \mu$, then $\mathcal{O}(A) \subset \mathcal{G}(A)$. Take any set $A \subset X$

with $|A| = \lambda$; we can find an increasing λ-sequence $\{B_\alpha : \alpha < \lambda\}$ such that $A = \bigcup\{B_\alpha : \alpha < \lambda\}$ and $|B_\alpha| < \lambda$ for any $\alpha < \lambda$. If $E \in \mathcal{O}(A)$, then there exists $\alpha < \lambda$ such that $E \in \mathcal{O}(B_\alpha)$. By our induction hypothesis, $\mathcal{O}(B_\alpha) \subset \mathcal{G}(B_\alpha) \subset \mathcal{G}(A)$, so $E \in \mathcal{G}(A)$, i.e., $\mathcal{O}(A) \subset \mathcal{G}(A)$. □

Theorem 2.10. *Given an infinite cardinal κ suppose that X is a monotonically κ-monolithic space such that $t(X) \leq \kappa$. Then X is monotonically monolithic.*

Proof. Take an operator $\mathcal{O}$ that witnesses monotonic κ-monolithity of X. Given a finite set $H \subset X$, let $\mathcal{A}_H = \mathcal{O}(H)$; this makes it possible to define the family $\mathcal{G}(A) = \bigcup\{\mathcal{A}_H : H \in [A]^{<\omega}\}$ for every $A \subset X$.

Now take an arbitrary set $A \subset X$, a point $x \in \overline{A}$ and $U \in \tau(x, X)$. There exists a set $B \subset A$ with $|B| \leq \kappa$ such that $x \in \overline{B}$. The family $\mathcal{O}(B)$ is an external network for the set $\overline{B} \ni x$ so there exists $G \in \mathcal{O}(B)$ with $x \in G \subset U$. It follows from Lemma 2.9 that $G \in \mathcal{G}(B) \subset \mathcal{G}(A)$, and hence $\mathcal{G}(A)$ is an external network for $\overline{A}$. Finally, apply Theorem 2.8 to conclude that X is monotonically monolithic. □

Corollary 2.11. *If X is a monotonically ω-monolithic countably compact space, then X is a monotonically monolithic Corson compact space.*

Proof. A countably compact monotonically ω-monolithic space is compact by [1, Corollary 2.24], so X is a monotonically monolithic Corson compact space by Theorem 2.10 and [12, Corollary 2.2]. □

Theorem 2.12. *A space X is strongly monotonically ω-monolithic if and only if it is strongly monotonically monolithic.*

Proof. We must only prove necessity, so assume that X strongly monotonically ω-monolithic. Then $\chi(X) \leq \omega$, and hence $t(X) \leq \omega$. Take an operator $\mathcal{O}$ that witnesses strong monotonic ω-monolithity of X. Let $\mathcal{A}_H = \mathcal{O}(H)$ for any finite $H \subset X$ and consider the family $\mathcal{G}(A) = \bigcup\{\mathcal{A}_H : H \in [A]^{<\omega}\}$ for every set $A \subset X$. Fix any set $A \subset X$, a point $x \in \overline{A}$, and $U \in \tau(x, X)$. There exists a countable set $B \subset A$ with $x \in \overline{B}$. The family $\mathcal{O}(B)$ is an external base for the set $\overline{B} \ni x$ so there exists $G \in \mathcal{O}(B)$ with $x \in G \subset U$. It follows from Lemma 2.9 that $G \in \mathcal{G}(B) \subset \mathcal{G}(A)$, and hence $\mathcal{G}(A)$ is an external base for $\overline{A}$. Finally, apply Theorem 2.8 to conclude that X is strongly monotonically monolithic. □

It follows from a general result of Chaber [7] that every Lindelöf Σ-space with a point-countable base is second countable. The following theorem shows that the assumption on a point-countable base can be weakened to strong monotonic monolithity.

