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by

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# GENERALIZED METRIC TOPOLOGIES OF THE EARTH

AKIO KATO

Abstract. We propose four new topologies of the Earth inspired by the propagation of seismic waves. All of them are Lindelöf and stratifiable, but not metrizable. The first one is symmetrizable, but turns out to be neither Fréchet nor simply connected. We simplify this topology to get three locally contractible topologies, one of which is first-countable.

## 1. Introduction

An earthquake propagates by two kinds of seismic waves, the *body waves* and the *surface waves*. The former travels through the Earth, while the latter over the Earth's surface. Both are known to have properties such that the propagation velocity of the body wave tends to increase with depth, but the surface wave is relatively slower than the body wave. We want to define new topologies on the closed unit disc $\overline{\mathbb{D}}$ or the cross section of the Earth so that these topologies characterize, to some extent, the above phenomena. Of course, we must greatly simplify the cross section of the Earth in order to discard geological details (see Figure 1).

We propose that the body wave takes the path of geodesic of the Poincaré disc $\mathbb{D}$ (the open unit disc) with the hyperbolic metric $\rho$ so that the topology of the interior of the Earth is governed by the hyperbolic metric $\rho$. On the other hand, the surface wave travels on the great circle of the surface of the Earth, which is the boundary circle of the cross section of the Earth. So, the surface wave is measured by the usual Euclidean arc length on the boundary circle. Hence, we see that our simplified model $\overline{\mathbb{D}}$ of the Earth has a multi-metric structure, the hyperbolic metric $\rho$ on $\mathbb{D}$ and the Euclidean metric $d$ on the boundary circle $S^1 = \partial\mathbb{D}$. Of course,

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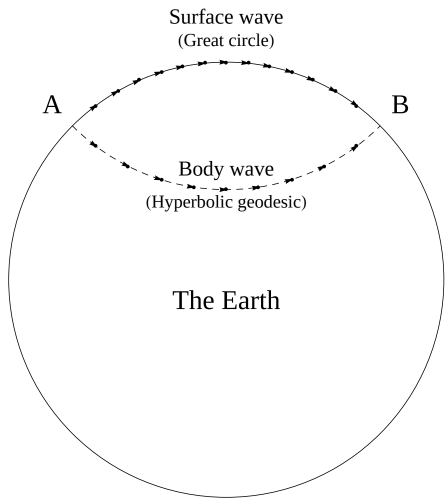


FIGURE 1. Simplified model of the Earth.

the Euclidean metric $d$ is defined not only on the boundary but also on the whole $\overline{\mathbb{D}}$. We want to define new topologies on this closed disc $\overline{\mathbb{D}}$ which will respect both of these two metrics.

Our first example $(X, \tau)$ in section 3 will be Lindelöf, symmetrizable, and stratifiable, but neither Fréchet nor simply connected. We study the structure of this topology $\tau$ in detail in section 4, and then in section 5 we will improve it to get three simplified topologies, all of which are locally contractible, with the last one even being first-countable.

## 2. Hyperbolic Tangent Disc Topology

Throughout this paper, $\rho$ denotes the hyperbolic metric, while $d$ is the Euclidean metric on the plane. Let us recall the definition of the *hyperbolic metric $\rho$ on $\mathbb{D}$* (see [2]), where $\mathbb{D}$ is supposed to be the open unit disc $|z| < 1$ in the complex plane $\mathbb{C}$. For simplicity, we put $X = \overline{\mathbb{D}} = \mathbb{D} \cup S$, where $S = \partial\mathbb{D}$ is the boundary unit circle $|z| = 1$. Any distinct points $z, w \in \mathbb{D}$ are contained in a unique hyperbolic line or a geodesic $L(\alpha, \beta)$ such that $L(\alpha, \beta)$ is an open circular arc (or a diameter) connecting the end points $\alpha, \beta \in S$ and is orthogonal to $S$ at both $\alpha$ and $\beta$. Then the *hyperbolic distance* $\rho(z, w)$ is defined as

$$\rho(z, w) = |\log(\alpha, z, w, \beta)|,$$

where $(\alpha, z, w, \beta)$ denotes the cross ratio $(\frac{w-\alpha}{w-\beta})/(\frac{z-\alpha}{z-\beta})$, and this cross ratio is a positive real since the four points $\alpha$, $z$, $w$, and $\beta$ are on the circle determined by the geodesic $L(\alpha, \beta)$. An important property of this hyperbolic distance is that it is invariant under Möbius transformations. So, for example, if a Möbius map transforms the geodesic $L(\alpha, \beta)$ to the diameter $L(-1, +1)$ so that the four points $\alpha$, $z$, $w$, and $\beta$ are mapped to $-1 < r_1 < r_2 < +1$ correspondingly in this order, then the distance $\rho(z, w)$ can be calculated as

$$(*) \qquad \rho(r_1, r_2) = |\log(-1, r_1, r_2, +1)| = \log\frac{1+r_2}{1-r_2} - \log\frac{1+r_1}{1-r_1}.$$

When $z$ or $w$ approaches the boundary point $\alpha$ or $\beta$, respectively, the distance $\rho(z, w)$ increases to infinity. So it is natural to extend $\rho$ to $X = \overline{\mathbb{D}}$ by defining $\overline{\rho}(x, y) = \infty$ when either of $x$ or $y$ is on the boundary $S$. This extension $\overline{\rho}$, though not a "metric," exhibits the idea that the boundary $S$ is located infinitely far away from inside and that the boundary points are "discretely" apart from each other.

Topologically, inside $\mathbb{D}$, the hyperbolic metric $\rho$ and the Euclidean metric $d$ induce the same topology. But, when we consider the corresponding open balls, we have to care about the positions of their centers. The hyperbolic open ball

$$B_\rho(z_0; r_0) = \{z \in \mathbb{D} : \rho(z_0, z) < r_0\}$$

of hyperbolic center $z_0 \in \mathbb{D}$ and hyperbolic radius $0 < r_0 < \infty$ is, as a set, identical with some Euclidean open ball

$$B_d(z_1; r_1) = \{z \in \mathbb{D} : d(z_1, z) < r_1\}$$

with Euclidean center $z_1 \in \mathbb{D}$ and Euclidean radius $0 < r_1 < 1$, but the hyperbolic center $z_0$ is always shifted away from the Euclidean center $z_1$ towards the boundary $S$; that is, $z_0 = t\,z_1$ for some $t > 1$ if $z_0 \in \mathbb{D}\backslash\{0\}$. So, if $z_0$ approaches the boundary, the concentric hyperbolic balls $B_\rho(z_0; r)$ with radii $0 < r < \infty$ look more like the horocycles or horoballs (See Figure 2).

