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SOME PROPERTIES OF MINIMAL $S(\alpha)$ AND $S(\alpha)FC$ SPACES

ALEXANDER V. OSIPOV

Abstract. An $S(n)$-space is $S(n)$-functionally compact ($S(n)FC$) if every continuous function onto an $S(n)$-space is closed. $S(n)$-closed, $S(n)$-θ-closed, minimal $S(n)$, and $S(n)FC$ spaces are characterized in terms of $\theta(n)$-complete accumulation points. In this paper we also give new characteristics of R-closed and regular functionally compact spaces. The obtained results answer some questions raised by D. Dikranjan and E. Giuli and Louis M. Friedler, Mike Girou, Dix H. Pettey, and Jack R. Porter.

1. Introduction

D. Dikranjan and E. Giuli [3] introduced a notion of the θ^n-closure operator and developed a theory of $S(n)$-closed and $S(n)$-θ-closed spaces. Shouli Jiang, Ivan Reilly, and Shuquan Wang [7] used the θ^n-closure in studying properties of minimal $S(n)$-spaces.

In [10] we continue the study of properties inherent in $S(n)$-closed and $S(n)$-θ-closed spaces, using the θ^n-closure operator; in addition, wider classes of spaces (weakly $S(n)$-closed and weakly $S(n)$-θ-closed spaces) are introduced.

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Key words and phrases. minimal $S(n)$ space, $S(n)$-closed space, $S(n)$-θ-closed space, θ^n-closure.

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In this paper we continue the investigation of $S(n)$-closed, $S(n)$-θ-closed, and minimal $S(n)$ spaces with the use of $\theta(n)$-complete accumulation points. As we define a new class of S(n)-spaces called S(n)-functionally compact spaces, we also answer some questions raised in [3] and [4].

Section 2 acquaints the reader with main definitions and known properties in the theory of $S(n)$-spaces. Section 3 is completely devoted to the study of weakly $S(n)$-closed and weakly $S(n)$-θ-closed spaces. It is proved that any $S(n)$-closed ($S(n)$-θ-closed) space is weakly $S(n)$-closed (weakly $S(n)$-θ-closed). In the remaining sections, we characterize $S(n)$-closed, $S(n)$-θ-closed, minimal $S(n)$, $S(n)$-functionally compact, R-closed, minimal regular, and regular functionally compact spaces with the use of $\theta(n)$-complete accumulation and $\theta(\omega)$-complete accumulation points.

2. Main Definitions and Notation

Let X be a topological space, $M \subseteq X$, and $x \in X$. For any $n \in \mathbb{N}$, we consider the θ^n-closure operator: $x \notin cl_{\theta^n} M$ if there exists a set of open neighborhoods U_1, U_2, ..., U_n of the point x such that $clU_i \subseteq U_{i+1}$ for $i = 1, 2, ..., n-1$ and $clU_n \bigcap M = \emptyset$ if $n > 1$; $cl_{\theta^0} M = clM$ if $n = 0$; and, for $n = 1$, we get the θ-closure operator, i.e., $cl_{\theta^1} M = cl_\theta M$. A set M is θ^n-closed if $M = cl_{\theta^n} M$. Denote by $Int_{\theta^n} M = X \setminus cl_{\theta^n}(X \setminus M)$ the θ^n-interior of the set M. Evidently, $cl_{\theta^n}(cl_{\theta^s} M) = cl_{\theta^{n+s}} M$ for $M \subseteq X$ and $n, s \in \mathbb{N}$. For $n \in \mathbb{N}$ and a filter $\mathcal{F}$ on X, denote by $ad_{\theta^n}\mathcal{F}$ the set of θ^n-adherent points, i.e., $ad_{\theta^n}\mathcal{F} = \bigcap\{cl_{\theta^n}\mathcal{F}_\alpha : F_\alpha \in \mathcal{F}\}$. In particular, $ad_{\theta^0}\mathcal{F} = ad\mathcal{F}$ is the set of adherent points of the filter $\mathcal{F}$. For any $n \in \mathbb{N}$, a point $x \in X$ is $S(n)$-separated from a subset M if $x \notin cl_{\theta^n} M$. For example, x is $S(0)$-separated from M if $x \notin clM$. For $n > 0$, the relation of $S(n)$-separability of points is symmetric. On the other hand, $S(0)$-separability may be not symmetric in some not T_1-spaces. Therefore, we say that points x and y are $S(0)$-separated if $x \notin cl_X\{y\}$ and $y \notin cl_X\{x\}$.

Let $n \in \mathbb{N}$ and X be a topological space.

1. X is called an $S(n)$-*space* if any two distinct points of X are $S(n)$-separated.

2. A filter $\mathcal{F}$ on X is called an $S(n)$-*filter* if every point, not being an adherent point of the filter $\mathcal{F}$, is $S(n)$-separated from some element of the filter $\mathcal{F}$.

3. An open cover $\{U_\alpha\}$ of the space X is called an $S(n)$-*cover* if every point of X lies in the θ^n-interior of some U_α.

It is obvious that $S(0)$-spaces are T_1-spaces, $S(1)$-spaces are Hausdorff spaces, and $S(2)$-spaces are Urysohn spaces. It is clear that every filter is an $S(0)$-filter, every open cover is an $S(0)$-cover, and every open filter is an

$S(1)$-filter. Open $S(2)$-filters are called Urysohn filters. For $n > 1$, open $S(n)$-filters are defined in [12]. $S(1)$-covers are called Urysohn covers. In a regular space, every filter (every cover) is an $S(n)$-filter ($S(n)$-cover) for any $n \in \mathbb{N}$.

$S(n)$-closed and $S(n)$-θ-closed spaces are $S(n)$-spaces, closed and, respectively, θ-closed in any $S(n)$-space containing them.