Theorem 2.13. *If X is a strongly monotonically monolithic Lindelöf Σ-space, then X is second countable.*

Proof. Let $\mathcal{O}$ be a strong monotonic monolithity operator on X and fix a countable network $\mathcal{N}$ with respect to a compact cover $\mathcal{C}$ of the space X. Take an arbitrary point $x_0 \in X$ and let $A_0 = \{x_0\}$. Proceeding by induction, assume that we have countable subsets $A_0, \dots, A_n$ of the space X with the following properties:

(1) $A_0 \subset \dots \subset A_n$;
(2) for every $i < n$, if there exists a finite family $\mathcal{V} \subset \mathcal{O}(A_i)$ and $N \in \mathcal{N}$ such that $N \setminus (\bigcup \mathcal{V}) \neq \emptyset$, then $(N \setminus (\bigcup \mathcal{V})) \cap A_{i+1} \neq \emptyset$.

For every finite $\mathcal{V} \subset \mathcal{O}(A_n)$, if $N \in \mathcal{N}$ and $N \setminus \bigcup \mathcal{V} \neq \emptyset$, then choose a point $a(\mathcal{V}, N) \in N \setminus \bigcup \mathcal{V}$. Let $A_{n+1} = A_n \cup \{a(\mathcal{V}, N) : \mathcal{V}$ is a finite subfamily of $\mathcal{O}(A_n)$ and, for $N \in \mathcal{N}$, we have $N \setminus \bigcup \mathcal{V} \neq \emptyset\}$. It is immediate that conditions (1) and (2) are now satisfied for all $i < n+1$, so we can construct a sequence $\{A_i : i \in \}$ of countable subsets of X such that (1) and (2) hold for all $n \in \omega$.

To see that $A = \bigcup\{A_n : n \in \omega\}$ is dense in X suppose not and pick a point $x \in X \setminus \overline{A}$. There exists $C \in \mathcal{C}$ with $x \in C$. The set $K = C \cap \overline{A}$ is compact and the family $\mathcal{O}(A)$ is an external base at all points of $\overline{A}$, so we can choose a finite family $\mathcal{V} \subset \mathcal{O}(A)$ such that $x \notin \bigcup \mathcal{V}$ and $K \subset \bigcup \mathcal{V}$. The set $G = (\bigcup \mathcal{V}) \cup (X \setminus \overline{A})$ is an open neighborhood of C so there exists $N \in \mathcal{N}$ such that $C \subset N \subset G$.

Since $\mathcal{O}(A) = \bigcup\{\mathcal{O}(A_n) : n \in \omega\}$ and the family $\{\mathcal{O}(A_n) : n \in \omega\}$ is increasing, we can find $n \in \omega$ such that $\mathcal{V} \subset \mathcal{O}(A_n)$. Observe that $x \in N \setminus (\bigcup \mathcal{V})$, so the point $y = a(\mathcal{V}, N)$ must belong to A_{n+1}. However, $y \in N \setminus (\bigcup \mathcal{V}) \subset X \setminus \overline{A}$, so $y \notin \overline{A}$, which is a contradiction. Therefore, A is a dense subset of X, so $\mathcal{O}(A)$ is a countable base of X. □

3. Collins–Roscoe Property and Condensations

It turns out that the Collins–Roscoe property is also "lifted" by condensations of Lindelöf Σ-spaces. This makes it possible to generalize Gruenhage's theorem on Collins–Roscoe property of Gul'ko compact spaces in the context of their mappings into Σ_s-products.

Theorem 3.1. *Suppose that a Lindelöf Σ-space X condenses onto a Collins–Roscoe space. Then X is a Collins–Roscoe space.*

Proof. Fix a countable network $\mathcal{N}$ with respect to a compact cover of the space X and take a condensation $f : X \to Y$ of X onto a Collins–Roscoe space Y; let $\{\mathcal{O}_y : y \in Y\}$ be the respective Collins–Roscoe family in Y. For any point $x \in X$, the family $\mathcal{G}(x) = \{f^{-1}(F) \cap N : F \in \mathcal{O}_{f(x)}$ and