Taking these facts into account, we introduce a topology $\tau_\rho$ on the closed disc $X = \overline{\mathbb{D}}$ as follows. For a point $x \in S$ and $0 < s < 2$ let $V(x; s)$, denote the union of the point $x$ with the Euclidean open ball, internally tangent at $x \in S$, of diameter of Euclidean length $s$. We call this $V(x; s)$ a *horoball at $x$ of size $s$*. So, the definition itself of horoballs completely depends on the Euclidean metric $d$:

$$V(x; s) = \{x\} \cup B_d((1 - s/2)x;\ s/2).$$

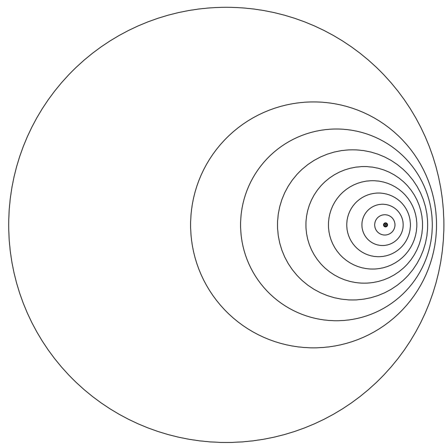

FIGURE 2. Hyperbolically concentric circles.

Define $\tau_\rho$ to be a topology on $X = \mathbb{D} \cup S$ generated by all of these horoballs, plus the usual (either of $\rho$ or $d$) metric topology on $\mathbb{D}$. We call this a *hyperbolic tangent disc topology* induced from the extended "metric" $\overline{\rho}$. This space $(X, \tau_\rho)$ is the union of the Poincaré disc $\mathbb{D}$ with the uncountable discrete closed space $S$, and this tangent disc topology is known to be completely regular Hausdorff, but is a bit wild, that is, not even normal. In the next section we will improve upon this defect.

Let $C_t = \partial V(x;t)$ denote the boundary circle of the horoball $V(x;t)$. Then $V(x;s)$ can be written as $\bigcup_{0<t<s} C_t$, and for $0 < t_1 < t_2 < s$, the hyperbolic distance between $C_{t_1}\setminus\{x\}$ and $C_{t_2}\setminus\{x\}$ is equal with $\rho(1-t_1, 1-t_2)$, and this depends only on $t_1$ and $t_2$ (see Lemma 3.2). Hence, we can see that the horoball $V(x;s)$ consists of the "center" $x$ and the " hyperbolically concentric" circles $C_t$ $(0 < t < s)$. So, suppose an earthquake happened at $x$; then the body wave will spread over $V(x;s)$ until the time $s$, forming the wavefronts $C_t$ $(0 < t < s)$. On the other hand, its surface wave will also propagate from $x$ to some arc $J$ containing $x$ on the boundary $S$. Hence, the set of the form $V(x;s) \cup J$ will represent the region over which the earthquake at $x$ will spread until the time $s$. Taking this fact into account, we will next introduce a new topology.

## 3. Stratifiable Topology

Let $J = a^\frown b$ be an open arc on the boundary $S$ with end points $a$ and $b$, and let $g$ be any function, not necessarily continuous, from the arc $J$

to the interval $(0,1]$. Put

$$V(g) = \bigcup_{x \in J} V(x; g(x)),$$

which is the union of horoballs at $x \in J$ of size $0 < g(x) \leqslant 1$. Define $\tau$ to be a topology on $X = \mathbb{D} \cup S$ generated by the usual topology on $\mathbb{D}$ and all sets of the form $V(g)$ where $J$ and $g$ are chosen arbitrarily (see Figure 3).

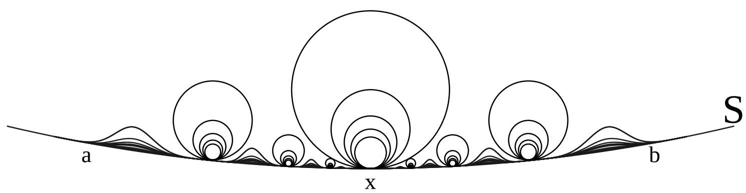


FIGURE 3. Some typical neighborhoods.

Observe that this new topology is coarser than $\tau_\rho$ (horoballs are no longer open), and that $(X, \tau)$ is the union of the Poincaré disc $\mathbb{D}$ and the Euclidean circle $S = S^1$. This topology is one that we wanted, and we will show $(X, \tau)$ is a regular space with a $\sigma$-closure preserving open base, that is, an $M_1$-space, consequently a stratifiable (or $M_3$) space.

**Remark 3.1.** In the theory of generalized metric spaces, a regular space with a $\sigma$-closure preserving open base is called an $M_1$*-space*, while a regular space with a $\sigma$-closure preserving quasi-base is called an $M_3$*-space* or a *stratifiable space*. (A collection $\mathcal{B}$ of subsets of a space $X$ is a *quasi-base* if whenever $x \in U$ with $U$ open, there is some $B \in \mathcal{B}$ with $x \in int(B) \subset B \subset U$.) So, $M_1$ is formally stronger than stratifiable. Though it is not yet known whether these two notions coincide or not, they are known to be equivalent for $F_\sigma$-metrizable spaces, that is, spaces which can be represented as a countable union of closed metrizable subspaces. All of our examples in this paper are $F_\sigma$-metrizable. The class of stratifiable spaces is well known as one of the most useful classes of generalized metric spaces. Indeed, this class is stable under the topological operations, such as taking "subsets," "countable products," and "closed maps." So, for example, we can take any two of our stratifiable examples and paste them together by any closed map; then the resultant space is also stratifiable. See [5], [6].

An alternative construction of this topology $\tau$ can be given using an "adjunction space" as follows. Let $(X, \tau_\rho)$ be the space in section 2 with the tangent disc topology $\tau_\rho$, and let us denote this space as $X_0 = \mathbb{D} \cup S_0$, where $S_0$ denotes the boundary $\partial\mathbb{D}$ with the discrete topology, while keeping the notation $S$ for the usual Euclidean circle. Let $\varphi : S_0 \to S$ be the identity map; then this map $\varphi$ can be seen as a continuous bijection from the discrete closed subspace $S_0$ of $X_0$ onto the unit circle $S$. Make an adjunction space $X_0 \cup_\varphi S$ and observe that this adjunction space topology is identical with our $\tau$. In the literature, various preservation theorems for adjunction spaces are proved, but unfortunately, we can not apply them here because they are mostly of the form that the adjunction space $Y \cup_h Z$ has a property $P$ if both $Y$ and $Z$ have $P$. As for such $P$, "normal," "monotonically normal," and "stratifiable" are known. But in our case of $X_0 \cup_\varphi S$, the space $X_0$ is neither normal nor stratifiable.

We will first show that $\tau$ is regular; this fact is not so obvious. The following lemma is important in showing the connection between the Euclidean and hyperbolic measurements via horoballs.