Jack R. Porter and Charles Votaw [12] characterize $S(n)$-closed spaces by means of open $S(n)$-filters and $S(n-1)$-covers.

Let $n \in \mathbb{N}^+$ and X be an $S(n)$-space. Then the following conditions are equivalent:

(1) $ad_{\theta^n}\mathcal{F} \neq \emptyset$ for any open filter $\mathcal{F}$ on X;

(2) $ad\mathcal{F} \neq \emptyset$ for any open $S(n)$-filter $\mathcal{F}$ on X;

(3) for any $S(n-1)$-cover $\{U_\alpha\}$ of the space X there exist $\alpha_1, \alpha_2, ..., \alpha_k$ such that $X = \bigcup_{i=1}^{k} \overline{U_{\alpha_i}}$;

(4) X is an $S(n)$-closed space.

Dikranjan and Giuli [3] characterize $S(n)$-θ-closed spaces in terms of $S(n-1)$-filters and $S(n-1)$-covers.

Let $n \in N^+$ and X be an $S(n)$-space. Then the following conditions are equivalent:

(1) $ad\mathcal{F} \neq \emptyset$ for any closed $S(n-1)$-filter $\mathcal{F}$ on X;

(2) any $S(n-1)$-cover of X has a finite subcover;

(3) $ad_{\theta^{(n-1)}}\mathcal{F} \neq \emptyset$ for any closed filter $\mathcal{F}$ on X;

(4) X is an $S(n)$-θ-closed space.

Note that, for $n = 1$, $S(1)$-closedness is H-closedness and $S(1)$-θ-closedness is compactness. For $n = 2$, $S(2)$-closedness is U-closedness and $S(2)$-θ-closedness is U-θ-closedness. From the characteristics themselves, it follows that any $S(n)$-θ-closed subspace of an $S(n)$-space is an $S(n)$-closed space.

Recall that an open cover $\mathcal{V}$ is a *shrinkable refinement of an open cover* $\mathcal{U}$ if and only if for each $V \in \mathcal{V}$, there is a $U \in \mathcal{U}$ such that $\overline{V} \subseteq U$. An open cover $\mathcal{V}$ is a *regular refinement of* $\mathcal{U}$ if and only if $\mathcal{V}$ refines $\mathcal{U}$ is a shrinkable refinement of itself. An open cover is *regular* if and only if it has an open refinement.

An *open filter base* $\mathcal{F}$ *in* X is a regular filter base if and only if for each $U \in \mathcal{F}$, there exists $V \in \mathcal{F}$ such that $\overline{V} \subseteq U$.

An *R-closed space* is a regular space closed in any regular space containing them.

Manuel P. Berri and R. H. Sorgenfrey [2] characterize R-closed spaces by means of regular filters and regular covers.

Let X be a regular space. The following are equivalent:

(1) X is R-closed;

(2) every regular filter base in X is fixed;
(3) every regular cover has a finite subcover.

For undefined notions and related theorems, we refer the readers to [3].

3. Weakly $S(n)$-Closed and Weakly $S(n)$-θ-Closed Spaces

In [1], P. S. Aleksandrov and P. S. Urysohn introduce the notion of a θ-complete accumulation point. A point x is called a θ-complete accumulation point of a set F if $|F \bigcap \overline{U}| = |F|$ for any neighborhood U of the point x. It was noted that any H-closed space has the following property:

(*) any infinite set of regular cardinality has a θ-complete accumulation point.

However, the converse is not true. The first example of a space possessing property (*) and not being H-closed was constructed by G. A. Kirtadze [8]. Simple examples in [10] and [11] also show the converse is not true.

Example 3.1 ([10, Example 1])**.** Let T_1 and T_2 be two copies of the Tychonoff plane $T = ((\omega_1 + 1) \times (\omega_0 + 1)) \setminus \{\omega_1, \omega_0\}$, whose elements will be denoted by $(\alpha, n, 1)$ and $(\alpha, n, 2)$, respectively. On the topological sum $T_1 \oplus T_2$, we consider the identifications $(\omega_1, k, 1) \sim (\omega_1, 2k, 2)$ for every $k \in \mathbb{N}$, and we identify all points $(\omega_1, 2k-1, 2)$ for any $k \in \mathbb{N}$ with the same point b. Adding, to the obtained space, a point a with the base of neighborhoods $U_{\beta,k}(a) = \{a\} \bigcup \{(\alpha, n, 1) : \beta < \alpha < \omega_1, k < n \leq \omega_0\}$ for arbitrary $\beta < \omega_1$ and $k < \omega_0$, we get a Urysohn space X. (See Figure 1.)

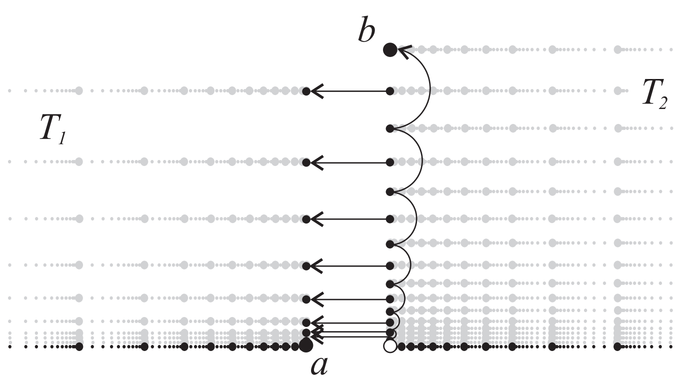

Figure 1. The space X

Note that space X is an example of a non-H-closed, Urysohn space with the property that for every chain of non-empty sets, the intersection of the θ-closures of the sets is nonempty; also, every infinite set has a θ-complete accumulation point. Porter investigated the space with the same properties in [11].

