TOPOLOGY
PROCEEDINGS
Volume 42, 2013
Pages 49–55
http://topology.auburn.edu/tp/

On a Problem of J. Stallings

by

A. Kalapodi and V. Tzannes

Electronically published on July 20, 2012

Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

http://topology.auburn.edu/tp/
TOPOLOGY PROCEEDINGS
Volume 42 (2013)
Pages 49-55
E-Published on July 20, 2012

ON A PROBLEM OF J. STALLINGS

A. KALAPODI AND V. TZANNES

Abstract. Let τ be a maximal connected topology finer than the usual topology on $[a,b]$. Let τ_R be the topology whose subbase consists of the open sets of τ and of the right-closed intervals $(x,y]$; let τ_L be the topology whose subbase consists of the open sets of τ and of the left-closed intervals $[x,y)$. We prove that for every open neighborhood R of b in τ_R and for every open neighborhood L of a in τ_L such that $R\cup L=[a,b]$, it holds $R\cap L\neq\emptyset$.

1. Introduction

J. Stallings [9] posed the following question: If τ is a topology on $I=[0,1]$, let τ_R be the topology whose subbase consists of the open sets of τ and of the right-closed intervals $(a,b]$; let τ_L be the topology whose subbase consists of the open sets of τ and of the left-closed intervals $[a,b)$. Suppose that τ is a connected topology for I and that τ is finer than the usual topology for I. Let R and L be subsets of I, $R\cup L=I$, $1\in R$, $0\in L$, R open in τ_R, and L open in τ_L. Is it necessarily true that $R\cap L\neq\emptyset$?

The answer to this question, given by S. K. Hildebrand [6], is negative.

We prove that if on the interval $[a,b]$ the finer than the usual topology is maximal connected, then Stallings' problem has an affirmative answer.

A space X is called (1) maximal if it has no isolated points and every finer topology has isolated points, (2) submaximal [2] if every dense subset is open, (3) extremally disconnected if the closure of every open set is open, and (4) nearly maximal connected [3] if it is connected and for

2010 *Mathematics Subject Classification.* Primary 54D05, 54G05; Secondary 54A10, 54D10.

Key words and phrases. extremally disconnected, maximal, maximal connected, nearly maximal connected, submaximal.

every regular open set V and for every x belonging to the boundary of V there exists an open neighborhood U of x and an open non-trivial set W such that $U \cap V \cap W = \emptyset$ and $W \cup \{x\}$ is closed.

2. Main Result

Let $\mathbb{R}$ be the set of real numbers and τ be a maximal connected topology finer than the usual topology on $\mathbb{R}$. The existence of such topologies is proved by Petr Simon [8] and by J. A. Guthrie, H. E. Stone, and M. L. Wage [5].

Let $x \in \mathbb{R}$ and $U(x)$ be an open neighborhood of x in τ. We set $U^-(x) = (-\infty, x) \cap U(x)$ and $U^+(x) = (x, +\infty) \cap U(x)$. Obviously, both sets $U^-(x)$ and $U^+(x)$ are non-empty.

The following lemma, except for property (2), is a corollary of Theorem 4.2 in [6], where it is proved that on the unit interval with the usual topology, every connected subset remains connected in every finer connected topology. Property (2) is expected by the construction of the extremal topology in [8]. Property (3) follows also by the fact that in a maximal connected space, the intersection of any two connected subsets is connected [7]. The proof here is based on the fact that a connected space is maximal connected if and only if it is nearly maximal connected and submaximal [3].

Lemma 2.1. *In the space* $(\mathbb{R}, \tau)$,

(1) *the interval* $[a, +\infty)$ *(*$(-\infty, a]$*, respectively), for all* $a \in \mathbb{R}$*, is connected;*
(2) *there do not exist disjoint open sets in* $[a, +\infty)$ *(*$(-\infty, a]$*, respectively) having the point* a *as a common accumulation point. Hence, the interval* $(a, +\infty)$ *(*$(-\infty, a)$*, respectively), for all* $a \in \mathbb{R}$*, is connected;*
(3) *the intervals* (a, b)*,* $[a, b), (a, b]$*, and* $[a, b]$*, where* $a < b$ *are connected.*

Proof. (1) Consider the open interval $(-\infty, a)$. Since τ is connected it follows that $Cl_\tau(-\infty, a) = (-\infty, a]$ and $Int_\tau Cl_\tau(-\infty, a) = (-\infty, a)$. Hence, the point a is a boundary point of $(-\infty, a)$. Since τ is nearly maximal connected, it follows that for the point a, there exists an open neighborhood $U(a)$ and an open non-trivial set W such that $U(a) \cap (-\infty, a) \cap W = \emptyset$ and $W \cup \{a\}$ is closed. Hence, the set $W \cup \{a\}$ is connected in τ, and therefore it is connected in the usual topology on $\mathbb{R}$. Since $U^-(a) \cap W = \emptyset$ and $W \cup \{a\}$ is closed, it follows that $W \cup \{a\} = [a, +\infty)$. For if $W \cup \{a\}$ is of the form $[a, x]$ for some $x \in \mathbb{R}$, then, since $(a, x]$ is open in τ, it follows that the set $(-\infty, x]$ is open in τ, which is impossible. Hence, both $(-\infty, a]$ and $[a, +\infty)$ are connected in τ.