**Lemma 3.2.** *For any point $x \in S$ and any $0 < s \leqslant 1$, the horoball $V(x; 3^{-1}s)$ is apart from $\mathbb{D} \backslash V(x; s)$ more than $1$ with respect to the hyperbolic metric:*

$$\rho(\, V(x; 3^{-1}s) \backslash \{x\},\ \mathbb{D} \backslash V(x; s)\,) > \log_e 3 = 1.098 \cdots > 1.$$

*Proof.* By rotation we can assume that $x = 1$. Then the distance between $V(x; 3^{-1}s)$ and $\mathbb{D} \backslash V(x; s)$ is measured by $\rho(r_1, r_2)$, where $r_1 = 1 - s$ and $r_2 = 1 - 3^{-1}s$. By the formula $(*)$ in section 2,

$$\rho(r_1, r_2) = \log \frac{1 + r_2}{1 + r_1} + \log \frac{1 - r_1}{1 - r_2}.$$

The first term is positive since $r_2 > r_1$, while the second term is $\log 3$. □

Let $V(g)$ be as defined above.

**Lemma 3.3.** *The open set $V(3^{-1}g)$ is apart from $\mathbb{D} \backslash V(g)$ more than $1$ with respect to the hyperbolic metric.*

*Proof.* By definition, $V(3^{-1}g)$ is the union of $V(x; 3^{-1}g(x))$ for all $x \in J$. Each $V(x; 3^{-1}g(x))$ is, by Lemma 3.2, apart from $\mathbb{D} \backslash V(g)$ more than 1. Consequently, we get Lemma 3.3. □

**Lemma 3.4.** *Let $cl_\tau$ denote the closure in $\tau$. Then*

$$cl_\tau\, V(g) = [J] \cup cl_\rho(V(g) \backslash J),$$

*where $[J]$ is the closed arc $J \cup \{a, b\}$ and $cl_\rho(V(g) \backslash J)$ is the set of all points $z$ in $\mathbb{D}$ such that $\rho(z, V(g) \backslash J) = 0$.*

*Proof.* Put $E = [J] \cup cl_\rho(V(g)\backslash J)$. All we have to show is that $E$ is closed with respect to the topology $\tau$. Indeed, we can even show that $E$ is closed with respect to the Euclidean metric topology. Take any point $z \in X\backslash E$. If $z$ belongs to $\mathbb{D}$, the set $\mathbb{D}\backslash cl_\rho(V(g)\backslash J)$ is obviously a $\rho$-open (hence, $d$-open) neighborhood of $z$ missing $E$. So, suppose $z \in S\backslash [J]$. Let $\overline{V}(x;1)$ denote the horoball $V(x;1)$ of size 1 together with its boundary circle, and put $F = \bigcup\{\overline{V}(x;1) : x \in [J]\}$. Then this $F$ is a Euclidean closed set including the set $V(g)$ and missing $z$. Hence, $X\backslash F$ is a $d$-open neighborhood of $z$ missing $E$. □

**Property 3.5.** *The topology $\tau$ is regular.*

*Proof.* We need to check the regularity only at a point $x$ on the boundary $S$. Let $x \in V(g)$ and $g : J \to (0,1]$. Take an arc $K = c^\frown d$ so that $x \in K \subset [K] = K \cup \{c,d\} \subset J$, and let $h$ be the restriction of $g$ to $K$. By Lemma 3.4, we get

$$x \in V(3^{-1}h) \subseteq cl_\tau\, V(3^{-1}h) = [K] \cup cl_\rho(V(3^{-1}h)\backslash K).$$

Lemma 3.3 implies that $cl_\rho(V(3^{-1}h)\backslash K) \subseteq V(h)$. Hence, $cl_\tau\, V(3^{-1}h)$ is included in $J \cup V(h) \subseteq V(g)$. □

Once we know that $\tau$ is regular, we can derive its nice separation property. Indeed, since $(X,\tau)$ is a countable union of compact sets ($S$ is compact!), it is Lindelöf. Hence, $\tau$ is regular Lindelöf and, consequently, paracompact.

Next we will show that $(X,\tau)$ has a $\sigma$-closure preserving open base. Choose a locally finite cover $\mathcal{B}$ of $\mathbb{D}$ consisting of open balls of diameter $\leqslant 1$ with respect to the hyperbolic metric $\rho$; we fix this open cover of $\mathbb{D}$ hereafter. For any subset $A \subseteq X$, define its enlargement $A^\star$ by $\mathcal{B}$ as

$$A^\star = A \cup \bigcup\{B \in \mathcal{B} : A \cap B \neq \emptyset\}.$$

Note that $V(x;s)^\star = \{x\} \cup (V(x;s)\backslash\{x\})^\star$ and $V(g)^\star = J \cup (V(g)\backslash J)^\star$. Lemma 3.2 and Lemma 3.3 show the inclusions

$$V(x;3^{-1}s) \subseteq V(x;3^{-1}s)^\star \subseteq V(x;s) \text{ and } V(3^{-1}g) \subseteq V(3^{-1}g)^\star \subseteq V(g)$$

hold, and this means that if some collection of $V(g)$'s form a neighborhood base at some point of $S$, so does the corresponding collection of $V(g)^\star$'s. Recall that a collection of sets is called *closure preserving* if for any subcollection, the union of the closures equals the closure of the union. Fix an open arc $J$ on the boundary $S$, and let $\mathcal{V}^\star(J)$ denote the collection of all open sets of the form $V(g)^\star$ where $g$ ranges over all functions from $J$ to the interval $(0,1/3]$. Then we have the following lemma.

**Lemma 3.6.** *$\mathcal{V}^\star(J)$ is closure preserving in $(X,\tau)$.*

*Proof.* Let $\mathcal{G}$ be any collection of functions $g : J \to (0, 1/3]$. We need to show that $cl_\tau(\bigcup_{g\in\mathcal{G}} V(g)^\star)$ is included in $\bigcup_{g\in\mathcal{G}} cl_\tau(V(g)^\star)$. Let $g \in \mathcal{G}$ and $x \in J$. Then, since $V(x; g(x))^\star \subseteq V(x; 3\,g(x)) \subseteq V(x; 1)$, we have that $V(g)^\star \subseteq F$ and $\bigcup_{g\in\mathcal{G}} V(g)^\star \subseteq F$, where $F = \bigcup\{\overline{V}(x; 1) : x \in [J]\}$ is the Euclidean closed set in the proof of Lemma 3.4. Hence, in the same way as in Lemma 3.4, we can show that

$$cl_\tau\,(V(g)^\star) = [J] \cup cl_\rho(V(g)^\star \setminus J)$$

and

$$cl_\tau\,(\bigcup_{g\in\mathcal{G}} V(g)^\star) = [J] \cup cl_\rho(\bigcup_{g\in\mathcal{G}} V(g)^\star \setminus J).$$

So we need only show that the open collection $\{V(g)^\star \setminus J : g \in \mathcal{G}\}$ is closure preserving in $(\mathbb{D}, \rho)$. But this is obvious because $V(g)^\star \setminus J$ consists of elements of a locally finite cover $\mathcal{B}$ of $\mathbb{D}$. $\square$