Definition 3.2. A neighborhood U of a set A is called an *n-hull of the set A* if there exists a set of neighborhoods U_1, U_2, ..., $U_n = U$ of the set A such that $clU_i \subseteq U_{i+1}$ for $i = 1, ..., n-1$.

Definition 3.3. A point x from X is called a *$\theta^0(n)$-complete accumulation ($\theta(n)$-complete accumulation) point of an infinite set F* if $|F \bigcap U| = |F|$ ($|F \bigcap \overline{U}| = |F|$) for any U, where U is an n-hull of the point x.

Note that, for $n = 1$, a $\theta^0(1)$-complete accumulation point is a point of complete accumulation, and a $\theta(1)$-complete accumulation point is a θ-complete accumulation point.

Definition 3.4. A topological space X is called *weakly $S(n)$-θ-closed* (*weakly $S(n)$-closed*) if any infinite set of regular cardinality of the space X has a $\theta^0(n)$-complete accumulation ($\theta(n)$-complete accumulation) point.

Note that any $\theta^0(n)$-complete accumulation point is a $\theta(n)$-complete accumulation point; hence, any weakly $S(n)$-θ-closed space is weakly $S(n)$-closed. Moreover, since a $\theta(n)$-complete accumulation point is a $\theta^0(n+1)$-complete accumulation point, it follows that a weakly $S(n)$-closed space will be weakly $S(n+1)$-θ-closed. For $n = 1$, weakly $S(1)$-θ-closed and weakly $S(1)$-closed spaces are compact Hausdorff spaces and spaces with property (*), respectively.

Theorem 3.5. *Let X be an $S(n)$-closed $S(n)$-space. Then X is weakly $S(n)$-closed.*

Proof. Suppose the contrary. Let X be $S(n)$-closed but not weakly $S(n)$-closed. Then in the space X there exists an infinite set F of regular power that has no $\theta(n)$-complete accumulation point. For any point $x \in X$, there exists an n-hull U of the point x with the property $|F \bigcap \overline{U}| < |F|$. If we take such n-hull for every point $x \in X$, we derive an $S(n-1)$-cover of the space X. By $S(n)$-closedness, there exists a finite family $\mathcal{U}$ of n-hulls such that $|F \bigcap \overline{U}| < |F|$ for all $U \in \mathcal{U}$ and $\bigcup \overline{\mathcal{U}} = X$. This contradicts the fact that F is an infinite set of regular power. □

Theorem 3.6. *Let X be an $S(n)$-θ-closed $S(n)$-space. Then X is weakly $S(n)$-θ-closed space.*

Proof. The proof of Theorem 3.6 is analogous to that of Theorem 3.5. □

It was proved in [3] that $S(n)$-closedness implies $S(n+1)$-θ-closedness. Thus, for $S(n)$-spaces, classes of the considered spaces are presented in the following diagram:

$$\begin{array}{ccc}
\text{compact Hausdorff space} & \Longleftrightarrow & \text{weakly } S(1)\text{-}\theta\text{-closed} \\
\Downarrow & & \Downarrow \\
H\text{-closed} & \Longrightarrow & \text{weakly } H\text{-closed} \\
\Downarrow & & \Downarrow \\
U\text{-}\theta\text{-closed} & \Longrightarrow & \text{weakly } U\text{-}\theta\text{-closed} \\
\Downarrow & & \Downarrow \\
U\text{-closed} & \Longrightarrow & \text{weakly } U\text{-closed} \\
\Downarrow & & \Downarrow \\
\dots & \dots & \dots \\
\Downarrow & & \Downarrow \\
S(n-1)\text{-}\theta\text{-closed} & \Longrightarrow & \text{weakly } S(n-1)\text{-}\theta\text{-closed} \\
\Downarrow & & \Downarrow \\
S(n-1)\text{-closed} & \Longrightarrow & \text{weakly } S(n-1)\text{-closed} \\
\Downarrow & & \Downarrow \\
S(n)\text{-}\theta\text{-closed} & \Longrightarrow & \text{weakly } S(n)\text{-}\theta\text{-closed} \\
\Downarrow & & \Downarrow \\
S(n)\text{-closed} & \Longrightarrow & \text{weakly } S(n)\text{-closed}
\end{array}$$

Note that all implications in the diagram are irreversible. Examples of $S(n)$-closed but not $S(n)$-θ-closed spaces and $S(n)$-θ-closed but not $S(n-1)$-closed spaces are considered in [3]. Examples showing that the remaining implications are irreversible are considered in [10].

Theorem 3.7. *Let X be a Lindelöf (finally compact) weakly $S(n)$-closed $S(n)$-space. Then X is an $S(n)$-closed space.*

Proof. Suppose the contrary. Let X be Lindelöf weakly $S(n)$-closed but not $S(n)$-closed. Then in the spaces X there exists an open filter $\mathcal{F}$ such that $ad_{\theta^n}\mathcal{F} = \emptyset$. For each point $x \in X$, there are $F_x \in \mathcal{F}$ and an n-hull U_x of F_x such that $x \notin \overline{U_x}$. Note that $\bigcap_{x\in X} \overline{U_x} = \emptyset$. Since X is Lindelöf, there exists a countable family $\{U_{x_i}\}$ such that $\bigcap \overline{U_{x_i}} = \emptyset$. Consider a sequence $\{y_j\}$ such that $y_j \in \bigcap_{i=1}^{j} F_{x_i}$. Clearly, the infinite set $\{y_j\}$ does not have a $\theta(n)$-complete accumulation point. This contradicts the fact that X is a weakly $S(n)$-closed space. □

Corollary 3.8. *Let X be a countable weakly $S(n)$-θ-closed $S(n)$-space. Then X is $S(n)$-closed.*

Corollary 3.9. *Let X be a second-countable weakly $S(n)$-θ-closed $S(n)$-space. Then X is $S(n)$-closed.*

Recall, that a space is linearly Lindelöf (finally compact in the sense of accumulation points) if every increasing open cover $\{U_\alpha : \alpha \in \kappa\}$ has a countable subcover (by increasing, we mean that $\alpha < \beta < \kappa$ implies $U_\alpha \subseteq U_\beta$).