(2) Suppose there exist two disjoint open sets U and V in $[a, +\infty)$, both having the point a as a common accumulation point; that is, a is a boundary point of the regular open set $Int_\tau Cl_\tau U$ (and of $Int_\tau Cl_\tau V$). Since every connected subspace of a maximal connected space is maximal connected [4], it follows that $[a, +\infty)$ is maximal connected, and hence it is nearly maximal connected. That is, there exists an open neighborhood $\{a\} \cup U^+(a)$ of a in $[a, +\infty)$ and an open non-trivial set W in $[a, +\infty)$ such that $(\{a\} \cup U^+(a)) \cap U \cap W = \emptyset$ and the set $W \cup \{a\}$ is closed, which is impossible because $W \cup \{a\} = [a, +\infty)$.

(3) That the open interval (a, b) is connected in τ is proved similarly. Obviously, $[a, b), (a, b]$, and $[a, b]$ are also connected. □

Lemma 2.2. *The space* $(\mathbb{R}, \tau_R)$ *(*$(\mathbb{R}, \tau_L)$*, respectively) is maximal.*

Proof. Obviously, $(\mathbb{R}, \tau_R)$ has no isolated points. The topology τ_R is submaximal because every topology finer than a submaximal is submaximal [1]. Since a space is maximal if and only if it is extremally disconnected submaximal without isolated points [2], it is enough to prove that $(\mathbb{R}, \tau_R)$ is extremally disconnected. Let $A \in \tau_R$ where $A \neq \emptyset$. Then $Int_\tau A \neq \emptyset$, for if $Int_\tau A = \emptyset$, then A is closed discrete in τ, hence closed discrete in τ_R, which is impossible. Let $a \in Cl_{\tau_R} A$. Since the set $Cl_{\tau_R} A \setminus A$ is closed discrete in both topologies τ and τ_R, it follows that, for the point a, either there exists an open neighborhood $U(a)$ (in τ) such that $U(a) \setminus \{a\} \subseteq Int_\tau A$ (that is $a \in Int_\tau Cl_\tau Int_\tau A$), or there exists an open neighborhood $U(a)$ and a right-closed interval $(x, a]$ such that $U^-(a) \cap (x, a] \subseteq Int_\tau A$. Hence, in both cases, the set $A \cup \{a\}$ is open in τ_R, and therefore the set $Cl_{\tau_R} A$ is open and closed in τ_R.

Similarly, it is proved that τ_L is maximal. □

Proposition 2.3. *On the interval* $[a, b]$ *let* τ *be a maximal connected topology finer than the usual topology on* $[a, b]$*. Let* τ_R *be the topology whose subbase consists of the open sets in* τ *and of the right-closed intervals* $(x, y]$*. Let* τ_L *be the topology whose subbase consists of the open sets in* τ *and of the left-closed intervals* $[x, y)$*. Then*

(1) *the space* $([a, b], \tau_R)$ *(*$([a, b], \tau_L)$*, respectively) is maximal;*
(2) *if* R *is an open neighborhood of* b *in* τ_R *and* L *is an open neighborhood of* a *in* τ_L *such that* $R \cup L = [a, b]$*, then* $R \cap L \neq \emptyset$*.*

Proof. (1) It is proved as in Lemma 2.2, where now $U(x)$, for every $x \in [a, b]$, denotes the open neighborhood of x in τ on $[a, b]$.

(2) Since all three topologies τ, τ_R, and τ_L on $[a, b]$ are submaximal and $\tau \subseteq \tau_R$ and $\tau \subseteq \tau_L$, it follows that they have the same dense subsets (which are open in all τ, τ_R, and τ_L), and hence the same closed discrete

subsets. Thus, if R is open-dense in τ_R, then R is open-dense in τ_L, and therefore $R \cap L \neq \emptyset$. Similarly, if L is open-dense in τ_L, then $R \cap L \neq \emptyset$.