**Property 3.7.** *The space $(X, \tau)$ has a $\sigma$-closure preserving open base, hence is stratifiable.*

*Proof.* Choose a countable open base $J_i$ $(i \in \omega)$ of $S$ consisting of open arcs. Then $\bigcup_{i\in\omega} \mathcal{V}^\star(J_i)$ forms a neighborhood base for points at $S$, and each $\mathcal{V}^\star(J_i)$ is closure-preserving by Lemma 3.6. Since $(\mathbb{D}, \rho)$ is second-countable, we can take an open base $\bigcup_{i\in\omega} \mathbb{C}_i$ such that $\mathbb{C}_i$ is a singleton. Thus, we get a $\sigma$-closure preserving open base $\bigcup_{i\in\omega} \mathcal{V}^\star(J_i) \cup \bigcup_{i\in\omega} \mathbb{C}_i$ for $\tau$. $\square$

We remark that our space $(X, \tau)$ cannot be represented as an image of any metric space by any closed map. To see this, it is enough to show that our space is not "Fréchet" because "being Fréchet" is preserved under closed maps. Recall that a space $Y$ is called *Fréchet at a point $y \in Y$* if, whenever $A \subseteq Y$ and $y \in cl\,A$, there exists a convergent sequence $a_n$ in $A$ such that $a_n \to y$.

**Property 3.8.** *The space $(X, \tau)$ is not Fréchet, hence not first-countable, at any point on the boundary $S$.*

*Proof.* Take any point $x_0$ on the boundary $S$ and let $\overline{V}(x_0; 1)$ be the closed horoball in the proof of Lemma 3.4. Consider a subset $A = \mathbb{D} \setminus \overline{V}(x_0; 1)$. Then it is obvious that $x_0 \in cl_\tau A$. Suppose there were a $\tau$-convergent sequence $\{a_n\}$ in $A$ such that $a_n \to x_0$. Put $C = \{a_n : n \in \omega\}$. Since this convergent sequence is also a convergent sequence in the coarser Euclidean topology, $C \cup \{x_0\}$ is a Euclidean closed set. Take any open arc $J$ containing the point $x_0$, and for each point $x \in J \setminus \{x_0\}$ choose a horoball $V(x; s_x)$ disjoint from $C \cup \{x_0\}$. Define a function $g : J \to (0, 1]$ as $g(x_0) = 1$ and $g(x) = s_x$ for $x \in J \setminus \{x_0\}$. (This $g$ can be chosen as being continuous on $J\backslash\{x_0\}$.) Then the neighborhood $V(g)$ of $x_0$ misses $C$, a contradiction. $\square$

**Remark 3.9.** In spite of Property 3.8, our space $(X,\tau)$ turns out to be *sequential*, that is, a subset of $X$ is closed if and only if it is sequentially closed. Indeed, we can remark here that $(X,\tau)$ is symmetrizable and that symmetrizable spaces are known to be sequential (see [5], [6]). According to A. V. Arhangel'skiĭ [1], a space $(X,\tau)$ is *symmetrizable* if there is a function $\delta : X \times X \to [0,\infty)$ (not necessarily continuous) such that

(1) $\delta(x,y) = 0$ iff $x = y$;
(2) $\delta(x,y) = \delta(y,x)$;
(3) $U \in \tau$ iff for each $x \in U$ there exists $\epsilon > 0$ such that $x \in B_\delta(x;\epsilon) \subset U$, where $B_\delta(x;\epsilon) = \{z \in X : \delta(x,z) < \epsilon\}$ denotes the "ball" with regard to $\delta$.

Here, note that, because of the lack of the triangle inequality, condition (3) does *not* declare $B_\delta(x;\epsilon) \in \tau$. For our space $X = \mathbb{D} \cup S$, define $\delta(x,y)$ in such a way that in case either $x,y \in \mathbb{D}$ or $x,y \in S$, let $\delta(x,y)$ be the Euclidean distance $d(x,y)$; if $x \in S$ and $y \in \mathbb{D}$, then define $\delta(x,y) = \delta(y,x) = s$, where $0 < s < 2$ is such that the point $y$ lies on the boundary of the horoball $V(x;s)$. It is easy to see that thus defined, $\delta$ satisfies conditions (1), (2), and (3), and that the " ball" at $x \in S$ of "radius" $s$ with regard to $\delta$ is

$$B_\delta(x;s) = V(x;s) \cup J,$$

where $J = a^\frown b, \quad d(a,x) = d(x,b) = s$, which is the same set as we mentioned at the end of section 2.

## 4. Shape of Neighborhoods at the Boundary

Here we study the shape of neighborhoods at points on the boundary $S$. Let us denote by $\tau_d$ the Euclidean topology on the closed disc $\overline{\mathbb{D}}$. If $h_c : J \to (0,1]$ is a constant function of value $c \in (0,1]$, the pillow-shaped neighborhood $V(h_c)$ obviously belongs to the Euclidean $\tau_d$. In general, for a neighborhood $V \in \tau$, we call a point $x \in V \cap S$ a *Euclidean point of* $V$ if $x$ has a Euclidean neighborhood $U \in \tau_d$ contained in $V$, while a point of $V \cap S$ other than Euclidean points, we call *non-Euclidean points of* $V$. We denote the set of all Euclidean points of $V$ by $Ec(V)$ and call it the *Euclidean part of* $V$, while the set of all non-Euclidean points by $Nec(V)$, we call the *non-Euclidean part of* $V$. (Incidentally, "$Nec$"$(V)$ represents the geometrical "neck" of $V$.) Obviously, $Ec(V)$ is an open set in $V \cap S$. We can say more.

**Property 4.1.** *$Ec(V)$ is an open dense subset in $V \cap S$.*

*Proof.* It suffices to prove the case $V = V(g)$ for some $g : J \to (0, 1]$. Put $G_n = g^{-1}((1/n, 1])$. Then $J$ is the countable union of $G_n$'s ($n \geqslant 2$). Since $J$ is a homeomorph of the real line, we can apply the Baire category theorem to get some $m$ such that the closure of $G_m$ has a non-empty interior in $J$. That is, we can find some proper open arc $I$ in $J$ such that $I \cap G_m$ is dense in $I$. By the definition of $G_m$, for every $x \in I \cap G_m$, the horoball $V(x; 1/m)$ of the fixed size $1/m$ is contained in $V(x; g(x)) \subseteq V(g)$. Since $I \cap G_m$ is dense in $I$, the set $I \cup \bigcup\{V(x; 1/m) : x \in I \cap G_m\}$ is the same as the open set $V(h)$ for a constant function $h : I \to (0, 1]$ of value $1/m$. Since $V(h) \in \tau_d$, we have $I \subset Ec(V)$. Now replacing $g$ by its restriction to any subarc of $J$, we can conclude that $Ec(V)$ is open dense in $J$. $\square$

Thus, in other words, the non-Euclidean part $Nec(V)$ is nowhere dense closed in $V \cap S$. If this set were countable, life would be easy. Example 4.3 presents a neighborhood with uncountable non-Euclidean part. For each point $a$ in $S$, let $h_a : S \to [0, 1]$ denote a function such that

$$h_a(x) = (1 - \cos\theta)/2 \text{ where } 0 \leqslant \theta \leqslant \pi \text{ is the arc length of } a^\frown x.$$

Let $J = a^\frown b$ be an open arc of length $< \pi$ on the boundary $S$, and define a function $h_J : J \to (0, 1]$ by $h_a \wedge h_b$, that is, $h_J(x) = \mathrm{Min}\{h_a(x), h_b(x)\}$. We call this function $h_J$ *the ceiling function on* $J$. When a function $h$ is defined on some disjoint union of open arcs and $h$ is the ceiling function on each open arc, we also call such a function $h$ the ceiling function.