Theorem 3.10. *Let $n > 1$ and X be a linearly Lindelöf weakly $S(n)$-θ-closed $S(n)$-space. Then X is weakly $S(n-1)$-closed.*

Proof. Suppose the contrary. Then there is a countable set S such that the set S has not a $\theta(n-1)$-complete accumulation point. Let $x \in X$ and U be an $(n-1)$-hull of x such that $\overline{U} \bigcap S = \emptyset$. For every $y \in \overline{U}$, there exists a neighborhood W_y of y such that $W_y \bigcap S = \emptyset$. Consider an open set $W = \bigcup_{y \in \overline{U}} W_y$. Then $\overline{U} \subseteq W$ and W is an n-hull of the point x. Note that $W \bigcap S = \emptyset$. It follows that x is not a $\theta^0(n)$-complete accumulation point of S. This contradicts the fact that X is weakly $S(n)$-θ-closed. $\square$

Corollary 3.11. *Let $n > 1$ and X be a Lindelöf weakly $S(n)$-θ-closed $S(n)$-space. Then X is $S(n-1)$-closed.*

Problem 5 in [3] is to prove or disprove that the product of U-θ-closed spaces is feebly compact. In particular, it was not known if every Lindelöf U-θ-closed space is H-closed.

In [9], two Urysohn U-θ-closed spaces whose product is not feebly compact are constructed. Thus, the question is negatively solved.

Corollary 3.12. *Let X be a Lindelöf U-θ-closed Urysohn space. Then X is H-closed.*

Remark 3.13. Observe that every H-closed space is feebly compact. By Corollary 3.12, the product of Lindelöf U-θ-closed spaces is feebly compact.

Corollary 3.14. *Let $n > 1$ and X be a Lindelöf weakly $S(n)$-θ-closed $S(n)$-space. Then X is $S(n)$-θ-closed.*

Thus, for Lindelöf $S(n)$-spaces, classes of the considered spaces are presented in the following diagram:

$$\begin{array}{ccc}
\text{compact Hausdorff space} & \Longleftrightarrow & \text{weakly } S(1)\text{-}\theta\text{-closed} \\
\Downarrow & & \Downarrow \\
H\text{-closed} & \Longleftrightarrow & \text{weakly } H\text{-closed} \\
\Updownarrow & & \Updownarrow \\
U\text{-}\theta\text{-closed} & \Longleftrightarrow & \text{weakly } U\text{-}\theta\text{-closed} \\
\Downarrow & & \Downarrow \\
U\text{-closed} & \Longleftrightarrow & \text{weakly } U\text{-closed} \\
\Updownarrow & & \Updownarrow \\
\dots & \dots & \dots \\
\Updownarrow & & \Updownarrow
\end{array}$$

$$\begin{array}{ccc}
S(n-1)\text{-}\theta\text{-closed} & \Longleftrightarrow & \text{weakly } S(n-1)\text{-}\theta\text{-closed} \\
\Downarrow & & \Downarrow \\
S(n-1)\text{-closed} & \Longleftrightarrow & \text{weakly } S(n-1)\text{-closed} \\
\Updownarrow & & \Updownarrow \\
S(n)\text{-}\theta\text{-closed} & \Longleftrightarrow & \text{weakly } S(n)\text{-}\theta\text{-closed} \\
\Downarrow & & \Downarrow \\
S(n)\text{-closed} & \Longleftrightarrow & \text{weakly } S(n)\text{-closed}
\end{array}$$

Question 3.15. For $n > 1$, is there a Lindelöf $S(n)$-closed space that is not $S(n)$-θ-closed?

4. Characterizations of $S(n)$-Closed and $S(n)$-θ-Closed Spaces

Now for every $n \in \mathbb{N}$ we introduce an operator of θ_0^n-closure; for $M \subseteq X$ and $x \in X$, $x \notin cl_{\theta_0^n} M$ if there is an n-hull U of x such that $U \bigcap M = \emptyset$. A set $M \subseteq X$ is θ_0^n-closed if $M = cl_{\theta_0^n} M$.

Definition 4.1. A subset M of a topological space X is an $S(n)$-θ_0^n-set if every $S(n)$-cover γ covering M ($M \subseteq \bigcup\{Int_{\theta^n} U_\alpha : U_\alpha \in \gamma\}$) by open sets of X has a finite subfamily which covers M with the θ_0^n-closures of its members.

Definition 4.2. A subset A of a space X weakly $\theta(n)$-converges to the set B if, for any $S(n-1)$-cover $\gamma = \{U_\alpha\}$ of B, there exists a finite family $\{U_{\alpha_i}\}_{i=1}^k \subseteq \gamma$ such that $|A \setminus Int(\bigcup_{i=1}^k \overline{U_{\alpha_i}})| < |A|$.