$N \in \mathcal{N}\}$ is countable. Take any set $A \subset X$; if $x \in \overline{A}$ and $y = f(x)$, then $\mathcal{E} = \bigcup\{\mathcal{O}_{f(z)} : z \in A\}$ is a network at the point $y = f(x) \in \overline{f(A)}$, so we can apply Lemma 2.3 to conclude that the family $\{f^{-1}(F) \cap N : F \in \mathcal{E}$ and $N \in \mathcal{N}\} = \bigcup\{\mathcal{G}(z) : z \in A\}$ is a network at the point x, i.e., $\{\mathcal{G}(x) : x \in X\}$ is a Collins–Roscoe family on X. □

The concept of Σ_s-product was introduced in [17] by G. A. Sokolov who proved that a compact space X is Gul'ko compact (i.e., $C_p(X)$ is a Lindelöf Σ-space) if and only if X embeds into a Σ_s-product of real lines.

Given a family of spaces $\{X_t : t \in T\}$, suppose that $s = \{T_n : n \in \omega\}$ is a sequence of subsets of T; let $X = \prod_{t\in T} X_t$ and fix a point $a \in X$. Given any $x \in X$, let $\operatorname{supp}(x) = \{t \in T : x(t) \neq a(t)\}$ and $N_x = \{n \in \omega : |\operatorname{supp}(x) \cap T_n| < \omega\}$. Then the set $S = \{x \in X : T = \bigcup\{T_n : n \in N_x\}\}$ is called the Σ_s-product centered at a with respect to the sequence s.

Theorem 3.2. *Any Σ_s-product of compact spaces is a Lindelöf Σ-space.*

Proof. Take a family $\{K_t : t \in T\}$ of compact spaces and an arbitrary sequence $s = \{T_n : n \in \omega\}$ of subsets of T; let $K = \prod_{t\in T} K_t$ and fix a point $a \in K$. Given any $x \in K$, let $\operatorname{supp}(x) = \{t \in T : x(t) \neq a(t)\}$ and $N_x = \{n \in \omega : |\operatorname{supp}(x) \cap T_n| < \omega\}$. We must prove that $S = \{x \in K : T = \bigcup\{T_n : n \in N_x\}\}$ is a Lindelöf Σ-space.

Given any set $E \subset T$, let $K_E = \prod\{K_t : t \in E\}$ and define a point $a_E \in K_E$ by the equality $a_E(t) = a(t)$ for every $t \in E$. For each $n \in \omega$, consider the σ-product $E_n = \{x \in K_{T_n} : |\{t \in T_n : x(t) \neq a(t)\}| < \omega\}$; it is standard to see that E_n is σ-compact so the set $H_n = E_n \times K_{T\setminus T_n}$ is also σ-compact. Fix any $x \in S$ and $y \in K \setminus S$; there exists $t \in T$ such that $t \notin \bigcup\{T_n : n \in N_y\}$. Take $n \in N_x$ with $t \in T_n$; then $\operatorname{supp}(y) \cap T_n$ is infinite and $\operatorname{supp}(x) \cap T_n$ is finite which shows that $x \in H_n$, while $y \notin H_n$. Thus, the family $\{H_n : n \in \omega\}$ of σ-compact subsets separates the points of S from the points of $K \setminus S$, so S is a Lindelöf Σ-space. □

A family $\mathcal{U}$ of subsets of a space X is called weakly σ-point-finite if there exists a sequence $\{\mathcal{U}_n : n \in \omega\}$ of subfamilies of $\mathcal{U}$ such that, for every $x \in X$, we have the equality $\mathcal{U} = \bigcup\{\mathcal{U}_n : \mathcal{U}_n$ has finite order at the point $x\}$. The family $\mathcal{U}$ is said to *T_0-separate the points of X* if, for any distinct points $x, y \in X$, there exists $U \in \mathcal{U}$ such that $U \cap \{x, y\}$ is a singleton.

Lemma 3.3. *Any Σ_s-product S of spaces of countable i-weight has a weakly σ-point-finite family of cozero sets that T_0-separates the points of S.*

Proof. Observe first that the existence of a weakly σ-point-finite T_0-separating family of cozero sets is preserved by stronger topologies. Every

Σ_s-product of spaces of countable i-weight condenses into a Σ_s-product of spaces of countable weight, so it suffices to prove our lemma for second countable spaces.