**Lemma 4.2.** *$V(h_J)$ is disjoint from $V(a; 1) \cup V(b; 1)$, where $J = a^\frown b$.*

*Proof.* Fix a point $x \in J$ and let $\theta$ be the arc length of $a^\frown x$. By symmetry we need only show that $V(x; h_a(x))$ is disjoint from $V(a; 1)$. Let $B_d(x; \varepsilon)$ be the Euclidean open ball (in the plane) of center $x$ and radius $\varepsilon$, which is tangent with $V(a; 1)$. Then

$$\varepsilon = \sqrt{5/4 - \cos\theta} - 1/2\,,$$

and this $\varepsilon$ is bigger than $(1 - \cos\theta)/2 = h_a(x)$. Hence, the horoball $V(x; h_a(x))$ is contained in $B_d(x; \varepsilon)$, which is disjoint from $V(a; 1)$. $\square$

Let $S_{a,b}$ be the circular sector of the unit disc $\mathbb{D}$ spanned by the arc $J = a^\frown b$, and put

$${}_a\mathrm{A}_b = S_{a,b} \setminus (V(a; 1) \cup V(b; 1) \cup V(h_J)).$$

This set is a curvilinear triangle of the form like "arbelos," and so let us call this ${}_a\mathrm{A}_b$ as the *arbelos* determined by $a$ and $b$. Our definition of topology $\tau$ implies that this ${}_a\mathrm{A}_b$ is closed with regard to $\tau$, but note that this set is not closed with regard to the Euclidean topology $\tau_d$, that is, it includes the two points $a$ and $b$ in its Euclidean closure.

**Example 4.3.** We present a typical example of a neighborhood with uncountable non-Euclidean part. Let $J = a^\frown b$ an open arc of length $< \pi$ on the boundary $S$ and let $C$ be a homeomorph of the Cantor set in $J$. Then the open dense subset $J \backslash C$ of $J$ can be written as a disjoint union $\cup_{n\in\omega} J_n$ of open arcs $J_n = a_n^\frown b_n$. Let $h : J \to (0, 1]$ be a function such that $h$ takes the constant value 1 on $C$ and $h$ is the ceiling function on $J \backslash C$. We show that $Nec(V(h)) = C$. Put $V_C = \bigcup_{y\in C} V(y; 1)$. Since $\{a_n, b_n \,|n \in \omega\}$ is dense in $C$, we have

$$V_C = C \cup \bigcup_{n\in\omega} (V(a_n; 1) \cup V(b_n; 1)).$$

Hence, Lemma 4.2 shows that $V(h)$ forms a disjoint union

$$V(h) = V_C \cup \bigcup_{n\in\omega} V(h_{J_n}).$$

The points $a_n$ and $b_n$ belong to the Euclidean closure of the arbelos ${}_{a_n}\mathrm{A}_{b_n}$, and this arbelos locates outside of $V(h)$. This means that $a_n, b_n \in Nec(V(h))$. Since $\{a_n, b_n \,|n \in \omega\}$ is dense in $C$ and $Nec(V(h))$ is closed, we get $C \subset Nec(V(h))$. Since our function $h$ is continuous on $J \backslash C$, it is obvious that $J \backslash C \subset Ec(V(h))$. Thus, we get $Nec(V(h)) = C$.

Note, in general, that if $V_1 \subset V_2$, then $V_1 \cap Nec(V_2) \subset Nec(V_1)$. Using this fact and Example 4.3, we show the following proposition.

**Property 4.4.** *Every neighborhood open base at every point of $S$ with regard to $\tau$ contains a member which has uncountably many non-Euclidean points.*

*Proof.* Let $\mathcal{V}(x_0) \subset \tau$ be any neighborhood base at $x_0 \in S$. Choose an open arc $J$ of length $< \pi$ and a copy of a Cantor set $C$ such that $x_0 \in C \subset J \subset S$. Consider the neighborhood $V(h)$, $h : J \to (0, 1]$ in Example 4.3 with $C = Nec(V(h))$. Choose any $W \in \mathcal{V}(x_0)$ included in this $V(h)$. Then $W \cap C = W \cap Nec(V(h)) \subset Nec(W)$. Since $W \cap C$ is uncountable, this $W$ has uncountably many non-Euclidean points. $\square$

**Property 4.5.** *Suppose $V \in \tau$ has uncountably many non-Euclidean points. Then $V$ contains a simple closed curve $l$ such that $l \cap S$ is a proper closed arc and the Euclidean domain enclosed by $l$ contains some point of $\mathbb{D} \backslash V$. Therefore, this $V$ is not simply connected.*

*Proof.* Suppose $V \in \tau$ has uncountably many non-Euclidean points. Then, since $V \cap S$ is a union of at most countably many disjoint open arcs, at least one of these open arcs contains uncountably many non-Euclidean points. Therefore, we can assume, without loss of generality, that $C \subset J \subset V(g) \subset V$ and $C \subset Nec(V) \subset Nec(V(g))$ for some Cantor set

$C$ and some function $g : J \to (0,1]$. Let $J \backslash C = \cup_{n \in \omega} a_n^\frown b_n$ and put $C_* = C \backslash \{a_n, b_n : n \in \omega\}$. Points in $C_*$ are usually called *inner points of* $C$. For each inner point $x \in C_*$ consider the horoball $V(x; g(x))$. Since $C_*$ is uncountable, and no uncountable collection of disjoint balls exists in the closed disc, we can find distinct points $x_1, x_2 \in C_*$ such that $V(x_1; g(x_1))$ meets $V(x_2; g(x_2))$. Put $B_i = V(x_i; g(x_i))$, where $i = 1, 2$, and denote by $z_i$ the Euclidean center of $B_i$. Join $z_1$ and $z_2$ by some arc $\widetilde{z_1 z_2}$ inside $B_1 \cup B_2$. Consider the simple closed curve $l$ which connects this $\widetilde{z_1 z_2}$ with $\overline{z_1 x_1}$ (the radius of $B_1$), $x_1^\frown x_2$ (the arc in $S$), and $\overline{z_2 x_2}$ (the radius of $B_2$). This simple closed curve $l$ is contained in $V(g) \subset V$. Consider the domain $D$ in the plane enclosed by $l$. To prove that $V$ is not simply connected, it is sufficient to show that the domain $D$ contains a point outside of $V$. Suppose not; then $x_1^\frown x_2 \cup D \in \tau_d$ is contained in $V$. Hence, $x_1^\frown x_2 \subset Ec(V)$. But, since points $x_1$ and $x_2$ are chosen from $C_*$, the open arc $x_1^\frown x_2$ contains some point $x_3 \in C$, which must be a non-Euclidean point of $V$ because of our choice of $C$. This contradiction shows that the domain $D$ is not entirely contained in $V$. □