Theorem 4.3. *For $n \in \mathbb{N}$, an $S(n)$-space X is $S(n)$-closed if and only if each infinite subset A of X weakly $\theta(n)$-converges to the set of the $\theta(n)$-complete accumulation points of A.*

Proof. Necessary. Let B denote the $\theta(n)$-complete accumulation points of A. Take any $S(n-1)$-cover γ of B where B is the set of $\theta(n)$-complete accumulation points of A. For each point $x \notin B$ we take an n-hull $O(x)$ such that $|\overline{O(x)} \bigcap A| < |A|$. Then we have an $S(n-1)$-cover $\gamma' = \gamma \bigcup \{O(x) : x \notin B\}$ of X. As the space X is $S(n)$-closed there are finite families $\{U_i\}_{i=1}^s \subseteq \gamma$ and $\{O(x_j)\}_{j=1}^k$ such that $(\bigcup_{i=1}^s \overline{U_i}) \bigcup (\bigcup_{j=1}^k \overline{O(x_j)}) = X$. Note that $A \setminus Int(\bigcup_{i=1}^s \overline{U_i}) \subseteq \bigcup_{j=1}^k \overline{O(x_j)}$. As $|A \bigcap (\bigcup_{j=1}^k \overline{O(x_j)}| < |A|$, we have $|A \setminus Int(\bigcup_{i=1}^s \overline{U_i})| < |A|$. Thus, A weakly $\theta(n)$-converges to the set B.

Note that B is an $S(n)$-θ_0^n-set. Also, $(A \bigcap Int(\bigcup_{i=1}^s \overline{U_i})) \bigcap \overline{S(x)} \neq \emptyset$ for every $x \in B$ and for any n-hull $S(x)$ of the point x. It follows that $S(x) \bigcap (\bigcup_{i=1}^s U_i) \neq \emptyset$ and x is contained in θ_0^n-closure of $\bigcup_{i=1}^s U_i$. Thus, $B \subseteq cl_{\theta_0^n} \bigcup_{i=1}^s U_i$.

Sufficiency. Let $\varphi = \{V_\alpha\}$ be an open $S(n)$-filter on X. Assume that $ad\ \varphi = \emptyset$. Choose $V_0 \in \varphi$ such that $|V_0| = inf\{|V_\alpha| : V_\alpha \in \varphi\}$. Since $\bigcap \overline{V_\alpha} = \emptyset$, we have $\xi = \{U_\alpha : U_\alpha = X \setminus \overline{V_\alpha}\}$ is an $S(n-1)$-cover of B where B is the set of $\theta(n)$-complete accumulation points of V_0. By the condition, there exists a finite family $\{U_{\alpha_i}\}_{i=1}^k \subseteq \xi$ such that $|V_0 \setminus Int(\bigcup_{i=1}^k \overline{U_{\alpha_i}})| < |V_0|$. Consider $V_{\alpha_i} \in \varphi$ such that $U_{\alpha_i} = X \setminus V_{\alpha_i}$. Let $V = \bigcap_{i=1}^k V_{\alpha_i}$; then $V \bigcap V_0 \subseteq V_0 \setminus Int(\bigcup_{i=1}^k \overline{U_{\alpha_i}})$ and $|V \bigcap V_0| < |V_0|$. This contradicts our choice of V_0. Thus, X is an $S(n)$-closed space. □

Corollary 4.4. *Let X be an $S(n)$-closed space and A be an infinite set of X. Then a set B of $\theta(n)$-complete accumulation points of A is an $S(n)$-θ_0^n-set.*

Definition 4.5. A subset A of a space X weakly $\theta^0(n)$-converges to the set B if, for any $S(n-1)$-cover $\gamma = \{U_\alpha\}$ of B, there exists a finite family $\{U_{\alpha_i}\}_{i=1}^k \subseteq \gamma$ such that $|A \setminus \bigcup_{i=1}^k U_{\alpha_i}| < |A|$.

Theorem 4.6. *For $n \in \mathbb{N}$, an $S(n)$-space X is $S(n)$-θ-closed if and only if each infinite subset A of X $\theta^0(n)$-converges to the set of $\theta^0(n)$-complete accumulation points of A.*

5. Characterization of Minimal $S(n)$-Spaces

A $\mathcal{P}$ space is minimal $\mathcal{P}$ if it has no strictly coarser $\mathcal{P}$ topology. The terms "minimal Urysohn" and "minimal regular" are abbreviated as MU and MR, respectively.

Definition 5.1. A subset A of a space X $\theta(n)$-converges to the set B if, for any $S(n-1)$-cover $\gamma = \{U_\alpha\}$ of B, there exists a finite family $\{U_{\alpha_i}\}_{i=1}^k \subseteq \gamma$ such that $|A \setminus \bigcup_{i=1}^k \overline{U_{\alpha_i}})| < |A|$.

Theorem 5.2. *For $n \in \mathbb{N}$, an $S(n)$-space X is a minimal $S(n)$-space if and only if each infinite subset A of X $\theta(n)$-converges to the set of the $\theta(n)$-complete accumulation points of A, and, if there exists a point x such that A does not $\theta(n)$-converge to $X \setminus \{x\}$, then x is a complete accumulation point of A.*

Proof. Necessary. Let X be a minimal $S(n)$-space and $A \subseteq X$. Then X is an $S(n)$-closed space [7, Corollary 2.3] and A (weakly) $\theta(n)$-converges to the set B of its $\theta(n)$-complete accumulation points. Let $x \in X$ such that A does not $\theta(n)$-converge to $X \setminus \{x\}$. Consider $S(n-1)$-cover $\gamma = \{U_\alpha\}$ of $X \setminus \{x\}$ such that $|A \setminus \bigcup_{i=1}^k \overline{U_{\alpha_i}}| = |A|$ holds for any $\gamma' = \{U_{\alpha_i}\}_{i=1}^k \subseteq \gamma$.

Let ω be an open $S(n)$-filter generated by $\{X \setminus \overline{U_\alpha} : U_\alpha \in \gamma\}$. Then ω has a unique adherent point x. Since X is a minimal $S(n)$-space, we have that the open $S(n)$-filter ω converges to x. Thus, for every open neighborhood $O(x)$ of x, there is $V \in \omega$ such that $V \subseteq O(x)$. So $|V \bigcap A| = |A|$ and x is a complete accumulation point of A.