Now let $R \in \tau_R$ and $L \in \tau_L$ be arbitrary open neighborhoods of b and a, respectively. We set

$$R_\tau = Int_\tau Cl_\tau Int_\tau R, \quad L_\tau = Int_\tau Cl_\tau Int_\tau L$$

and

$$N = Cl_{\tau_R} Int_\tau R \setminus Int_\tau R, \quad M = Cl_{\tau_L} Int_\tau L \setminus Int_\tau L.$$

Every point of N (of M, respectively) will be called right accumulation point of $Int_\tau R$ (left accumulation point of $Int_\tau L$, respectively). Every point of $N \setminus R_\tau$ (of $M \setminus L_\tau$, respectively) will be called strictly-right accumulation point of $Int_\tau R$ (strictly-left accumulation point of $Int_\tau L$, respectively). It is obvious that every point of $R_\tau \setminus Int_\tau R$ (of $L_\tau \setminus Int_\tau L$, respectively) is simultaneously right and left accumulation point of $Int_\tau R$ (of $Int_\tau L$, respectively) and that every accumulation point of $Int_\tau R$ is a strictly-right accumulation point of $Int_\tau R$ if and only if it is a strictly-left accumulation point of $[a, b] \setminus Cl_{\tau_R} Int_\tau R$.

Consequently, since both topologies τ_R and τ_L are also extremally disconnected, we have the following cases for the set R.

$R(1)$. The set R is open in τ.

$R(2)$. The set R is open and closed in τ_R.

$R(3)$. The set $R = Int_\tau R \cup N'$, where $N' \subset N$.

And we have the following corresponding cases for the set L.

$L(1)$. The set L is open in τ.

$L(2)$. The set L is open and closed in τ_L.

$L(3)$. The set $L = Int_\tau L \cup M'$, where $M' \subset M$.

Case $R(1)$ and $L(1)$: Since $[a, b]$ is connected, it follows that $R \cap L \neq \emptyset$.

Case $R(2)$ and $L(1)$: First, we observe that the set R contains a strictly-right accumulation point. For if not, then $R = R_\tau$; that is, R is an open neighborhood of b in τ. Therefore, if $R \cap L = \emptyset$, the interval $[a, b]$ is not connected. Also, R cannot contain a unique strictly-right accumulation point because if x is this point, then the interval $[x, b]$ is not connected. Let x and y, where $x < y$, be two strictly-right accumulation points of R. There exist open neighborhoods $U(x)$ and $V(y)$ of x and y, respectively, such that $U(x) \cap V(y) = \emptyset$, and hence $U^-(x) \subseteq Int_\tau R$, $V^+(y) \subseteq L$. Let $a_1 \in U^-(x)$ and $b_1 \in V^+(y)$. In the interval $[a_1, b_1]$, the set $L \cap [a_1, b_1]$ is an open neighborhood of b_1 in τ, hence open in τ_R, and its complement in $[a_1, b_1]$ is $R \cap [a_1, b_1]$ which cannot be open in τ_L, unless it has no strictly-right accumulation points which then implies that $[a_1, b_1]$ is not connected.

Case $R(3)$ and $L(1)$: Let $x \in N \setminus N'$. Then either

(i) $x \in R_\tau \setminus Int_\tau R$, and therefore there exists an open neighborhood $U(x)$ such that $U(x) \subseteq R_\tau$

or

(ii) x is a strictly-right accumulation point of $Int_\tau R$, and therefore there exists an open neighborhood $V(x)$ such that $V^-(x) \subseteq Int_\tau R$.

Since $x \in L$ and L is open in τ, there exists an open neighborhood $W(x)$ such that $W(x) \subseteq L$. Thus, in (i) the point x is isolated in τ and in (ii) the set $V(x) \cap W(x) = \{x\} \cup (V^+(x) \cap W^+(x))$ is open in τ, which is impossible because τ is connected.

Case $R(2)$ and $L(2)$: The set R must contain a strictly-right accumulation point. For if not, then the set L does not have a strictly-left accumulation point. Hence, $R = R_\tau$ and $L = L_\tau$, and therefore if $R \cap L = \emptyset$, then $[a, b]$ is not connected. Let x be a strictly-right accumulation point of R. Since x is a strictly-left accumulation point of L and since L contains all its left accumulation points, it follows that $R \cap L \neq \emptyset$.

Case $R(2)$ and $L(3)$: Let $x \in M'$. If x is a strictly-left accumulation point of $Int_\tau L$, then x is a strictly-right accumulation point of R. Hence, $x \in R$, and therefore $R \cap L \neq \emptyset$. If $Int_\tau L$ has no strictly-left accumulation point, then $M' \subseteq L_\tau \setminus Int_\tau L$. If $M' = L_\tau \setminus Int_\tau L$, then $L = L_\tau$; that is, L is open in τ, and hence this case is reduced to case $R(2)$ and $L(1)$. Let $x \notin L$ and $x \in L_\tau \setminus Int_\tau L$. Hence, if we suppose that $R \cap L = \emptyset$, it follows that $x \in R$, and therefore either

(i) $x \in Int_\tau R$

or

(ii) $x \in R_\tau \setminus Int_\tau R$

or

(iii) x is a strictly-right accumulation point of R.