**Remark 4.6.** In general, the geometric shape of $V(g)$, where $g : J \to (0,1]$, is completely determined by some countable subset of $J$. Indeed, since the Euclidean open set $V(g) \backslash J$ is Lindelöf, its open cover $\bigcup_{x \in J} (V(x; g(x)) \backslash \{x\})$ has some countable subcover $\bigcup_{i \in \omega} (V(x_i; g(x_i)) \backslash \{x_i\})$. So we have

$$V(g) = J \cup \bigcup_{i \in \omega} V(x_i; g(x_i)).$$

Of course, such a countable subset $\{x_i : i \in \omega\}$ of $J$ depends on $V(g)$. We also note here that, for $V(h_1)$ and $V(h_2)$ where $h_1, h_2 : J \to (0,1]$, the inequality $h_1 \leqslant h_2$ implies $V(h_1) \subseteq V(h_2)$, but the converse is not necessarily true. For example, in the above formula of $V(g)$, define $h : J \to (0,1]$ to be any function such that $h(x_i) = g(x_i)$, where $i \in \omega$, and $0 < h(x) \leqslant g(x)$ for other $x$'s. Then it is always true that $V(h) = V(g)$.

## 5. Simplified Topologies

In the former section we have studied the shape of neighborhoods of the topology $\tau$, and it turned out that their shape is fairly complicated especially when uncountably many non-Euclidean points are involved. Reflecting upon these results, we want to simplify a bit the topology $\tau$. Of course our idea is simply to avoid neighborhoods having uncountably many non-Euclidean points. But we have to be a bit careful technically, because it is possible to find a function $g$ such that the shrunken $V(3^{-1}g)$ has uncountably many non-Euclidean points even though the original $V(g)$ has not. In other words, the collection of $V(g)$'s with at most countably many non-Euclidean points does not behave well for the operation $g \mapsto 3^{-1}g$. To get around this difficulty, we need to examine more about the structure of neighborhoods in $\tau$.

For a function $g : J \to (0,1]$, let $D\langle g\rangle$ denote the *discontinuous part of* $g$, that is, the set of all points $x \in J$ such that $g$ is not continuous at $x$. Since every point where $g$ is continuous belongs to the Euclidean part $Ec(V(g))$, we get

$$Nec(V(g)) \subset D\langle g\rangle.$$

But, in general, these two sets do not coincide, since $D\langle g\rangle$ is not necessarily closed in $J$. So, now we seek some condition which assures that these two sets do coincide. Let us consider a function $g : J \to (0,1]$ such that

$(*)$ $g$ is continuous on some open dense subset $U$ of $J$, and is bounded by the ceiling function $h_U$ on this $U$.

This means that if $U$ is a disjoint union of open arcs $J_n$ ($n \in \omega$), then $g$ is continuous on each $J_n$ and $g \leqslant h_{J_n}$. Of course, $U$ depends on $g$. Let us denote by $\mathbb{G}(*)$ the set of all functions $g$ satisfying $(*)$, where $J$ ranges over all open arcs on $S$, while $\mathbb{G}(J;*)$ denotes all such $g$'s with the fixed $J$.

**Property 5.1.** *$Nec(V(g)) = D\langle g\rangle$ for every $g \in \mathbb{G}(*)$.*

*Proof.* Let $g \in \mathbb{G}(*)$ and put $F = J\backslash U$ where $U$ is as in $(*)$. We can show that $F \subset Nec(V(g))$ in the same way as shown in Example 4.3. Consequently, $F \subset Nec(V(g)) \subset D\langle g\rangle$. On the other hand, since $g$ is continuous on $U$, we have $D\langle g\rangle \cap U = \emptyset$, that is, $D\langle g\rangle \subset F$. Hence, we get $F = Nec(V(g)) = D\langle g\rangle$. □

**Property 5.2.** (1) *For every $f : J \to (0,1]$, we can choose $g \in \mathbb{G}(J;*)$ such that $V(g) \subset V(f)$ and $Nec(V(g)) = Nec(V(f))$, though we cannot always require that $g \leqslant f$.*

(2) *For any $g_1, g_2 \in \mathbb{G}(J;*)$, define $g_1 \wedge g_2(x) = \mathrm{Min}\{g_1(x), g_2(x)\}$; then we have $g_1 \wedge g_2 \in \mathbb{G}(J;*)$ and $V(g_1 \wedge g_2) \subset V(g_1) \cap V(g_2)$.*

*Proof.* (1) Put $U = Ec(V(f))$, and express $U = \cup_{n\in\omega} J_n$ by a disjoint union of open arcs. Let $f_n$ be the restriction of $f$ on $J_n$. Then, since $V(f_n)$ is a Euclidean open set containing $J_n$, we can choose a continuous function $g_n \leqslant h_{J_n}$ such that $J_n \subset V(g_n) \subset V(f_n)$. (We cannot require that $g_n \leqslant f_n$ as $f_n$ is not necessarily continuous on $J_n$.) Let $g$ be a function such that $g = g_n$ on each $J_n$ and $g = f$ outside of $U$. This $g$ is what we wanted.

(2) Let $U_i (i = 1, 2)$ be the open dense set in $(*)$ for $g_i (i = 1, 2)$. Then the open dense set for $g_1 \wedge g_2$ is $U_1 \cap U_2$. Let the ceiling function $h_i (i = 1, 2)$

bound $g_i (i = 1, 2)$ on $U_i (i = 1, 2)$. Then the ceiling function $h_1 \wedge h_2$ on $U_1 \cap U_2$ bounds $g_1 \wedge g_2$. $\square$

Properties 5.1 and 5.2 tell us that $\mathbb{G}(*)$ provides a neighborhood base $\{V(g) : g \in \mathbb{G}(*)\}$ of $\tau$ at points of $S$, with the additional property that the non-Euclidean part of $V(g)$ coincides with the discontinuous part of $g$. Now we are ready to define a simplified topology. From $\mathbb{G}(*)$ choose $g$'s such that

($\star$) $D\langle g \rangle$ is countably infinite,

and denote the set of all such $g$'s by $\mathbb{G}(\text{ctbl})$; the set of $g \in \mathbb{G}(\text{ctbl})$ with the fixed $\text{dom}(g) = J$ will be denoted by $\mathbb{G}(J; \text{ctbl})$. Because of Property 5.1, condition ($\star$) is equivalent to saying

$$D\langle g \rangle \text{ is countably infinite and closed in } \text{dom}(g).$$