Sufficiency. We need to show that any open $S(n)$-filter φ with unique adherent point x is convergent.

Suppose that $ad\ \varphi = \{x\}$, but φ does not converge. Then there is an open neighborhood $O(x)$ of x such that $W_\alpha = V_\alpha \setminus O(x) \neq \emptyset$ for any $V_\alpha \in \varphi$. Choose W_{α_0} such that $|W_{\alpha_0}| = inf\{|W_\alpha| : V_\alpha \in \varphi\}$. Let B be the set of $\theta(n)$-complete accumulation points of W_{α_0}. Note that $x \in B$. To the contrary, assume that $x \notin B$, then for every $y \in B$, there are n-hull neighborhoods $O(y)$ of y and W_α such that $\overline{O(y)} \bigcap W_\alpha = \emptyset$. Consider $S(n-1)$-cover $\gamma = \{O(y) : y \in B\}$ of B. For every finite family $\{O(y_i)\}_{i=1}^k \subseteq \gamma$, there is W_α such that

$$(\bigcup_{i=1}^{k} \overline{O(y_i)}) \bigcap (W_\alpha \bigcap W_{\alpha_0}) = \emptyset.$$

By the choice of W_{α_0}, we have $|W_\alpha \bigcap W_{\alpha_0}| = |W_{\alpha_0}|$. Thus, W_{α_0} does not $\theta(n)$-converge to B. It follows that $x \in B$ and W_{α_0} does not $\theta(n)$-converge to $B \setminus \{x\}$. For each point $y \in X \setminus B$, we take an n-hull $O_1(y)$ such that $|\overline{O_1(y)} \bigcap A| < |A|$. Consider $S(n-1)$-cover $\gamma_1 = \gamma \bigcup \{O_1(y) : y \in X \setminus B\}$ of $X \setminus \{x\}$. For every finite family $\{V_i\}_{i=1}^k \subseteq \gamma_1$, there is W_α such that $(\bigcup_{i=1}^k \overline{V_i}) \bigcap (W_\alpha \bigcap W_{\alpha_0}) = \emptyset$. Thus, W_{α_0} does not $\theta(n)$-converge to $X \setminus \{x\}$. By the condition, x is a complete accumulation point of W_{α_0}. This contradicts the fact that $W_{\alpha_0} = V_{\alpha_0} \setminus O(x)$. □

Clearly, the weakly $\theta(n)$-convergence implies $\theta(n)$-convergence. By Theorem 4.3 and Theorem 5.2, we have the following.

Theorem 5.3. *For $n \in \mathbb{N}$, an $S(n)$-space X is a minimal $S(n)$-space if and only if X is $S(n)$-closed, and, if there exists a point x such that an infinite set A does not $\theta(n)$-converge to $X \setminus \{x\}$, then x is a complete accumulation point of A.*

In [4, Q40], we are asked to find a property $\mathcal{Q}$ which does not imply U-closed such that a space is MU if and only if it is U-closed and has property $\mathcal{Q}$.

The following theorem answers this question.

Theorem 5.4. *A Urysohn space X is MU if and only if X is U-closed, and, if there exists a point x such that infinite A does not $\theta(2)$-converge to $X \setminus \{x\}$, then x is a complete accumulation point of A.*

In [4, Q35] it is asked, Does every MU space have a base of open sets with U-closed complements?

Note that the negative answer to this question is the following example [6]. This is an example of an MU space that has no open base with U-closed complements.

Example 5.5 (Herrlich)**.** For any ordinal number α, let $W(\alpha)$ be the set of all ordinals strictly less than α. Let ω_0 be the first infinite ordinal and ω_1 the first uncountable ordinal. Let $R = (W(\omega_1 + 1) \times W(\omega_0 + 1)) \setminus \{(\omega_1, \omega_0)\}$ and $R_n = R \times \{n\}$ where $n = 0, \pm 1, \pm 2, ...$ Denote the elements of R_n by (x, y, n). Identify (ω_1, y, n) with $(\omega_1, y, n+1)$ if n is odd and (x, ω_0, n) with $(x, \omega_0, n+1)$ if n is even. Call the resulting space T. To the subspace $E = R_1 \bigcup R_2 \bigcup R_3$ of T, add two points a and b, and let $X = E \bigcup \{a, b\}$. A set $V \subset X$ is open if and only if

(1) $V \bigcap E$ is open in E,
(2) $a \in V$ implies there exists $\alpha_0 < \omega_0$ such that $\{(\alpha, \beta, 1) : \beta_0 < \beta \leq \omega_0, \alpha_0 < \alpha < \omega_1\} \subset V$, and
(3) $b \in V$ implies there exist $\alpha_0 < \omega_1$ and $\beta_0 < \omega_0$ such that $\{(\alpha, \beta, 3) : \beta_0 < \beta < \omega_0, \alpha_0 < \alpha \leq \omega_1\} \subset V$.

Really, for any open $V \ni a$, if $a \in O(a) = \{(\alpha, \beta, 1) : \beta_0 < \beta \leq \omega_0, \alpha_0 < \alpha < \omega_1\} \subset V$, then $X \setminus O(a)$ is not U-closed.

For $A = \{(\alpha, \omega_0, 2) : \alpha_0 < \alpha < \omega_1\}$, A does not weakly $\theta(2)$-converge to the set of its $\theta(2)$-complete accumulation points.

6. Characterization $S(n)FC$ Spaces

Definition 6.1. An $S(n)$-space is *$S(n)$-functionally compact* ($S(n)FC$) if every continuous function onto an $S(n)$-space is closed.

A set C will be called a *complete accumulation set of a set A* if $|U \bigcap A| = |A|$ for any open set $U \supseteq C$.