In (i) and (ii), obviously the point x is isolated in τ.

In (iii), for the point x, there exist open neighborhoods $V(x)$ and $U(x)$ such that $V(x) \setminus \{x\} \subseteq Int_\tau L$ and $U^-(x) \subseteq Int_\tau R$. Therefore, the set $V(x) \cap U(x) = \{x\} \cup (V^+(x) \cap U^+(x))$ is open in τ, which is impossible.

Case $R(3)$ and $L(3)$: Let $x \in M \setminus M'$. Then either

(1) $x \in L_\tau \setminus Int_\tau L$

or

(2) x is a strictly-left accumulation point of $Int_\tau L$.

In either case, if we suppose that $R \cap L = \emptyset$, then, at the same time, $x \in R$, and hence either

(i) $x \in Int_\tau R$

or

(ii) $x \in R_\tau \setminus Int_\tau R$

or

(iii) x is a strictly-right accumulation point of R.

In (1)(i) and (1)(ii), obviously the point x is isolated in τ.

In (1)(iii), for the point x, there exist open neighborhoods $V(x)$ and $U(x)$ such that $V(x) \subseteq L_\tau$ and $U^-(x) \subseteq Int_\tau R$. Hence, the set $V(x) \cap U(x) = \{x\} \cup (V^+(x) \cap U^+(x))$ is open in τ, which is impossible.

In (2)(i), for the point x, there exist open neighborhoods $V(x)$ and $U(x)$ such that $V^+(x) \subseteq Int_\tau L$ and $U(x) \subseteq Int_\tau R$, and in (2)(ii) such that $V^+(x) \subseteq Int_\tau L$ and $U(x) \subseteq R_\tau$. In both these cases the set $V(x) \cap U(x) = \{x\} \cup (V^+(x) \cap U^+(x))$ is open in τ, which is impossible. Therefore, it does not exist $x \in L_\tau \setminus Int_\tau L$.

Consequently, in (2)(iii), $L = Int_\tau L \cup M'$, where now M' contains all the strictly-left accumulation points of $Int_\tau L$ which do not belong to $Int_\tau R$. Since all these points are simultaneously strictly-right accumulation points of $Int_\tau R$, if we remove the set M' from L and we add it in R, it follows that the set $R \cup M'$ is open and closed in τ_R. Therefore, the intersection of $R \cup M'$ and $Int_\tau L$ cannot be empty (Case $R(2)$ and $L(1)$).

The remaining cases are symmetric to the above and their proofs are omitted. □

Acknowledgment. We thank the referee for pointing out a mistake in the proof of Proposition 2.3 as well as for his/her useful suggestions and comments.

References

[1] A. V. Arhangel'skiĭ and P. J. Collins, *On submaximal spaces*, Topology Appl. **64** (1995), no. 3, 219–241.

[2] Nicolas Bourbaki, *Elements of Mathematics. General Topology. Part 1*. Reading, Mass.-London-Don Mills, Ont.: Addison-Wesley Publishing Co., 1966.

[3] Bradd Clark and Victor Schneider, *A characterization of maximal connected spaces and maximal arcwise connected spaces*, Proc. Amer. Math. Soc. **104** (1988), no. 4, 1256–1260.

[4] J. A. Guthrie, D. F. Reynolds, and H. E. Stone, *Connected expansions of topologies*, Bull. Austral. Math. Soc. **9** (1973), 259–265.

[5] J. A. Guthrie, H. E. Stone, and M. L. Wage, *Maximal connected expansions of the reals*, Proc. Amer. Math. Soc. **69** (1978), no. 1, 159–165.

[6] S. K. Hildebrand, *A connected topology for the unit interval*, Fund. Math. **61** (1967), 133–140.

[7] V. Neumann-Lara and R. G. Wilson, *Some properties of essentially connected and maximally connected spaces*, Houston J. Math. **12** (1986), no. 3, 419–429.

[8] Petr Simon, *An example of maximal connected Hausdorff space*, Fund. Math. **100** (1978), no. 2, 157–163.

[9] J. Stallings, *Fixed point theorems for connectivity maps*, Fund. Math. **47** (1959), 249–263.

(Kalapodi) Department of Business Planning and Information Systems; T.E.I. of Patras; Patras 26334, Greece
E-mail address: kalapodi@math.upatras.gr

(Tzannes) Department of Mathematics; University of Patras; Patras 26504, Greece
E-mail address: tzannes@math.upatras.gr