Define $\tau(\text{ctbl})$ to be a topology generated by all neighborhoods $V(g)$ such that $g \in \mathbb{G}(\text{ctbl})$ plus the usual topology on the open disc $\mathbb{D}$. This is one of the simplified topologies we have sought. We will show that this modified topology $\tau(\text{ctbl})$ has a structure simpler than $\tau$. First, observe that $g_1, g_2 \in \mathbb{G}(J; \text{ctbl})$ implies $g_1 \wedge g_2 \in \mathbb{G}(J; \text{ctbl})$. This follows from Property 5.2(2) and from the inclusion $D\langle g_1 \wedge g_2 \rangle \subset D\langle g_1 \rangle \cup D\langle g_2 \rangle$ which assures that $D\langle g_1 \wedge g_2 \rangle$ is countable. Hence, $\{V(g) : g \in \mathbb{G}(\text{ctbl})\}$ forms a neighborhood base at points of $S$ for the topology $\tau(\text{ctbl})$. Next, observe that $\mathbb{G}(J; \text{ctbl})$ has the property that if $g \in \mathbb{G}(J; \text{ctbl})$, then $t \cdot g \in \mathbb{G}(J; \text{ctbl}))$ and $D\langle t \cdot g \rangle = D\langle g \rangle$ for any real $t \in (0, 1]$. Due to this property, we can prove, similarly as in section 3, that the space $(X, \tau(\text{ctbl}))$ is also regular and stratifiable.

Now, let $g \in \mathbb{G}(J; \text{ctbl})$ and enumerate $D\langle g \rangle = \{c_n : n \in \omega\}$ by distinct points. Then, by induction, we can choose the sizes $0 < s_n < g(c_n)$ of horoballs so that $V(c_n; s_n)$ $(n \in \omega)$ become disjoint. Let $\widetilde{g}$ be a function which takes the value $s_n$ at $c_n$ and coincides with $g$ on $J \backslash D\langle g \rangle$. Then this $\widetilde{g}$ also belongs to $\mathbb{G}(J; \text{ctbl})$. Letting $J \backslash D\langle g \rangle = \cup_{i \in \omega} J_i$ be a disjoint union of open arcs, the corresponding neighborhood $V(\widetilde{g})$ is expressed in the form of disjoint union

$$\bigcup_{n \in \omega} V(c_n; s_n) \ \cup \ \bigcup_{i \in \omega} V(g \restriction J_i),$$

where each $V(g \restriction J_i)$ is a Euclidean open set. Think of $V(t \cdot \widetilde{g})$, and vary $t$ from 1 to 0; then $V(\widetilde{g})$ contracts to the arc $J$, which obviously contracts to one point. Hence, $V(\widetilde{g})$ is contractible in itself. In other words, we have the property below.

**Property 5.3.** *The space* $(X, \tau(\text{ctbl}))$ *is locally contractible.*

Since any contractible set is simply connected, Property 5.3 contrasts with Property 4.5, and this fact proves that this modified topology $\tau(\mathrm{ctbl})$ is different from $\tau$, truly coarser than $\tau$.

We can consider two more topologies still coarser than $\tau(\mathrm{ctbl})$. First, think of the following condition instead of $(\star)$,

$(\star\star)$ $D\langle g\rangle$ is finite.

Denote the set of all such $g$'s by $\mathbb{G}(\mathrm{fn})$ and let $\tau(\mathrm{fn})$ be the corresponding topology. Obviously, we can replace condition $(\star\star)$ by "$D\langle g\rangle$ is a singleton," and then, at a point $x \in S$, this topology has a neighborhood base consisting of $V(g)$'s such that

$$V(g) = V(x; g(x)) \cup V(g \restriction J_0) \cup V(g \restriction J_1),$$

where $g$ is continuous on $J \backslash \{x\} = J_0 \cup J_1$, and the sets $V(x; g(x))$, $V(g \restriction J_0)$, and $V(g \restriction J_1)$ are disjoint. Similar to the case of $\tau(\mathrm{ctbl})$, we can show that $\tau(\mathrm{fn})$ is regular, Lindelöf, stratifiable, and locally contractible. Note that no countable subset can divide the above $V(g)$ into more than three components, while the neighborhood $V(\widetilde{g})$ described before Property 5.3 has a countably infinite subset which separates $V(\widetilde{g})$ into countably infinite components. Hence, we can conclude that this topology $\tau(\mathrm{fn})$ is strictly coarser than $\tau(\mathrm{ctbl})$.

Next, we will define a first-countable topology coarser than $\tau(\mathrm{fn})$. Let $x \in S$ and $0 < \varepsilon \leqslant 1$. Let $J_{x,\varepsilon}$ be an open arc containing $x$ such that $J_{x,\varepsilon} \backslash \{x\}$ is a disjoint union of two arcs of the same length $\varepsilon/2$. Define a function $h_{x,\varepsilon}$ on $J_{x,\varepsilon}$ by setting $h_{x,\varepsilon}(x) = 1$, and $h_{x,\varepsilon}$ is the ceiling function on $J_{x,\varepsilon} \backslash \{x\}$. Consider a new topology $\tau(\mathrm{N})$ on $X = \overline{\mathbb{D}}$ which is generated by the usual topology on $\mathbb{D}$ and all sets of the form

$$V(t \cdot h_{x,\varepsilon}) \ \text{ for } \ x \in S, \ \ 0 < t, \, \varepsilon \leqslant 1.$$

Since the sets $V(t \cdot h_{x,\varepsilon})$ only for $t = \varepsilon = 1/n$ $(n = 1, 2 \cdots)$ form a neighborhood base at the point $x$, this space $(X, \tau(\mathrm{N}))$ is first-countable. It is easy to see that this space is regular, Lindelöf, and stratifiable. A first-countable, stratifiable space is often called a *Nagata space*; this is why we used the notation $\tau(\mathrm{N})$. Note that the same proof as that for Property 3.8 shows that neither $\tau(\mathrm{ctbl})$ nor $\tau(\mathrm{fn})$ is Fréchet (see Remark 5.5). But the topology $\tau(\mathrm{N})$ is Fréchet because it is first-countable. Nevertheless, we can show the following property.