To do so, take a family $\{X_t : t \in T\}$ of second countable spaces and a sequence $s = \{T_n : n \in \omega\}$ of subsets of T; let $X = \prod_{t\in T} X_t$ and fix a point $a \in X$. Given any $x \in X$, let $\mathrm{supp}(x) = \{t \in T : x(t) \neq a(t)\}$ and $N_x = \{n \in \omega : |\mathrm{supp}(x) \cap T_n| < \omega\}$. We must show that the space $S = \{x \in X : T = \bigcup\{T_n : n \in N_x\}\}$ has a weakly σ-point-finite T_0-separating family of cozero sets.

Given any $t \in T$, let $\mathcal{B}_t$ be a countable base in $X_t \setminus \{a(t)\}$ such that $a(t) \notin \overline{U}$ for any $U \in \mathcal{B}_t$. Choose an enumeration $\{B^t_n : n \in \omega\}$ of the family $\mathcal{B}_t$. Denote by p_t the projection of X onto X_t and observe that the family $\{p_t^{-1}(B^t_n) : t \in T,\ n \in \omega\}$ is T_0-separating in X and consists of cozero sets. It is easy to see that the countable union of weakly σ-point-finite families is weakly σ-point-finite, so it suffices to show that the family $\mathcal{U}_m = \{p_t^{-1}(B^t_m) \cap S : t \in T\}$ is weakly σ-point-finite for every $m \in$.

We have $\mathcal{U}_m = \bigcup\{\mathcal{V}_n : n \in \omega\}$ where $\mathcal{V}_n = \{p_t^{-1}(B^t_m) \cap S : t \in T_n\}$ for each $n \in \omega$; let us show that this decomposition of $\mathcal{U}_m$ witnesses that it is weakly σ-point-finite. Take any $x \in S$, then $T = \bigcup\{T_n : n \in N_x\}$ which shows that $\mathcal{U}_m = \bigcup\{\mathcal{V}_n : n \in N_x\}$. For any $n \in N_x$, the set $Q = \{t \in T_n : x(t) \neq a(t)\}$ is finite; if $x \in p_t^{-1}(B^t_m)$, then $x(t) = p_t(x) \in B^t_m \subset X_t \setminus \{a(t)\}$ which shows that $x(t) \neq a(t)$, and therefore $t \in Q$. This proves that the cardinality of the family $\{V \in \mathcal{V}_n : x \in V\}$ does not exceed the number of the elements of the set $\{t \in T_n : x \in p_t^{-1}(B^t_m)\} \subset Q$, so $\mathcal{V}_n$ has a finite order at x, and hence the family $\mathcal{U}_m$ is weakly σ-point-finite for every $m \in \omega$. □

Corollary 3.4. *Any Σ_s-product of second countable spaces has the Collins–Roscoe property.*

Proof. Take a family $\{X_t : t \in T\}$ of second countable spaces and a sequence $s = \{T_n : n \in \omega\}$ of subsets of T; let $X = \prod_{t\in T} X_t$ and fix a point $a \in X$. Given any $x \in X$, let $\mathrm{supp}(x) = \{t \in T : x(t) \neq a(t)\}$ and $N_x = \{n \in \omega : |\mathrm{supp}(x) \cap T_n| < \omega\}$. We have to prove that $S = \{x \in X : T = \bigcup\{T_n : n \in N_x\}\}$ has the Collins–Roscoe property.

Choose a second countable compactification K_t of the space X_t for all $t \in T$. Denote by K the space $\prod_{t\in T} K_t$ and let S_K be the Σ_s-product in K centered at a with respect to the same sequence s. The space S_K has the Lindelöf Σ-property by Theorem 3.2; observe that Lemma 3.3 implies that S_K has a weakly σ-point-finite T_0-separating family of cozero sets, so we can apply [19, Theorem 2.14] to see that S_K is a Collins–Roscoe space. Since $S \subset S_K$, the space S also has the Collins–Roscoe property. □

A compact space X is Gul'ko compact if and only if it embeds into a Σ_s-product of real lines; therefore, the following corollary gives one more generalization of Gruenhage's result on the Collins–Roscoe property of Gul'ko compact spaces.