Theorem 6.2. *For $n \in \mathbb{N}$, an $S(n)$-space X is $S(n)FC$ if and only if each infinite subset A of X $\theta(n)$-converges to the set of the $\theta(n)$-complete accumulation points of A, and, if there exists a θ^n-closed set C such that A does not $\theta(n)$-converge to $X \setminus C$, then C is a complete accumulation set of A.*

Proof. Necessary. Let X be $S(n)FC$ and $A \subseteq X$. Then X is an $S(n)$-closed space and A (weakly) $\theta(n)$-converges to the set B of its $\theta(n)$-complete accumulation points. Let C be a θ^n-closed set such that A does not $\theta(n)$-converge to $X \setminus C$.

Consider $S(n-1)$-cover $\gamma = \{U_\alpha\}$ of $X \setminus C$ such that $|A \setminus \bigcup_{i=1}^{k} \overline{U_{\alpha_i}}| = |A|$ holds for any $\gamma' = \{U_{\alpha_i}\}_{i=1}^{k} \subseteq \gamma$.

Consider the ω open $S(n)$-filter generated by $\{X \setminus \overline{U_\alpha} : U_\alpha \in \gamma\}$.

Suppose that there exists an open set $W \supseteq C$ such that $|A \bigcap W| < |A|$. Consider the quotient space $(X/C, \tau)$ of X with C identified to a point

c. Now $\tau_1 = \{V \in \tau : c \in V \text{ implies } V \in \omega\}$ is a topology on X/C. In $(X/C, \tau_1)$, we have $ad_{\theta^n} N_x$ for any x where N_x is the neighborhood filter at the point x, and thus, $(X/C, \tau_1)$ is an $S(n)$-space. It is clear that τ_1 is strictly coarser than τ. The quotient function from X to X/C is denoted as p_C, and q_C denotes $s \circ p_C$ where $s : (X/C, \tau) \to (X/C, \tau_1)$ is the identity function. Note that $q_C(X \setminus W)$ is not closed in $(X/C, \tau_1)$. Thus, q_C is continuous but is not closed. This is a contradiction that X is an $S(n)FC$ space.

Sufficiency. Suppose that X is not an $S(n)FC$ space. Then there is a continuous function f from X onto an $S(n)$-space Y such that f is not closed. Consider the closed set $A \subseteq X$ such that $f(A)$ is not closed. Let $y \in \overline{f(A)} \setminus f(A)$ and $N_y = \{V_\alpha\}$ is the neighborhood $S(n)$-filter at the point y. Then $B = f^{-1}(y) = \bigcap\{f^{-1}(V_\alpha)\}$ and B is a θ^n-closed subset of X. Note that $X \setminus A$ is an open set containing B such that $W_\alpha = U_\alpha \setminus (X \setminus A) \neq \emptyset$ for any $U_\alpha \in \{f^{-1}(V_\alpha)\}$.

Choose W_{α_0} such that $|W_{\alpha_0}| = \inf\limits_{\alpha}\{|W_\alpha|\}$.

Let D be the set of $\theta(n)$-complete accumulation points of W_{α_0}. By the condition, set W_{α_0} $\theta(n)$-converges to the set D. We claim that θ^n-closed set B such that W_{α_0} does not $\theta(n)$-converge to $X \setminus B$. Indeed, for any $x \in X \setminus B$, there is U_{α_x} such that x is $S(n)$-separated from U_{α_x}. Let $O(x)$ be an n-hull neighborhood of x such that $\overline{O(x)} \bigcap U_{\alpha_x} = \emptyset$. Consider an $S(n-1)$-cover $\gamma = \{O(x) : x \in X \setminus B\}$ of $X \setminus B$. For any finite family $\{O(x_i)\}_{i=1}^k \subseteq \gamma$, there is $U = \bigcap\limits_{i=1}^{k} U_{\alpha_{x_i}}$ such that $\bigcup\limits_{i=1}^{k} \overline{O(x_i)} \bigcap (U \bigcap W_{\alpha_0}) = \emptyset$. By the choice of W_{α_0}, we have $|U \bigcap W_{\alpha_0}| = |W_{\alpha_0}|$. It follows that W_{α_0} does not $\theta(n)$-converge to $X \setminus B$. By the condition, B is a complete accumulation set of W_{α_0}. This contradicts the fact that $X \setminus A$ is an open set containing B. $\square$

Corollary 6.3. *A Urysohn space X is UFC if and only if X is U-closed, and, if there exists a θ^2-closed set C such that infinite set A does not $\theta(2)$-converge to $X \setminus C$, then C is a complete accumulation set of A.*

Definition 6.4. An $S(n)$-space X is $S(n)FFC$ ($S(n)CFC$) if every continuous function f onto an $S(n)$-space Y with $f^{-1}(y)$ finite (compact) is a closed function.

Theorem 6.5. *For $n \in \mathbb{N}$, an $S(n)$-space X is $S(n)FFC$ ($S(n)CFC$) if and only if X is $S(n)$-closed, and, if there exists a finite (compact) set C such that an infinite set A does not $\theta(n)$-converge to $B \setminus C$, then C is a complete accumulation set of A.*

Proof. A proof of Theorem 6.5 is analogous to that of Theorem 6.2. $\square$

Question 6.6. Is every $S(n)FC$ ($S(n)FFC$; $S(n)CFC$) space necessarily compact ($n > 1$)?

7. $S(\omega)$-Closed and Minimal $S(\omega)$ Spaces

Two filters $\mathcal{F}$ and $\mathcal{Q}$ on a space X are $S(\omega)$-separated if there are open families $\{U_\beta : \beta < \omega\} \subseteq \mathcal{F}\}$ and $\{V_\beta : \beta < \omega\} \subseteq \mathcal{Q}\}$ such that $U_0 \bigcap V_0 = \emptyset$ and for $\gamma + 1 < \omega$, $clU_{\gamma+1} \subseteq U_\gamma$ and $clV_{\gamma+1} \subseteq V_\gamma$. A space X is $S(\omega)$ if for distinct points $x, y \in X$, the neighborhood filters $\mathcal{N}_x$ and $\mathcal{N}_y$ are $S(\omega)$-separated.