**Property 5.4.** *The space $(X, \tau(\mathrm{N}))$ cannot be represented as a closed image of a metric space.*

*Proof.* The Hanai–Morita–Stone theorem (see [4, 4.4.17]) shows that any closed image of a metric space is metrizable if and only if it is first-countable. So, we need only show that $\tau(\mathrm{N})$ is not metrizable. Suppose

$\tau(\mathrm{N})$ were metrizable; then, since it is Lindelöf, $\tau(\mathrm{N})$ must be second countable. So, we would have a countable collection $V(t_n \cdot h_{x_n,\varepsilon_n})$ where $x_n \in S$, $0 < t_n$, $\varepsilon_n \leqslant 1$, and $n \in \omega$, which forms an open base at points of $S$ (see [4, 1.1.15]). Choose some point $z \in S$ other than $x_n$'s, and consider its neighborhood $V(h_{z,1})$. Then there must be some $k \in \omega$ such that $z \in V(t_k \cdot h_{x_k,\varepsilon_k}) \subset V(h_{z,1})$. Since $z$ is a non-Euclidean point of $V(h_{z,1})$, it must be also a non-Euclidean point of $V(t_k \cdot h_{x_k,\varepsilon_k})$, and consequently, we get $z = x_k$. This contradicts our choice of the point $z$. $\square$

**Remark 5.5.** In Remark 3.9 we pointed out that the topology $\tau$ is symmetrizable, and consequently, sequential. We note here that neither $\tau(\mathrm{ctbl})$ nor $\tau(\mathrm{fn})$ is sequential (hence, neither of them is symmetrizable). To see this, we first observe that the notion of the "convergent sequence" in $X = \mathbb{D} \cup S$ does not depend on the choice of three topologies $\tau$, $\tau(\mathrm{ctbl})$, and $\tau(\mathrm{fn})$. Let

$$x_n \,(n \in \omega) \to x$$

be an arbitrary convergent sequence in $X$ with regard to any of the three topologies $\tau \supset \tau(\mathrm{ctbl}) \supset \tau(\mathrm{fn})$, and put $C = \{x_n \,|\, n \in \omega\}$. In case $x \in \mathbb{D}$, the convergent sequence $x_n \to x$ is obviously identical with that of the Euclidean closed disc $\overline{\mathbb{D}}$. So, let us consider the case $x \in S$. Since $C \backslash S$ and $S \backslash \{x\}$ are disjoint closed subsets in the Euclidean space $\overline{\mathbb{D}} \backslash \{x\}$, we can find an open arc $J$ in $S$ containing $x$ and a continuous function $g : J \backslash \{x\} \to (0,1]$ such that $V(g) = V(g \restriction J_0) \cup V(g \restriction J_1)$ misses $C \backslash S$ where $J \backslash \{x\} = J_0 \cup J_1$. Extend $g$ to $\tilde{g} : J \to (0,1]$ by setting $\tilde{g}(x) = 1$. Then $V(\tilde{g}) = V(g) \cup V(x;1)$ belongs to $\tau(\mathrm{fn}) \subset \tau(\mathrm{ctbl}) \subset \tau$. Since $\tau(\mathrm{fn})$ is the coarsest among the three topologies, $C \cup \{x\}$ is always a convergent sequence with regard to $\tau(\mathrm{fn})$. Hence, almost all (except finitely many) points of $C$ are included in the neighborhood $V(\tilde{g})$ of $x$. This means that almost all of $C$ are included in $S \cup V(x;1)$. Put $F_k = \overline{V}(x; \frac{1}{k}) \backslash V(x; \frac{1}{k+1})$, $k = 1, 2, \cdots$, where $\overline{V}(x; \frac{1}{k})$ means the Euclidean closure of $V(x; \frac{1}{k})$. Then $V(x;1) \backslash \{x\} \subset F_1 \cup F_2 \cup \cdots$. Since $F_k$ is closed in $X$ with regard to any of the three topologies, each $F_k$ can include only finitely many points of $C$. Summarizing, we can say that $C = (C \cap S) \cup (C \backslash S)$ has the property that

(1) if $C \cap S$ is infinite, $(C \cap S) \cup \{x\}$ is a convergent sequence in the Euclidean circle $S$;
(2) if $C \backslash S$ is infinite, $(C \backslash S) \cup \{x\}$ is a convergent sequence such that every horoball $V(x;s)$ of any size $0 < s \leqslant 1$ at $x$ contains almost all of $C \backslash S$.

Thus, the convergent sequences in $X$ are the same in the three topologies $\tau \supset \tau(\mathrm{ctbl}) \supset \tau(\mathrm{fn})$. Now consider any set $U \in \tau$ with uncountably many non-Euclidean points, for example, $U = V(h)$ in Example 4.3. Then $U$, as an open set in $\tau$, is obviously sequentially open with regard to $\tau$. (A subset $U$ is *sequentially open* if every convergent sequence with a limit point in $U$ is almost contained in $U$.) Since the convergent sequences are the same in the three topologies $\tau$, $\tau(\text{ctbl})$, and $\tau(\text{fn})$, this $U$ is sequentially open also with regard to $\tau(\text{ctbl})$ or $\tau(\text{fn})$. It is easy to see, using Lindelöfness, that every open set in $\tau(\text{ctbl})$ or $\tau(\text{fn})$ has only countably many non-Euclidean points. Therefore, $U$ is not open with regard to $\tau(\text{ctbl})$ and $\tau(\text{fn})$. This proves that neither $\tau(\text{ctbl})$ nor $\tau(\text{fn})$ is sequential.

## 6. Concluding Remarks

Inspired by the propagation of seismic waves, we have found four different new topologies $\tau$, $\tau(\text{ctbl})$, $\tau(\text{fn})$, and $\tau(\text{N})$ on the cross section of the Earth, which respect the two well-known metrics $\rho$ and $d$:

$$\tau_\rho \supset \tau \supset \tau(\text{ctbl}) \supset \tau(\text{fn}) \supset \tau(\text{N}) \supset \tau_d.$$

And these topologies have turned out to be both

(1) not so far away from "metrizable," being regular Lindelöf and stratifiable

and

(2) a bit far away from "metrizable," being not a closed image of a metric space.

Moreover, $\tau$ is symmetrizable; $\tau(\text{ctbl})$, $\tau(\text{fn})$, and $\tau(\text{N})$ are locally contractible; and $\tau(\text{N})$ is first-countable. We believe these new topologies embody some essence of the multi-metric structure of our Earth, and it will not be so difficult to extend our results to a more complicated model of the Earth. The real Earth is three-dimensional and it is known that its interior consists mainly of five layers each having its own metric: inner core, outer core, lower mantle, upper mantle, and crust.

Finally, we want to remark that all of our new topologies respect very well the Euclidean topology on figures made by geodesics. Let $L(\alpha, \beta)$ be a geodesic connecting two points $\alpha, \beta \in S$, and put

$$L[\alpha, \beta] = L(\alpha, \beta) \cup \{\alpha, \beta\}.$$

Since the geodesic is orthogonal to $S$ at both ends, this closed arc $L[\alpha, \beta]$ with any of our four topologies is the same as the Euclidean closed arc. Let $P$ be a polyhedron enclosed by a finite number of such geodesics, usually called an "ideal hyperbolic polyhedron." Then, the figure $P$ as well as its join $P \cup S$ with $S$, having any of our four topologies, is the same as the one with the Euclidean topology.

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