Corollary 3.5. *If a Lindelöf Σ-space X condenses into a Σ_s-product of spaces of countable i-weight, then X has the Collins–Roscoe property.*

Proof. Since every Σ_s-product of spaces of countable i-weight can be condensed into a Σ_s-product of spaces of countable weight, we can apply Theorem 3.1 and Corollary 3.4. □

It is still an open question whether every strong Collins–Roscoe space has a point-countable base. The Corollary to Theorem 5 of [14, p. 76] establishes that every strong Collins–Roscoe space X has a dense subspace Y with a point-countable base. It is possible to extract from the proof of Theorem 5 of [14] that Y actually has a point-countable external base $\mathcal{B}$ in X. It is clear that $\mathcal{B}$ is a point-countable π-base of X.

We will show that a weaker condition suffices to derive existence of a point-countable π-base. Given a space X and a family $\mathcal{A}$ of subsets of X say that a family $\mathcal{E} = \{O_A : A \in \mathcal{A}\}$ is an *open expansion of* $\mathcal{A}$ if $\mathcal{E} \subset \tau(X)$ and $A \subset O_A$ for any $A \in \mathcal{A}$. A family $\mathcal{E}$ is an open expansion of a set $A \subset X$ if $\mathcal{E} = \{U_x : x \in A\}$ and $U_x \in \tau(x, X)$ for each $x \in A$.

Theorem 3.6. *Let X be a Collins–Roscoe space. Then every left-separated subspace $Y \subset X$ has a point-countable open expansion.*

Proof. Let $\{\mathcal{P}_x : x \in X\}$ be a Collins–Roscoe collection in X. Take a well-order $<$ on Y which witnesses that Y is left-separated. For every $y \in Y$ the set $L_y = \{x \in Y : x < y\}$ is closed in Y, so we can find a set $U_y \in \tau(y, X)$ such that, for every point $z \in U_y$, there exists $P \in \mathcal{P}_z$ with $y \in P$ and $P \cap \overline{L}_y = \emptyset$. The family $\mathcal{U} = \{U_y : y \in Y\}$ is an open expansion of Y; we claim that $\mathcal{U}$ is point-countable.

To see this, suppose that $x \in X$ and there exists an uncountable set $A \subset Y$ such that $x \in \bigcap\{U_y : y \in A\}$. For each $y \in A$ there is a set $P_y \in \mathcal{P}_x$ such that $y \in P_y \subset X \setminus \overline{L}_y$. The family $\mathcal{P}_x$ being countable, we can find $y, z \in A$ such that $y < z$ and $P_y = P_z$. Observe that $y \in \overline{L}_z$ while $y \in P_y = P_z \subset X \setminus \overline{L}_z$, which is a contradiction. □

Corollary 3.7. *If X is a Collins–Roscoe space and $\pi\chi(X) \leq \omega$, then X has a point-countable π-base.*

Proof. Take a dense left-separated subspace Y of the space X and fix a countable π-base $\mathcal{B}_x$ at every point $x \in X$. By Theorem 3.6 we can find a point-countable open expansion $\{U_x : x \in Y\}$ of the set Y in X. It is easy

to see that the family $\mathcal{B} = \{B \cap U_x : x \in Y,\ B \in \mathcal{B}_x, \text{ and } B \cap U_x \neq \emptyset\}$ is a point-countable π-base of X. □

Our next statement is an easy consequence of Corollary 3.7; it is worth noting that in [8] it was mentioned after Theorem 14 that every strong Collins–Roscoe space has a dense subspace with an external point-countable base, so the following corollary was known to the authors.