An $S(\omega)$-closed space is an $S(\omega)$ space closed in any $S(\omega)$ space containing them.

In 1973, Porter and Votaw [12] established next results.

(1) A minimal $S(\omega)$ space is $S(\omega)$-closed and semiregular.
(2) A minimal $S(\omega)$ space is regular.
(3) A space is R-closed if and only if it is $S(\omega)$-closed and regular.
(4) A space is MR if and only if it is minimal $S(\omega)$.

Definition 7.1. A neighborhood U of a point x is called an *ω-hull of the point x* if there exists a set of neighborhoods $\{U_i\}_{i=1}^{\infty}$ of the point x such that $clU_i \subseteq U_{i+1}$ and $U_i \subseteq U$ for every $i \in \mathbb{N}$.

Definition 7.2. A point $x \in X$ is called a *$\theta(\omega)$-complete accumulation point of an infinite set F* if $|F \bigcap U| = |F|$ for any U, where U is an ω-hull of the point x.

Definition 7.3. The set A *$\theta(\omega)$-converges* to the set B if, for any regular cover $\gamma = \{U_\alpha\}$ of B, there exists a finite family $\{U_{\alpha_i}\}_{i=1}^{s} \subseteq \gamma$ such that $|A \setminus \bigcup_{i=1}^{s} U_{\alpha_i}| < |A|$.

Theorem 7.4. *A regular space X is R-closed if and only if each infinite subset A of X $\theta(\omega)$-converges to the set of the $\theta(\omega)$-complete accumulation points of A.*

Proof. A proof of Theorem 7.4 is analogous to that of Theorem 4.3. □

In [4], Q39 asks us to find a property $\mathcal{P}$ which does not imply R-closed such that a space is MR if and only if it is R-closed and has property $\mathcal{P}$.

The following theorem answers this question.

Theorem 7.5. *A regular space X is an MR space if and only if X is R-closed, and, if there exists a point $x \in B$ such that infinite set A does not $\theta(\omega)$-converge to $X \setminus \{x\}$, then x is a complete accumulation point of A.*

Proof. A proof of Theorem 7.5 is analogous to that of Theorem 5.2. □

We introduce an operator of θ^ω-closure; for $M \subseteq X$ and $x \in X$ $x \notin cl_{\theta^\omega} M$ if there is an ω-hull U of x such that $U \bigcap M = \emptyset$. A set $M \subseteq X$ is θ^ω-closed if $M = cl_{\theta^\omega} M$.

Definition 7.6. A regular space is *regular functionally compact* (RFC) if every continuous function onto a regular space is closed.

Theorem 7.7. *A regular space X is RFC if and only if X is R-closed, and, if there exists a θ^ω-closed set C such that infinite set A does not $\theta(\omega)$-converge to $X \setminus C$, then C is a complete accumulation set of A.*

Proof. A proof of Theorem 7.7 is analogous to that of Theorem 6.2. □

Question 7.8. Is every RFC space necessarily compact?

Remark 7.9. Note that any non-compact H-closed Urysohn space is U-closed and the closure of any open set is also U-closed. This provides a negative answer to question Q27 in [4].

Question 7.10 ([4, Q26])**.** Is an R-closed space in which the closure of every open set is R-closed necessarily compact?

References

[1] P. S. Aleksandrov and P. S. Urysohn, *Memuar o Kompaktnykh Topologicheskikh Prostranstvakh.* [Memoir on Compact Topological Spaces]. Third edition. Moscow: Izdat. "Nauka," 1971.

[2] Manuel P. Berri and R. H. Sorgenfrey, *Minimal regular spaces*, Proc. Amer. Math. Soc. **14** (1963), 454–458.

[3] D. Dikranjan and E. Giuli, *$S(n)$-θ-closed spaces*, Topology Appl. **28** (1988), no. 1, 59–74.

[4] Louis M. Friedler, Mike Girou, Dix H. Pettey, and Jack R. Porter, *A survey of R-, U-, and CH-closed spaces*, Topology Proc. **17** (1992), 71–96.

[5] Louis M. Friedler and Jack R. Porter, *Compactly functionally compact spaces*, Houston J. Math. **22** (1996), no. 4, 775–785.

[6] Horst Herrlich, *T_v-abgeschlossenheit und T_v-minimalität*, Math. Z. **88** (1965) 285–294.

[7] Shouli Jiang, Ivan Reilly, and Shuquan Wang, *Some properties of $S(n)$-θ-closed spaces*, Topology Appl. **96** (1999), no. 1, 23–29.

[8] G. A. Kirtadze, *Different types of completeness of topological spaces* (Russian), Mat. Sb. (N.S.) **50 (92)** (1960), 67–90.

[9] A. V. Osipov, *An example of the nonfeebly compact product of U-θ-closed spaces*, Proc. Steklov Inst. Math. **2001** Algebra. Topology, suppl. 2, S186–S188.

[10] ——— *Different kinds of closedness in $S(n)$-spaces*, Proc. Steklov Inst. Math. **2003** Asymptotic Expansions. Approximation Theory. Topology, suppl. 1, S155–S160.

[11] Jack Porter, *Almost H-closed*, Topology Proc. **38** (2011), 301–308.

[12] Jack R. Porter and Charles Votaw, *$S(\alpha)$ spaces and regular Hausdorff extensions*, Pacific J. Math. **45** (1973), no. 1, 327–345.

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