Corollary 3.8. *Every strong Collins–Roscoe space has a point-countable π-base.*

Proof. If X is a strong Collins–Roscoe space, then $\chi(X) \leq \omega$, so Corollary 3.7 is applicable. □

Corollary 3.9. *Suppose that X is a Collins–Roscoe space and $s(X) \leq \omega_1$. If $\pi\chi(X) \leq \omega$, then $d(X) \leq \omega_1$.*

Proof. By Corollary 3.7, we can find a point-countable base $\mathcal{B}$ in the space X. It is a theorem of B. É. Shapirovskiĭ (see [16, Lemma 3.1]) that there exists a family $\mathcal{D} = \{D_\alpha : \alpha < \omega_1\}$ of discrete subspaces of X such that $B \cap (\bigcup \mathcal{D}) \neq \emptyset$ for any $B \in \mathcal{B}$. In particular, the set $D = \bigcup\{D_\alpha : \alpha < \omega_1\}$ is dense in X. It follows from $s(X) \leq \omega_1$ that $|D| \leq \omega_1$, so $d(X) \leq \omega_1$. □

Corollary 3.10. *Suppose that X is a hereditarily Lindelöf strong Collins–Roscoe space. Then X has a point-countable base.*

Proof. We have the inequalities $s(X) \leq hl(X) \leq \omega$ and $\chi(X) \leq \omega$ which show that Corollary 3.9 is applicable to conclude that $d(X) \leq \omega_1$, and hence the space X has a point-countable base by [14, Theorem 7]. □

In [10], A. Dow, H. Junnila, and J. Pelant considered several properties implied by the existence of a stronger metric topology on function spaces. Among other things, they introduced spaces with point-countably expandable networks. Recall that a family $\mathcal{A}$ of subsets of a space X is *point-countably expandable* if it has a point-countable open expansion in X. The class of spaces with a point-countably expandable network is important because it contains all Gul'ko compact spaces. The following proposition provides an alternative way to prove that every Gul'ko compact space has the Collins–Roscoe property.

Proposition 3.11. *If a space X has a point-countably expandable network, then X is a Collins–Roscoe space.*

Proof. Suppose that $\mathcal{N}$ is a network in X and $\{O_N : N \in \mathcal{N}\}$ is its point-countable open expansion. Given any point $x \in X$, let $\mathcal{G}(x) = \{N \in \mathcal{N} : x \in O_N\}$; it is clear that the collection $\mathcal{G}(x)$ is countable. Take any set $A \subset X$ and a point $x \in \overline{A}$; for any $U \in \tau(x, X)$, there exists $N \in \mathcal{N}$

such that $x \in N \subset U$. Pick a point $a \in A \cap O_N$. Then $N \in \mathcal{G}(a)$, so the family $\bigcup\{\mathcal{G}(a) : a \in A\}$ contains an external network at all points of $\overline{A}$, and therefore $\{\mathcal{G}(x) : x \in X\}$ is a Collins–Roscoe family. □

Gruenhage proved in [12] that every compact space X which is monotonically ω-monolithic must be Corson compact. Therefore, it is a natural question ([12, Question 3.4]) whether every compact space with the Collins–Roscoe property must be Gul'ko compact. The last result of this paper is an example that answers this question in the negative. Recall that a family $\mathcal{N}$ of subsets of a space X is called *relatively discrete* if, for any $N \in \mathcal{N}$ and $x \in N$, we can find a set $U \in \tau(x, X)$ such that $U \cap M = \emptyset$ for any $M \in \mathcal{N} \setminus \{N\}$. In other words, $\mathcal{N}$ is relatively discrete if it is discrete in $\bigcup\mathcal{N}$. If $\mathcal{N} = \bigcup_{n\in\omega}\mathcal{N}_n$ and every $\mathcal{N}_n$ is relatively discrete, then $\mathcal{N}$ is called *σ-relatively discrete*.

Example 3.12. *There exists a compact space K which is not Gul'ko compact but has the Collins–Roscoe property.*

Proof. We will make use of the space constructed for other purposes by S. Argyros and S. Mercourakis (Lemma 3.4 of [2]), and S. Argyros, S. Mercourakis, and S. Negrepontis (Example 4.4 of [3]). Take a faithfully indexed set $T = \{x_\alpha : \alpha < \omega_1\} \subset \mathbb{I}$. A finite set $A \subset T$ is called *admissible* if $A = \{x_{\alpha_1}, \ldots, x_{\alpha_n}\}$ where $\alpha_1 < \ldots < \alpha_n$ and $|x_{\alpha_i} - x_{\alpha_j}| \leq 1/i$ whenever $1 \leq i \leq j \leq n$.

Let us define an almost disjoint family $\{E_\alpha : \alpha < {}_1\}$ of subsets of ω as follows. Choose a countably infinite set $E_n \subset \omega$ for every $n \in \omega$ in such a way that $\{E_n : n \in \omega\}$ is a partition of ω. If $\beta < \omega_1$ and an almost disjoint family $\{E_\alpha : \alpha < \beta\}$ has been defined, take a faithful enumeration $\{\beta_n : n \in \omega\}$ of the ordinal β and choose a set $F_n \subset E_{\beta_n} \setminus \bigcup\{E_{\beta_i} : i < n\}$ with $|F_n| = n$ for each $n \in \omega$. Finally, let $E_\beta = \bigcup\{F_n : n \in \omega\}$.

Once we have the almost disjoint family $\{E_\alpha : \alpha < {}_1\}$, call a finite set $A \subset T$ *appropriate* if $A = \{x_{\alpha_1}, \ldots, x_{\alpha_n}\}$ where $\alpha_1 < \ldots < \alpha_n$ and we have the inequality $|E_{\alpha_i} \cap E_{\alpha_j}| \geq \max\{i, j - i\}$ whenever $1 \leq i \leq j \leq n$. The family $\mathcal{A} = \{A \subset T :$ every finite subset of A is both admissible and appropriate$\}$ is easily checked to be adequate in the sense of [6, IV.6] so the space $\{\chi_A : A \in \mathcal{A}\} \subset \mathbb{D}^T$ is compact. It was proved in [3, Theorem 4.4] that K is Corson compact but fails to be Gul'ko compact.

L. Oncina and M. Raja (see [15, Lemma 4.6]) proved that the set $F = \{\chi_A : A \in \mathcal{A}$ and $|A| \leq 1\}$ is Eberlein compact and the space K_F obtained from K by contracting F to a point is also Eberlein compact. Every Eberlein compact space must have a σ-relatively discrete network [13] so we can find such networks $\mathcal{N}_0$ and $\mathcal{N}_1$ in F and $K \setminus F$, respectively. Then $\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_1$ is a σ-relatively discrete network in K. Any Corson compact space is hereditarily metalindelöf and it is standard that in a hereditarily

metalindelöf space every relatively discrete family has a point-countable open expansion. Therefore, the network $\mathcal{N}$ has a point-countable open expansion so the space K has the Collins–Roscoe property by Proposition 3.11. □

4. Open Questions

Now that we have a considerable amount of information about monotonically monolithic spaces and Collins–Roscoe spaces, it is important to find out what we can expect of the respective properties in the presence of compactness. At the present moment we don't even know whether they coincide in the class of compact spaces.

Question 4.1. Suppose that X is a monotonically ω-monolithic Lindelöf Σ-space. Must X be monotonically monolithic?

Question 4.2. Suppose that X is a monotonically monolithic compact space. Must X have the Collins–Roscoe property?

Question 4.3. Suppose that X is a monotonically monolithic Lindelöf Σ-space. Must X have the Collins–Roscoe property?

Question 4.4. Is it true that every Σ_s-product of Collins–Roscoe spaces has the Collins–Roscoe property?

Question 4.5. Is it true that every Σ_s-product of Collins–Roscoe compact spaces has the Collins–Roscoe property?

Question 4.6. Is it true that every Σ_s-product of monotonically monolithic spaces is monotonically monolithic?

Question 4.7. Is it true that every Σ_s-product of monotonically monolithic compact spaces is monotonically monolithic?

Question 4.8. Suppose that X is a Collins–Roscoe compact space. Must X have a dense metrizable subspace?

Question 4.9. Suppose that X is a monotonically monolithic compact space. Must X have a dense metrizable subspace?

Question 4.10. Suppose that X is a monotonically monolithic compact space with $c(X) \leq \omega$. Must X be metrizable?

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